MEASURES OF INTERMEDIATE ENTROPIES FOR SKEW PRODUCT DIFFEOMORPHISMS

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ABSTRACT. In this paper we study a skew product map F with a measure μ of positive entropy. We show that if on the fibers the map are $C^{1+\alpha}$ diffeomorphisms with nonzero Lyapunov exponents, then there are ergodic measures of intermediate entropies. To construct these measures we find a set on which the return map is a skew product with horseshoes along fibers. We can control the average return time and show the maximum entropy of these measures can be arbitrarily close to $h_\mu(F)$.

CONTENTS

1. INTRODUCTION

Entropy has been the centerpiece in dynamics. It is related to the exponential growth rate of divergence of orbits and reflects the complexity of the system. In general, positive entropy implies some (partially) hyperbolic structure of the system, which usually guarantees the existence of plenty of periodic orbits or invariant measures. We would like to characterize this aspect of entropy for a large class of systems. The work presented here is just another chapter of this story.

Let us consider a compact Riemannian manifold M and a $C^{1+\alpha}$ ($\alpha > 0$, this means the derivative is α-Hölder continuous) diffeomorphism f on M. f preserves an ergodic measure μ. Then there are real numbers λ_k, $k = 1, 2, \ldots, l$, such that for μ-almost every point x, there are subspaces $E_k(x)$ of the tangent space T_xM such that for any vector $v \in E_k(x)\backslash\{0\}$, we have $\lambda(v) := \lim_{n\to\infty} \log \|df^n v\|/n = \lambda_k$. l is some integer number no more than the dimension of M and these subspaces are invariant of df, the derivative of f. These numbers λ_k are called Lyapunov exponents. We say μ is a hyperbolic measure if all Lyapunov exponents are nonzero.

Years ago, A. Katok established a remarkable result as following:

Theorem 1.1. (Katok, [5, 6]) *If the metric entropy $h_\mu(f) > 0$ and μ is an ergodic hyperbolic measure, then for any $\epsilon > 0$, there is a hyperbolic horseshoe $\Lambda \subset M$ such that $h(f|\Lambda) > h_\mu(f) - \epsilon$.*

This theorem has an interesting corollary: Under the conditions of the theorem, there are ergodic measures μ_β such that $h_{\mu_\beta}(f) = \beta$ for any real number $\beta \in [0, h_\mu(f)]$. Since the horseshoe map is a full shift, these measures can be constructed by taking sub-shifts or properly assigning weights to different symbols. Existence of these measures of intermediate entropies exhibits the complicated structure of the system with positive entropy.

So far it is not known whether every smooth system ($C^{1+\alpha}$ diffeomorphism) on compact manifold with positive (topological) entropy has this property. In general, such a system may not have any hyperbolic measure. Herman constructed a well-known example, which is a minimal C^∞ diffeomorphism with positive topological entropy [3]. So even a closed invariant subset would not be expected. Fortunately, minimality does not prevent the system from having measures of intermediate entropies, at least in Herman's example. Herman suggested the question that whether positive topological entropy smooth systems are not uniquely ergodic. Katok conjectured a stronger statement: they have measures of arbitrary intermediate entropies. We are working towards this statement.

In this paper we deal with skew product maps with nonzero Lyapunov exponents along fiber directions. In precise, let $F = (g, f_x)$ be a skew product map on the space $X \times Y$ preserving a measure $\mu = \int \sigma_x d\nu$. By taking an ergodic component we may assume μ is ergodic. g is an invertible (mod 0) measure preserving transformation on the probability space (X, ν). g is ergodic by the assumption. For every $x \in X$, f_x is a $C^{1+\alpha}$ diffeomorphism on the compact smooth manifold Y.

Theorem 1.2. (Main Theorem) *Assume that $h_\mu(F) > 0$ and $h_\nu(g) = 0$. If for almost every $z = (x, y) \in X \times Y$ and every $v \in T_y(\{x\} \times Y)\backslash\{0\}$, the Lyapunov exponent*

$$\lambda(v) = \lim_{n\to\infty} \frac{\log \|df_{g^{n-1}(x)} \cdots df_{g(x)} df_x v\|}{n} \neq 0,$$

then F has ergodic invariant measures of arbitrary intermediate entropies.

Our argument also works for $h_\nu(g) > 0$. In this case it concludes that there are ergodic measures of entropies between $h_\nu(g)$ and $h_\mu(F)$. We assume g has no periodic point, or else the problem is reduced to Theorem 1.1. We have shown in [8] that under the conditions there are measures of zero entropy. Some lemmas are generalized and adapted in this paper.

