

Final Exam Study Guide

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1 First Order Equations

1.1 First order linear equations $y' + p(t)y = g(t)$

Integrating factor: $\mu(t) = e^{\int p(t)dt}$

Multiply the equation by μ : $(\mu y)' = \mu g(t)$

1.2 Separable equations

These equations can be written in the form $M(x)dx + N(y)dy = 0$ (or $P(y)y' = Q(x)$). Integrate both sides to get the solution. Sometimes the solution may be implicit. You should find the correct explicit solution.

1.3 Autonomous equations $y' = f(y)$

Equilibrium solutions (critical points): can be solved from $f(y) = 0$.

On the phase line: right arrow if $f(y) > 0$; left if $f(y) < 0$.

Determine types of critical points and end behaviors of solutions accordingly.

1.4 Exact equations

An equation in the form $M(x, y)dx + N(x, y)dy = 0$ (compare this form with separable equations) is exact if $\partial M/\partial y = \partial N/\partial x$.

Solution: $F(x, y) = c$

where $F(x, y)$ satisfies $\partial F/\partial x = M(x, y)$ and $\partial F/\partial y = N(x, y)$. To find F , integrate, for example, the first equation with respect to x . Instead of a constant the result should contain some function of y . Apply the second equation to determine this function.

2 Homogeneous Linear Equations

2.1 Solution to Equations with Constant Coefficients

Use characteristic equation. Each root corresponds to a solution of the form e^{rt} (real root, $te^{rt}, \dots, t^{n-1}e^{rt}$ if repeated n times), or $e^{at} \cos(bt)$ and $e^{at} \sin(bt)$ (complex roots $a \pm bi$, $te^{at} \cos / \sin(bt), \dots, t^{n-1}e^{at} \cos / \sin(bt)$ if repeated n times).

The general solution is the linear combination of these solutions.

2.2 End Behavior of a Solution

End behavior is determined by the sign of the real root or the real part of the complex root. As $t \rightarrow \infty$, $y \rightarrow \infty$ if the sign is positive; $y \rightarrow 0$ if the sign is negative. The solution has no limit but oscillates if the roots are pure imaginary.

3 Fundamental set of solutions

Two functions $f(x), g(x)$ are linearly DEPENDENT if $f(x) = cg(x)$ for some constant $c \neq 0$. Two solutions y_1 and y_2 of a second order linear homogeneous equation can form a fundamental set of solutions if they are linearly INDEPENDENT. In this case the general solution is $y = c_1y_1 + c_2y_2$.

4 Existence and uniqueness theorems

For INITIAL VALUE problems

$$y' + p(t)y = g(t), y(t_0) = y_0$$

or

$$y'' + p(t)y' + q(t)y = g(t), y(t_0) = y_0, y'(t_0) = y'_0$$

The longest interval on which a unique solution is guaranteed to exist, is the longest interval CONTAINING t_0 , on which $p(t)$, $q(t)$ and $g(t)$ are continuous.

5 Spring-Mass Systems

5.1 Undamped Free Vibrations

$$mu'' + ku = 0$$

General solution: $u = A \cos \omega_0 t + B \sin \omega_0 t = R \cos(\omega_0 t - \delta)$

Amplitude: $R = \sqrt{A^2 + B^2}$

Natural Frequency: $\omega_0 = \sqrt{k/m}$

Period: $T = 2\pi/\omega_0$

5.2 Damped Free Vibrations

$$mu'' + \gamma u' + ku = 0$$

The system is losing its energy so every solution asymptotically approaches zero as $t \rightarrow \infty$, regardless of the type of damping.

5.2.1 Over-damping and Critical damping

The characteristic equation may have two (negative) real roots, repeated (negative) real roots or two complex roots (with negative real parts).

Over-damping: $\gamma^2 - 4km > 0$, two real roots, solution does not oscillate

Critical damping: $\gamma^2 - 4km = 0$, repeated roots, solution does not oscillate

In both cases, the solution does not oscillate and can cross the equilibrium position at most once.

5.2.2 Under-damping

$\gamma^2 - 4km < 0$, two complex roots, solution oscillates with decreasing amplitude and crosses the equilibrium position infinitely many times.

5.3 Undamped Forced Vibration

$$mu'' + ku = F_0 \cos \omega t$$

Resonance: $\omega_0 = \omega$, where $\omega_0 = \sqrt{k/m}$ is the natural frequency

6 Laplace Transform

6.1 Definition

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

Additivity: $\mathcal{L}\{af(t) \pm bg(t)\} = a\mathcal{L}\{f(t)\} \pm b\mathcal{L}\{g(t)\}$.

