Final Exam Study Guide

Peng Sun

1 Autonomous Equations $y' = f(y)$

Equilibrium solutions (critical points): can be solved from $f(y) = 0$.
On the phase line: right arrow if $f(y) > 0$; left if $f(y) < 0$.
Determine types of critical points and end behaviors of solutions accordingly.

2 Homogeneous Linear Equations

2.1 Solution to Equations with Constant Coefficients

Use characteristic equation. Each root corresponds to a solution of the form e^{rt}(real root, te^{rt},...,$t^{n-1}e^{rt}$ if repeated n times), or $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$ (complex roots $a \pm bi$, $te^{at}\cos/\sin(bt), \ldots, t^{n-1}e^{at}\cos/\sin(bt)$ if repeated n times).
The general solution is the linear combination of these solutions.

2.2 End Behavior of a Solution

End behavior is determined by the sign of the real root or the real part of the complex root. As $t \to \infty$, $y \to \infty$ if the sign is positive; $y \to 0$ if the sign is negative. The solution has no limit but oscillates if the roots are pure imaginary.

2.3 Wronskian

2.3.1 Abel's Theorem

For any two solutions y_1, y_2 of the equation $y'' + p(t)y' + q(t)y = 0$, The Wronskian $W(y_1, y_2) = y_1y_2' - y_1'y_2 = c\exp[-\int p(t)dt]$.

2.3.2 Linear Independence

Two functions $f(t), g(t)$ are linearly dependent on the interval I if $W(f, g) = 0$ for every t in I. Otherwise they are linearly independent.

2.4 The Existence and Uniqueness of the solution

3 Exact Equations

An equation $M(x,y)dx + N(x,y)dy = 0$ is an exact equation if
$\partial M/\partial y = \partial N/\partial x$

4 Spring-Mass Systems

4.1 Modeling

$$mu'' + \gamma u' + ku = F(t)$$

where u is the displacement from the equilibrium position, m the mass, γ the damping constant and F(t) the external force. k is the spring constant which can be determined from the equilibrium condition $mg = kL$, where mg is the weight and L is the elongation of the spring at equilibrium position.

4.2 Undamped Free Vibrations

$$mu'' + ku = 0$$

General solution: $u = A\cos\omega_0 t + B\sin\omega_0 t$
Natural Frequency: $\omega_0 = \sqrt{k/m}$
Period: $T = 2\pi/\omega_0$

4.3 Damped Free Vibrations

$$mu'' + \gamma u' + ku = 0$$

The system is losing its energy so every solution asymptotically approaches zero as $t \to \infty$, regardless of the type of damping.

4.3.1 Over-damping and Critical damping

The characteristic equation may have two (negative) real roots, repeated (negative) real roots or two complex roots (with negative real parts).
Over-damping: $\gamma^2 - 4km > 0$, two real roots
Critical damping: $\gamma^2 - 4km = 0$, repeated roots
In both cases, the solution does not oscillate and can cross the equilibrium position at most once.

4.3.2 Under-damping

$\gamma^2 - 4km < 0$, two complex roots, solution oscillates and crosses the equilibrium position infinitely many times
General solution: $u = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t)$

4.4 Undamped Forced Vibration

$$mu'' + ku = F_0 \cos \omega t$$

Resonance: $\omega_0 = \omega$, where $\omega_0 = \sqrt{k/m}$ is the natural frequency

5 Laplace Transform

5.1 Definition

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

Additivity: $\mathcal{L}\{af(t) \pm bg(t)\} = a\mathcal{L}\{f(t)\} \pm b\mathcal{L}\{g(t)\}$.

5.2 Solving Initial Value Problems

$$y'' + py' + qy = g(t), y(0) = y_0, y'(0) = y_0'$$

Step 1: Laplace transform:

$$s^2 Y(s) - sy(0) - y'(0) + p(sY(s) - y(0)) + qY(s) = \mathcal{L}\{g(t)\}$$

where $Y(s) = \mathcal{L}\{y\}$
Step 2: Plug in initial conditions and solve for $Y(s)$ from the equation above.
Step 3: Take inverse transform and find $y = \mathcal{L}^{-1}\{Y(s)\}$

5.3 Step Functions

5.3.1 Unit Step Functions

$$u_c(t) = u(t-c) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

5.3.2 Laplace Transform of Step Functions

$$\mathcal{L}\{u(t-c)\} = e^{-cs}/s$$

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$$

5.3.3 Inverse Transform

To find the inverse transform for $e^{-cs}F(s)$, first find $f(t) = \mathcal{L}^{-1}\{F(s)\}$. Then