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2. Entropy and Separated Sets on Fibers

This section is devoted to the discussion on the entropy of a skew product diffeomorphism. We are aiming to get an estimate on the cardinality of the (m, ϵ)-separated set on each fiber, which is analogous to the definition of metric entropy by Katok [4].

Let η be a measurable partition of the fiber Y with finite entropy $H_x(\eta) < \infty$ for almost every $x \in X$, where $H_x(\eta) = \sum_{\mathcal{C}\in\eta} \sigma_x(\mathcal{C}) \log \sigma_x(\mathcal{C})$. Let us put

$$\eta_x^n = \bigvee_{k=0}^{n-1} f_x^{-1} f_{g(x)}^{-1} \cdots f_{g^{k-1}(x)}^{-1} \eta$$

Theorem 2.1. (Abramov and Rohlin, [1]) *For every η, put*

$$h^g(f,\eta) = \lim_{n\to\infty} \frac{1}{n} \int_X H_x(\eta_x^n) d\nu$$

The limit exists and it is finite. Let

$$h^g(f) = \sup_\eta h^g(f,\eta)$$

$h^g(f)$ is called the fiber entropy. We have

$$h_\mu(F) = h_\nu(g) + h^g(f)$$

Since for the skew product diffeomorphisms we considered, g is ergodic and, for almost every $x \in X$, $\sigma_x \circ f_x^{-1} = \sigma_{f(x)}$, i.e. f_x preserves the conditional measure on the fiber, we have for almost every x,

$$h_x^g(f,\eta) = \lim_{n\to\infty} \frac{1}{n} H(\eta_x^n) = h^g(f,\eta)$$

There is a version of Shannon-McMillan-Breiman Theorem from Random Dynamics theory which works for our setting.

Theorem 2.2. ([7], Theorem 1.1.4) *Let $\mathcal{C}_x^n(y)$ be the element of η_x^n containing y. Then for almost every $(x,y) \in X \times Y$,*

$$\lim_{n\to\infty} \frac{1}{n} \nu_x(\mathcal{C}_x^n(y)) = -h_x^g(f,\eta) = -h^g(f,\eta)$$

Let $d_n^F(z,z')$ be an increasing system of metrics defined on $X \times Y$ by:

$$d_n^F(z,z') = \max_{0\le i\le n-1} d(F^i z, F^i z')$$

If z and z' are on the same fiber $\{x\} \times Y$, then this metric induces a metric on the fiber. For $x \in X$ and $\delta > 0$, on the fiber $\{x\} \times Y$, let $\mathcal{N}_x^F(n,\epsilon,\delta)$ be the minimal number of ϵ-balls in the d_n^F-metric needed to cover a set of σ_x-measure at least $1-\delta$, and let $\mathcal{S}_x^F(n,\epsilon,\delta)$ be the maximal cardinality of a (d_n^F,ϵ)-separated set inside every set of σ_x-measure at least $1-\delta$.

We can follow exactly Katok's argument for [4, Theorem 1.1] and get the analogous result:

Theorem 2.3. *If F is ergodic, then for almost every $x \in X$ and every $\delta > 0$,*

$$h^g(f) = \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{\log \mathcal{N}_x^F(n,\epsilon,\delta)}{n} = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log \mathcal{N}_x^F(n,\epsilon,\delta)}{n}$$

And we also have

$$h^g(f) = \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{\log \mathcal{S}_x^F(n,\epsilon,\delta)}{n} = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log \mathcal{S}_x^F(n,\epsilon,\delta)}{n}$$

3. Properties of Recurrence

In this section we discuss some properties related to nontrivial recurrence of the map. For the skew product, recurrence of points in a set other than unions of whole fibers, can be very complicated. For example, points on same fiber may return to different fibers. We need to control the recurrence as we wish.

Theorem 3.1 is a generalization of [8, Proposition 3.2]. We still use only the special version in this paper.

Theorem 3.2 is crucial and is the most technical part of this paper. It reveals a fact that, for a complicated return map on some set of positive measure, if the return time is integrable, then this return map is one-to-one and measure preserving on a subset of positive measure. Moreover, we have an estimate of the integral of the return time on this invariant subset, which can be used to estimate the size (measure) of the subset.