6.2 Solving Initial Value Problems

$$y'' + py' + qy = g(t), y(0) = y_0, y'(0) = y'_0$$

Step 1: Laplace transform:

$$s^2Y(s) - sy(0) - y'(0) + p(sY(s) - y(0)) + qY(s) = \mathcal{L}\{g(t)\}$$

where $Y(s) = \mathcal{L}\{y\}$

Step 2: Plug in initial conditions and solve for $Y(s)$ from the equation above.

Step 3: Take inverse transform and find $y = \mathcal{L}^{-1}\{Y(s)\}$

6.3 Step Functions

6.3.1 Unit Step Functions

$$u_c(t) = u(t - c) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

6.3.2 Laplace Transform of Step Functions

$$\begin{aligned}\mathcal{L}\{u(t-c)\} &= e^{-cs}/s \\ \mathcal{L}\{u(t-c)f(t)\} &= e^{-cs}\mathcal{L}\{f(t+c)\}\end{aligned}$$

6.3.3 Inverse Transform

To find the inverse transform for $e^{-cs}F(s)$, first find $f(t) = \mathcal{L}^{-1}\{F(s)\}$. Then

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c)$$

7 Homogeneous Linear Systems with Constant Coefficients

7.1 Solving the Equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$

Step 1: Look for eigenvalues: set $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and solve for λ .

Step 2: Look for eigenvectors: Plug in the λ you get in step 1, find a nonzero vector ξ such that $(\mathbf{A} - \lambda\mathbf{I})\xi = 0$

Step 3: 3 cases here depending on the roots of the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

7.1.1 Different Real Roots

General solution: $\mathbf{x} = c_1e^{\lambda_1 t}\xi_1 + c_2e^{\lambda_2 t}\xi_2$

Phase portrait: Two straight lines; Critical point (origin) is a node or a saddle point.

7.1.2 Complex Roots

Take one complex root $a + bi$ and its eigenvector ξ . Find real valued vectors ξ_1 and ξ_2 such that

$$e^{(a+bi)t}\xi = e^{at}(\cos bt + i \sin bt)\xi = \xi_1 + i\xi_2$$

Then the general solution is $\mathbf{x} = c_1\xi_1 + c_2\xi_2$

Phase portrait: Spiral or center (if the complex roots have zero real parts)

7.1.3 Repeated Roots

Find a vector η such that $(\mathbf{A} - \lambda\mathbf{I})\eta = \xi$

The general solution is $\mathbf{x} = c_1e^{\lambda t}\xi + c_2e^{\lambda t}(t\xi + \eta)$

Phase portrait: Improper node (one straight line) or proper node (many straight lines. This happens if $A - \lambda I=0$)

7.2 Phase Portrait and Stability

A detailed summary with pictures is provided in Section 9.1. You may find Table 9.1.1 on page 494 and Figure 9.1.9 on page 497 helpful.

8 Nonlinear Systems $\begin{cases} x' = F(x, y) \\ y' = G(x, y) \end{cases}$

Critical points: can be solved from $\begin{cases} F(x, y) = 0 \\ G(x, y) = 0 \end{cases}$.

Type of the critical point (x_0, y_0) : determined by the linear system with the matrix $\begin{pmatrix} \partial F/\partial x(x_0, y_0) & \partial F/\partial y(x_0, y_0) \\ \partial G/\partial x(x_0, y_0) & \partial G/\partial y(x_0, y_0) \end{pmatrix}$. For correspondence between linear and nonlinear systems, see Table 9.3.1 on Page 513.

9 Two-point Boundary Value Problems

Solve the equation and find the general solution. Then determine the constants according to the give boundary conditions.

Eigenvalues: values of λ such that the problem has nontrivial (nonzero) solutions.

Eigenfunctions: nontrivial solutions corresponding to eigenvalues.

10 Fourier Series

10.1 Definition

Fourier series for the periodic function $f(x)$ with period $2L$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

with coefficients

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

10.2 Fourier Convergence Theorem

The Fourier series of $f(x)$ converges to $f(x)$ at all points where $f(x)$ is continuous, and to $[f(x+) + f(x-)]/2$ (the average of the limits from both sides) at all points where f is discontinuous.

10.3 Odd Functions

If $f(x)$ is an odd function, then the coefficients $a_n = 0$ (including a_0) and

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

The series $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$ is called the Fourier sine series.

10.4 Even Functions

If $f(x)$ is an even function, then the coefficients $b_n = 0$ and

$$a_0 = \frac{2}{L} \int_0^L f(x) dx, a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

The series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$ is called the Fourier cosine series.