Theorem 3.1. (Integrability of Return Time) *Let $P \subset X \times Y$ be a measurable subset. $B = \pi(P) \subset X$ is the projection of P on the base. For $x \in B$, denote $P \cap (\{x\} \times Y)$ by $P(x)$. Assume that $\nu(B) = \nu_0 > 0$ and there is $\sigma_0 > 0$ such that for (almost) every $x \in B$, $\sigma_x(P(x)) > \sigma_0$. Hence $\mu(P) = \mu_0 > \nu_0\sigma_0 > 0$. For (almost) every $z \in P$, denote by $\tilde{n}_l(z)$ the l-th return time of z. Let*

$$P_n^l(x) = \{z \in P(x) | \tilde{n}_l(z) \geq n\},\ N_l(x) = \max\{n | \nu_x(P_n^l(x)) > \sigma_0\}$$

Then

$$\int_B N_l(x) d\nu \leq \frac{l}{\sigma_0} < \infty$$

Remark. $N_l(x)$ is the longest return time for the l-th returns of the points in a subset of conditional measure no less than $\sigma_x(P(x)) - \sigma_0$ in $P(x)$. Or equivalently, $N_l(x)$ is the smallest number such that the set of points in $P(x)$ with l-th return times greater than $N_l(x)$ has conditional measure at most σ_0.

Proof. Since μ is F-invariant and $\mu(P) > 0$, we have:

$$0 < \int_P \tilde{n}_1(z) d\mu = \mu(\bigcup_{j \geq 0} F^j(P')) \leq 1$$

Hence

$$0 < \int_P \tilde{n}_l(z) d\mu = l \int_P \tilde{n}_1(z) d\mu \leq l$$

Note

$$\int_P \tilde{n}_l(z) d\mu = \sum_{j=1}^{\infty} \mu(P_j)$$

where $P_j = \{z \in P | \tilde{n}_l(z) \geq j\}$.

For every $x \in B$, let $B_j = \{x \in B | N_l(x) \geq j\}$. By definition, $P_n^l(x) = P_n \cap P(x)$, which implies that $x \in B_j$ iff $\sigma_x(P_j \cap P(x)) > \sigma_0$. So we must have $\mu(P_j) > \nu(B_j) \cdot \sigma_0$, and

$$\int_B N_l(x) d\nu = \sum_{j=1}^{\infty} \nu(B_j) < \sum_{j=1}^{\infty} \frac{1}{\sigma_0} \mu(P_j) \leq \frac{l}{\sigma_0}$$

□

Theorem 3.2. *Let g be a measure preserving transformation on a probability space (X,ν). g is invertible and has no periodic point. B is a subset of X with $\nu(B) = \nu_0 > 0$. $N : B \to \mathbb{N}$ is a measurable function such that $\tilde{g}(x) := g^{N(x)}(x) \in B$ for almost every $x \in B$. Assume $\int_B N(x)d\nu = \Sigma_B < \infty$. Then there is a subset B_1 of B such that the following holds: $\nu(B_1) = \nu_1 > 0$, $\tilde{g}(B_1) \subset B_1$ and $g_* = \tilde{g}|_{B_1}$ is invertible and ν-preserving. Moreover,*

$$\int_{B_1} N(x)d\nu \geq \nu(\bigcup_{k\in\mathbb{Z}} g^k(B))$$

Remark. This theorem is nontrivial because the map $\tilde{g}$ is not necessarily the first return, but just some return. So $\tilde{g}$ is just a measurable transformation on B which may be neither injective nor surjective. However, we are able to find a subset of B on which it is invertible, provided the integrability of return times.

We define a partial order on B: $x_1 \prec x_2$ iff there is $n \geq 0$ such that $g_B^n(x_1) = x_2$, i.e. x_2 is an image of x_1 under iterates of g. Since g is invertible and has no periodic point, this partial order is well defined.

Let $O^+(x) = \{\tilde{g}^k(x) | k \in \mathbb{N} \cup \{0\}\}$ be the positive $\tilde{g}$-orbit of x. We define an equivalence relation on B: $x_1 \sim x_2$ iff $Q(x_1,x_2) := O^+(x_1) \cap O^+(x_2) \neq \emptyset$, i.e. there are $n_1, n_2 > 0$ such that $\tilde{g}^{n_1}(x_1) = \tilde{g}^{n_2}(x_2)$. Note $x_2 \in O^+(x_1)$ or $x_2 \in O^+(x_1)$ implies $x_1 \sim x_2$, but the converse is not true. Also note within an equivalence class the partial order we defined is a total order. We showed in [8]:

Proposition 3.3. *For almost every $x \in B$, there is an element $x' \precneqq x$ such that $x' \sim x$. Consider the set*

$$G(x) = \bigcup_{x_1 \sim x} (\bigcap_{x_2 \sim x \text{ and } x_2 \prec x_1} O^+(x_2))$$

Then $G(x)$ is a nonempty subset of B.

Proof of Theorem 3.2. For almost every $x \in B$ and $\tilde{x} \in G(x)$, there are infinitely many elements in B such that their positive orbits contain $\tilde{x}$. However, by integrability of return time ($\int_B N(x)d\nu = \Sigma_B < \infty$), the pre-image $\tilde{g}^{-1}(\tilde{x})$ consists of finite number of elements. Let $H(\tilde{x}) = \{x' | \tilde{g}(x') = \tilde{x}$ and there are infinitely many elements $\{x_n\}_{n\in\mathbb{N}}$ such that $x' \in O^+(x_n)\}$. Then $H(\tilde{x}) \neq \emptyset$. Define $\bar{g}(\tilde{x}) = \hat{x}$, if there is such $\hat{x} \in G(x)$ that $\tilde{g}(\hat{x}) = \tilde{x}$; otherwise, $\bar{g}(\tilde{x}) = \min H(\tilde{x})$. Such $\hat{x}$ must be unique since any element in $G(x)$ "behind" (or "greater" according to the partial order) $\hat{x}$ must be contained in $O^+(\hat{x})$. So $\bar{g}$ is a well defined and $\tilde{g} \circ \bar{g} = Id$. With the same argument we have $H(\bar{g}(\tilde{x})) \neq \emptyset$, hence $\bar{g}^n(x)$ can be well defined for any $n \in \mathbb{N}$.

Define $\tilde{G}(x)$ as following: $\tilde{G}(x) = G(x)$ if $G(x)$ has no minimal element; $\tilde{G}(x) = G(x) \cup (\bigcup_{n=1}^\infty \{\bar{g}^n(\bar{x})\}$ if $\bar{x}$ is the minimal element of $G(x)$. Note for every $x' \in \tilde{G}(x)$, $\{x'' \in \tilde{G}(x) | x' \prec x''\} = O^+(x')$. Let $B_1 = \bigcup_{x\in B} \tilde{G}(x)$.

B_1 is invariant under $\tilde{g}$. For $x' \in B_1$, there is $x \in B$ such that $x' \in \tilde{G}(x)$. We have $\tilde{g}(x') \in \tilde{G}(x) \in B_1$ by definition of $\tilde{G}(x)$.

$\tilde{g}|_{B_1}$ is surjective. For $x' \in B_1$, $x' \in \tilde{G}(x)$ for some x. If $x' \in G(x)$ and x' is not the minimal element of $G(x)$ ($G(x)$ may have no minimal element), then there must be $x'' \in G(x)$ and $x'' \precneqq x'$. By definition of $G(x)$ we must have $x' \in O^+(x'')$ since they are both on the positive orbit of some element equivalent to x. So $x' = \tilde{g}^k(x'')$ and $\tilde{g}^{k-1}(x'') \in B_1$. If $G(x)$ has a minimal element $\bar{x}$ and $x' = \bar{g}^k(\bar{x})$ for some $k \geq 0$, then $x' = \tilde{g}(\bar{g}^{k+1}(\bar{x}))$ and $\bar{g}^{k+1}(\bar{x}) \in B_1$.

$\tilde{g}|_{B_1}$ is injective. If for $x_1,\, x_2 \in B_1$, we have $\tilde{g}(x_1) = \tilde{g}(x_2)$, then $x_1 \sim x_2$. We may assume $x_1 \prec x_2$. Note by definition if $x_1 \sim x_2$ then $\tilde{G}(x_1) = \tilde{G}(x_2)$. So we must have $x_2 \in O^+(x_1)$. $\tilde{g}(x_1) = \tilde{g}(x_2) = \tilde{g}(\tilde{g}^k(x_1))$. k must be zero and $x_1 = x_2$.

B_1 has positive measure. Note $B_2 = \bigcup_{x\in B} G(x) = \bigcap_{i=0}^{\infty} \tilde{g}^i(B)$ is measurable. $B_1 = \bigcup_{i=0}^{\infty} \bar{g}^i(B_2)$ is measurable. But $\bigcup_{n\in\mathbb{Z}} g^n(B_1) \supset B$ and B has positive measure.