10.5 Periodic Extension

Suppose a given function $f(x)$ is defined on $(0, L)$

10.5.1 Odd Periodic Extension with Period $2L$

If looking for sine series, apply the formula in 9.3

10.5.2 Even Periodic Extension with Period $2L$

If looking for cosine series, apply the formula in 9.4

10.5.3 Periodic Extension with Period L

Find the expression for $f(x)$ on $(-L/2, 0)$ then apply the formulas in 9.1 (Note in this case L in the formulas should be replaced by $L/2$)

10.5.4 Sketch the Graph of the Extended Function Over a Few Periods

11 Partial Differential Equations

11.1 Separation of Variables

To solve a partial differential equation, assume there are solutions of the form $u(x, t) = X(x)T(t)$. Plug it into the partial differential equation to see if the two variables x and t can be separated. If they can be separated, then they must be equal to the same constant number, assumed as λ . Then the partial differential equation can be separated into two ordinary differential equations.

11.2 Heat Conduction Problem with Homogeneous Boundary Conditions

$$\alpha^2 u_{xx} = u_t, u(0, t) = 0, u(L, t) = 0, u(x, 0) = f(x)$$

Step 1: Apply the method of separation of variables to separate the equation into two ODEs

$$X''(x) + \lambda X(x) = 0, T'(t) + \alpha^2 \lambda T(t) = 0$$

Find the boundary conditions for the ODE: $X(0) = 0, X(L) = 0$

Step 2: Solve the two-point boundary value problem and get the eigenvalues $\lambda_n = (n\pi/L)^2$ and the corresponding eigenfunction $X_n(x) = \sin \frac{n\pi x}{L}$.

Step 3: Plug in the eigenvalues and solve the other ODE,

$T_n(t) = c_n e^{-n^2 \pi^2 \alpha^2 t / L^2}$. For each eigenvalue λ_n , the product $u_n(x, t) = X_n(x)T_n(t)$ is a solution to the PDE satisfying the homogeneous boundary conditions.

Step 4: Find the solution as the linear combination of the solutions obtained in Step 3 which also satisfies the initial condition $u(x, 0) = f(x)$. (Plug in the linear combination you will see the sine series for $f(x)$).

Solution:

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t / L^2} \sin \frac{n\pi x}{L}$$

where

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

11.3 Heat Conduction Problem with Nonhomogeneous Boundary Conditions

$$\alpha^2 u_{xx} = u_t, u(0, t) = T_1, u(L, t) = T_2, u(x, 0) = f(x)$$

11.3.1 Steady-state Solution

The steady-state solution is the solution to the equation satisfying the nonhomogeneous boundary conditions that has the form $v(x, t) = v(x)$ (independent of t). For the above problem, the steady-state solution is $v(x) = \frac{T_2 - T_1}{L} x + T_1$.

11.3.2 solution of the problem

The solution is the sum of the steady-state solution and the solution to the problem with homogeneous boundary condition.

$$u(x, t) = v(x) + \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t / L^2} \sin \frac{n\pi x}{L}$$

where

$$c_n = \frac{2}{L} \int_0^L (f(x) - v(x)) \sin \frac{n\pi x}{L} dx$$

11.4 Heat Conduction with Insulated Ends

$$\alpha^2 u_{xx} = u_t, u_x(0, t) = 0, u_x(L, t) = 0, u(x, 0) = f(x)$$

Follow the steps in 10.2. Note in this case the boundary condition for the ODE is $X'(0) = 0, X'(L) = 0$. Hence the eigenvalues and eigenfunctions for the two-point boundary value problem are $\lambda_n = (n\pi/L)^2$ and $X_n(x) = \cos \frac{n\pi x}{L}$. Also note here n can take the value zero.

Solution:

$$u(x, t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t / L^2} \cos \frac{n\pi x}{L}$$

where

$$c_0 = \frac{2}{L} \int_0^L f(x) dx, c_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

11.5 The Wave Equations

11.5.1 1st Type

$$a^2 u_{xx} = u_{tt}, u(0, t) = 0, u(L, t) = 0, u(x, 0) = f(x), u_t(x, 0) = 0$$

Solution:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L}$$

where

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

11.5.2 2nd Type

$$a^2 u_{xx} = u_{tt}, u(0, t) = 0, u(L, t) = 0, u(x, 0) = 0, u_t(x, 0) = g(x)$$

Solution:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \sin \frac{n\pi at}{L}$$

where

$$\frac{n\pi a}{L} c_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

This study guide may contain mistakes or typos. If you find any, please tell me as soon as possible.

This study guide may be revised before the final exam.

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