In fact, let $D_k = \{x \in B_1 | N(x) = k\}$ for $k = 1, 2, \cdots$. Then consider

$$B_2 = \bigcup_{k=1}^{\infty}(\bigcup_{j=0}^{k-1} g^j(D_k))$$

We must have $B_2 = \bigcup_{k\in\mathbb{Z}} g^k(B)$. First note $g^k(D_k) \subset B_1 \subset B_2$, so B_2 is g-invariant. $g(B_2) = B_2$. By construction of B_1, for every $x \in B$, there is $l \subset \mathbb{N}$ such that $g^l(x) \in B_1$. Hence $B \subset B_2$. and $\bigcup_{k\in\mathbb{Z}} g^k(B) \subset B_2$. On the other hand, $D_k \subset B_1$ and $B_2 \subset \bigcup_{k\in\mathbb{Z}} g^k(B_1) \subset \bigcup_{k\in\mathbb{Z}} g^k(B)$. Furthermore,

$$\int_{B_1} N(x)d\nu = \sum_{k=1}^{\infty} k \cdot \nu(D_k) \geq \nu(B_2) = \nu(\bigcup_{k\in\mathbb{Z}} g^k(B))$$

$g_* = \tilde{g}|_{B_1}$ preserves ν. $D_i \bigcap D_j = \emptyset$ and $g_*(D_i) \bigcap g_*(D_j) = \emptyset$ for $i \neq j$, and $g_*|_{D_k} = g^k$ preserves ν. For any measurable subset $B' \subset B_1$, $B' = \bigcup_{1\leq n<\infty} B'_n$, where $B'_n = B' \bigcap D_n \subset D_n$. We have

$$\nu(g_*(B')) = \sum_{1\leq n<\infty} \nu(g_*(B'_n)) = \sum_{1\leq n<\infty} \nu(B'_n) = \nu(B')$$

□

4. Proof of the Main Theorem

4.1. **Regular Tube.** If on the fiber direction we have no zero Lyapunov exponents, then from Pesin theory [2, 6] we know for almost every point z there is regular neighborhood on the fiber. Inside each regular neighborhood we can introduce a local chart and identify a "rectangle" with the square $[-1,1]^2$ (z with 0) in Euclidean space (in higher dimension this should be recognized as the product of balls of radii 1 in dimensions corresponding to contracting and expanding directions).

Fix some small number $\gamma > 0$, we can define admissible (s,γ)-curves as the graphs $\{(\theta, \psi(\theta))|\theta \in [-1,1]\}$ and admissible (u,γ)-curves as $\{(\psi(\theta), \theta)|\theta \in [-1,1]\}$, where $\psi : [-1,1] \to [-1,1]$ is a C^1 map with $|\psi'| < \gamma$. These admissible curves are preserved under iterations of F^{-1} and F respectively, we mean, there is some $0 < h < 1$ such that if in addition $|\psi(0)| < h$ then their images are also admissible curves of the same types, respectively.

Consider admissible (s,γ)-rectangles defined as the set of points

$$\{(u,v) \in [-1,-1]^2 | v = \omega\psi_1(u) + (1-\omega)\psi_2(u), 0 \leq \omega \leq 1\}$$

where ψ_1 and ψ_2 are admissible (s,γ)-curves, and (u,γ)-rectangles defined analogously. Then these rectangles are also preserved by F^{-1} and F, respectively (in the sense described above).

Let us fix small numbers $\epsilon > 0$ and $r > 0$.

Proposition 4.1. *There is a "Regular Tube" P, which is a measurable subset of $X \times Y$ satisfying the following properties:*

(1) $\mu(P) = \mu_0 > 0$.
(2) *Let* $\pi : P \to X$ *be the projection to the base and let* $B = \pi(P)$. *Then* $\nu(B) = \nu_0 > 0$.
(3) *For every* $x \in B$, *there is a rectangle* $R(x)$ *on the fiber* $\{x\} \times Y$ *whose diameter is less than* $\epsilon/2$. $R(x) \subset \mathcal{R}(z)$ *where* $\mathcal{R}(z)$ *is the Lyapunov regular neighborhood of some point* $z = (x, y) \in X \times Y$ *on the fiber* $\{x\} \times Y$. *Let* $P(x) = P \cap (\{x\} \times Y)$. *Then* $P(x) \subset R(x)$ *for every* $x \in B$.
(4) *There is some number* $\sigma_0 > 0$ *such that, for every* $x \in B$, $\sigma_0 < \sigma_x(P(x)) < \sigma_0(1 + r)$.
(5) *Applying Theorem 2.3, we may assume that there is some* $m_1 > 0$ *such that for every* $m > m_1$ *and* $x \in B$, *inside any set of* σ_x*-measure at least* $\sigma_0/2$ *on the fiber* $\{x\} \times Y$, *we can find a* (d^F_m, ϵ)*-separated set with cardinality at least* $\exp m(h_\mu(F) - r)$.
(6) *For every* $x \in B$ *and* $z \in P(x)$, *if for some* $n > 0$, $F^n(z)$ *returns to* P, *i.e.* $F^n(z) \in P(g^n(x))$, *then the connected component of the intersection* $F^n(R(x)) \cap R(g^n(x))$ *containing* $F^n(z)$, *denoted by* $CC(F^n(R(x)) \cap R(g^n(x)), F^n(z))$, *is an admissible* (u, γ)*-rectangle in* $R(g^n(x))$ *and* $CC(F^{-n}(R(g^n(x))) \cap R(x), z)$ *is an admissible* (s, γ)*-rectangle in* $R(x)$. *Moreover, for* $j = 0, 1, ..., n$, *on the fiber* $\{g^j(x)\} \times Y$ *we have*

$$\mathrm{diam} F^j(CC(F^{-n}(R(g^n(x))) \cap R(x), z)) < \epsilon$$

Proof. This regular tube can be obtained with the following steps.
(1) On almost every fiber, find a regular point $z \in \{x\} \times Y$ and its regular neighborhood $\mathcal{R}(z)$. Take $R(x) \in \mathcal{R}(z)$ with diameter less than $\epsilon/2$.
(2) Find $P(x) \subset R(x)$ satisfying property (6). There is some $\sigma_0 > 0$ such that $B_0 = \{x | \sigma_x(P(x)) > \sigma_0\} > 0$. For $x \in B_0$, shrink the size of $P(x)$ properly such that $\sigma_0 < \sigma_x(P(x)) < \sigma_0(1 + r)$.
(3) Find m_1 and $B \subset B_0$ such that $P = \bigcup_{x \in B} P(x)$ also satisfies property (5) and $\nu(B) > 0$. P is as required. □

4.2. **Control of Return Time.** Now let us start with a regular tube P. Apply Theorem 3.1, we can find a measurable section $q : B \to P$, $\pi \circ q = Id$ such that

$$\int_B N_1(x) d\nu \leq \frac{1}{\sigma_0}$$

where $N_1(x)$ is the first return time of $q(x)$.

Denote by χ_P the characteristic function of the measurable set P. Consider the sets

$$\mathcal{A}_n = \{z \in X \times Y | \text{ For every } k \geq n,$$
$$\sum_{i=1}^{k} \chi_P(F^i z) < k\mu_0(1 + \frac{r}{3}) \text{ and } \sum_{i=1}^{k(1+r)} \chi_P(F^i z) > k\mu_0(1 + \frac{2r}{3})\}$$

(throughout this paper, numbers like $k(1 + r)$ are rounded to the nearest integer, if needed). Since μ is ergodic, by Birkhoff Theorem we have for almost every z,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_P(F^i z) = \mu(P) = \mu_0$$

which implies

$$\lim_{n\to\infty} \mu(\mathcal{A}_n) = 1$$

Let $\mathcal{B}_n = \{x | \sigma_x(P(x) \cap \mathcal{A}_n) > \sigma_0(1-r)\}$. Then

$$\nu(B \backslash \mathcal{B}_n) \to 0 \text{ and } \int_{B \backslash \mathcal{B}_n} N(x) d\nu \to 0$$

So there is m_0 and a measurable subset $B_1 \subset B$ with the following properties

1. $\nu(B_1) > \nu_0(1-r)$. Let $P_1 = \pi^{-1}(B_1) \cap P$. $\mu(P_1) > \mu_0(1-r)$.
2. For $x \in B_1$, let $P_1(x) = P_1 \cap (\{x\} \times Y)$ and $P_1^{m_0}(x) = P_1(x) \cap \mathcal{A}_{m_0}$. Then $\sigma_x(P_1^{m_0}(x)) > \sigma_0(1-r)$.
3. Let $B_2 = B \backslash B_1$. Then $\int_{B_2} N_1(x) < r$.

Now let us fix $m > \max\{m_1, m_0\}$. For convenience, denote by $\mathcal{K} = [m\mu_0(1+\frac{r}{2})]$ the integer part of $m\mu_0(1+\frac{r}{2})$. For large m,

$$m\mu_0(1+\frac{r}{3}) < \mathcal{K} < m\mu_0(1+\frac{2r}{3})$$

For every $x \in B_1$, $P_1^{m_0}(x) > \sigma_0(1-r) > \sigma_0/2$, by property (5) of the regular tube, there is a (d_m^F, ϵ)-separated set $E(x) \subset P_1^{m_0}(x)$ with cardinality

$$|E(x)| > \exp m(h_\mu(F) - r)$$

For $z \in E(x) \subset \mathcal{A}_m$, the $\mathcal{K}$-th return time of z to P is an integer number between $m+1$ and $m(1+r)$. So there is $V(x) \subset E(x)$ with cardinality

$$|V(x)| = [\frac{1}{mr} \exp m(h_\mu(F) - r)]$$

and the $\mathcal{K}$-th return times for points in $V(x)$ are the same, denoted by $N(x)$. The set $\bigcup_{x \in B_1} V(x)$ can be viewed as the union of $[\frac{1}{mr} \exp m(h_\mu(F) - r)]$ measurable sections over B_1.

$N(x)$ is measurable function on B_1. We extend $N(x)$ to a measurable function on B: for $x \in B_2$, let $N(x) = N_1(x)$. Consider the map $g_*(x) = g^{N(x)}(x)$. $g_*(x)$ is well defined on B and $g_*(B) \subset B$. Moreover,

$$\int_B N(x) d\nu \le \int_{B_1} N(x) d\nu + \int_{B_2} N_1(x) d\nu < m(1+r) \cdot \nu_0 + r < \infty$$

Applying Theorem 3.2 we can find a set $B_3 \subset B$ of positive measure such that the map g_* restricted on B_3 is invertible and preserves μ. We can assume that $g_*|_{B_3}$ is ergodic (with respect to the measure induced by ν).

Let $B_4 = B_3 \cap B_1$ and $B_5 = B_3 \cap B_2$. Let $\mathcal{G}(x)$ be the first return map on B_4 with respect to g_*. Then $\mathcal{G}$ is invertible and preserves ν. Define the measurable function ρ and l on B_4 such that $\mathcal{G}(x) = g^{\rho(x)}(x) = g_*^{l(x)}(x)$. For $x \in B_4$,

$$\rho(x) = \sum_{j=0}^{l(x)-1} N(g_*^j(x))$$

So we have

$$\int_{B_4} \rho(x) d\nu = \int_{B_3} N(x) d\nu$$

Let $C_k = \{x \in B_1 | N(x) = k\}$. Similar to the argument in the proof of Theorem 3.2, we know

$$C = \bigcup_{k=1}^{\infty} (\bigcup_{j=0}^{k-1} g^j(C_k))$$

is g-invariant. g is ergodic, so $\nu(C)=1$. Hence

$$\int_{B_4}\rho(x)d\nu=\int_{B_3}N(x)=\sum_{k=1}^{\infty}k\cdot\nu(C_k)\geq\nu(C)=1$$

But

$$\int_{B_3}N(x)\leq\int_{B_4}N(x)+\int_{B_2}N(x)\leq m(1+r)\cdot\nu(B_4)+r$$

So

$$\nu(B_4)\geq\frac{1-r}{m(1+r)}$$

and the average return time

$$\frac{1}{\nu(B_4)}\int_{B_4}\rho(x)d\nu\leq\frac{m(1+r)\cdot\nu(B_4)+r}{\nu(B_4)}$$

$$=m(1+r)+\frac{r}{\nu(B_4)}\leq\frac{m(1+r)}{1-r}$$

4.3. **Construction of Horseshoe.** We are going to construct a skew product map with base $\mathcal{G}$ on B_4 and horseshoes on fibers.

For every $x\in B_4$, define the sets $S_k(x)$, $k=1,2,\ldots,l(x)$ by induction as follows:

$$S_{l(x)}=R(\mathcal{G}(x))=R(g_*^{l(x)}(x))$$

$$S_{l(x)-1}=CC(R(g_*^{l(x)-1}(x)\cap F^{-N(g_*^{l(x)-1}(x))}(S_{l(x)}),q(g_*^{l(x)-1}(x)))$$

For $k\leq l(x)-1$, if S_k is defined, then

$$S_{k-1}=F^{-N(g_*^{k-1}(x))}(CC(R(g_*^k(x))\cap F^{N(g_*^{k-1}(x))}(R(g_*^{k-1}(x))),$$

$$F^{N(g_*^{k-1}(x))}(q(g_*^{k-1}(x))))\cap S_k)$$

Now we can find a map $u_x:V(x)\to P(x)$ such that for $z\in V(x)$, $u_x(z)$ is a point inside the set

$$F^{-N(x)}(CC(R(g_*(x))\cap F^{N(x)}(R(x)),F^{N(x)}(z))\cap S_1)$$

Note

$$F^{N(x)}(u_x(z))\subset S_1$$

and

$$F^{N(g_*^k(x))}(S_k)\subset S_{k+1}$$

for $k=1,2,...,l(x)-1$, which implies

$$F^{\rho(x)}(u_x(z))\subset P(\mathcal{G}(x))$$

Let $U(x)=u_x(V(x))$.

For every $x\in B_4$, $z\in V(x)$, consider the connected component of $P(g_*(x))$ containing $F^N(x)(z)$. $F^N(x)(z)$ is the $\mathcal{K}$-th return of z to P and $N(x)>m$. Since points in $V(x)$ are (d_m^F,ϵ)-separated, with property (6) of the set P, we can conclude that the connected component

$$CC(R(g_*(x))\cap F^{N(x)}(R(x)),F^{N(x)}(z))$$

contains exactly one point in the set $F^{N(x)}(V(x))$. Analogously, the connected component

$$CC(R(x)\cap F^{-N(x)}(R(g_*(x)),z)$$

contains exactly one point in $V(x)$.

So for $z \in U(x)$, the connected component

$$CC(R(\mathcal{G}(x)) \cap F^{\rho(x)}(R(x)), F^{\rho(x)}(z))$$

contains exactly one point in the set $F^{\rho(x)}(U(x))$ and the connected component

$$CC(R(x) \cap F^{-\rho(x)}(R(\mathcal{G}(x))), z)$$

contains exactly one point in $U(x)$. We may choose the points in $U(x)$ such that the union of these points form the union of $[\frac{1}{mr}\exp m(h_\mu(F) - r)]$ measurable sections over B_4.

Let $\mathcal{F}(z) = F^{\rho(\pi(z))}(z)$. Then for $x \in B_4$, $R(\mathcal{G}(x)) \cap \mathcal{F}(R(x))$ consists of $[\frac{1}{mr}\exp m(h_\mu(F) - r)]$ connected components and so dos $R(x) \cap \mathcal{F}^{-1}(R(\mathcal{G}(x))$. Consider the set

$$\Lambda = \bigcup_{x\in B_4} (\bigcap_{n\in\mathbb{Z}} \mathcal{F}^n(R(\mathcal{G}^{-n}(x))))$$

Then Λ is invariant of $\mathcal{F}$ and $\mathcal{F}|_\Lambda = (\mathcal{G}, \mathcal{H})$, with the base $\mathcal{G}$ on B_5 and $\mathcal{H}$ on the fiber conjugate to the full shift on $[\frac{1}{mr}\exp m(h_\mu(F) - r)]$ symbols.

4.4. **Estimate of Entropy.** Λ carries many ergodic invariant measures for $\mathcal{F}|_\Lambda$ of the form

$$\frac{1}{\nu(B_4)} \int_{B_4} \tau_x d\nu$$

where τ_x is supported on $\Lambda(x) = \bigcap_{n\in\mathbb{Z}} \mathcal{F}^n(R(\mathcal{G}^{-n}(x)))$. Entropies of these measures vary from 0 to the topological entropy of the full shift which equals

$$\log[\frac{1}{mr}\exp m(h_\mu(F) - r)]$$

Measures of arbitrary intermediate entropies can be obtained by carefully assigning weights to different symbols for the shift. These measures induce ergodic invariant measures of F. The average return time is

$$\frac{1}{\nu(B_4)} \int_{B_4} \int_{\Lambda(x)} \rho(x) d\tau_x d\nu = \frac{1}{\nu(B_4)} \int_{B_4} \rho(x) d\nu \leq m(1+r)/(1-r)$$

So the measures we constructed has the maximum entropy no less than

$$\log[\frac{1}{mr}\exp m(h_\mu(F) - r)] \cdot \frac{1-r}{m(1+r)}$$

which is arbitrarily close to $h_\mu(F)$ as $r \to 0$ and $m \to \infty$. This completes the proof of Theorem 1.2.

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