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c)$$

6 Homogeneous Linear Systems with Constant Coefficients

6.1 Solve the Equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$

Step 1: Look for eigenvalues: set $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and solve for λ.
Step 2: Look for eigenvectors: Plug in the λ you get in step 1, find a vector ξ such that $(\mathbf{A} - \lambda\mathbf{I})\xi = 0$
Step 3: 3 cases here depending on the roots of the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

6.1.1 Different Real Roots

General solution: $\mathbf{x} = c_1 e^{\lambda_1 t}\xi_1 + c_2 e^{\lambda_2 t}\xi_2$
Phase portrait: Two straight lines; Critical point (origin) is a node or a saddle point.

6.1.2 Complex Roots

Take one complex root $a + bi$ and its eigenvector ξ. Find real valued vectors ξ_1 and ξ_2 such that

$$e^{(a+bi)t}\xi = e^{at}(\cos bt + i \sin bt)\xi = \xi_1 + i\xi_2$$

Then the general solution is $\mathbf{x} = c_1\xi_1 + c_2\xi_2$
Phase portrait: Spiral or center (if the complex roots have zero real parts)

6.1.3 Repeated Roots

Find a vector η such that $(\mathbf{A} - \lambda\mathbf{I})\eta = \xi$
The general solution is $\mathbf{x} = c_1 e^{\lambda t}\xi + c_2 e^{\lambda t}(t\xi + \eta)$
Phase portrait: Improper node (one straight line) or proper node (many straight lines)

6.2 Phase Portrait and Stability

A detailed summary with pictures is provided in Section 9.1. You may find Table 9.1.1 on Page 492 and Figure 9.1.9 on Page 495 helpful.

7 Nonlinear Systems $\begin{cases} x' = F(x, y) \\ y' = G(x, y) \end{cases}$

Critical points: can be solved from $\begin{cases} F(x, y) = 0 \\ G(x, y) = 0 \end{cases}$.

Type of the critical point (x_0, y_0): determined by the linear system with the matrix $\begin{pmatrix} \partial F/\partial x(x_0,y_0) & \partial F/\partial y(x_0,y_0) \\ \partial G/\partial x(x_0,y_0) & \partial G/\partial y(x_0,y_0) \end{pmatrix}$. For correspondence between linear and nonlinear systems, see Table 9.3.1 on Page 508.

8 Two-point Boundary Value Problems

Solve the equation and find the general solution. Then determine the constants according to the give boundary conditions.
Eigenvalues: values of λ such that the problem has nontrivial (nonzero) solutions.
Eigenfunctions: nontrivial solutions corresponding to eigenvalues.

9 Fourier Series

9.1 Definition

Fourier series for the periodic function $f(x)$ with period $2L$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L})$$

with coefficients

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

9.2 Fourier Convergence Theorem

The Fourier series of $f(x)$ converges to $f(x)$ at all points where $f(x)$ is continuous, and to $[f(x+) + f(x-)]/2$ (the average of the limits from both sides) at all points where f is discontinuous.

9.3 Odd Functions

If $f(x)$ is an odd function, then the coefficients $a_n = 0$ (including a_0) and

$$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

The series $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$ is called the Fourier sine series.

9.4 Even Functions

If $f(x)$ is an even function, then the coefficients $b_n = 0$ and

$$a_0 = \frac{2}{L}\int_0^L f(x)dx, a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

The series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$ is called the Fourier cosine series.

9.5 Periodic Extension

Suppose a given function $f(x)$ is defined on $(0, L)$

9.5.1 Odd Periodic Extension with Period $2L$

If looking for sine series, apply the formula in 9.3

9.5.2 Even Periodic Extension with Period $2L$

If looking for cosine series, apply the formula in 9.4

9.5.3 Periodic Extension with Period L

Find the expression for $f(x)$ on $(-L/2, 0)$ then apply the formulas in 9.1 (Note in this case L in the formulas should be replaced by $L/2$)

9.5.4 Sketch the Graph of the Extended Function Over a Few Periods

10 Partial Differential Equations

10.1 Separation of Variables

To solve a partial differential equation, assume there are solutions of the form $u(x,t) = X(x)T(t)$. Plug it into the partial differential equation to see if the two variables x and t can be separated. If they can be separated, then they must be equal to the same constant number, assumed as λ. Then the partial differential equation can be separated into two ordinary differential equations.

10.2 Heat Conduction Problem with Homogeneous Boundary Conditions

$$\alpha^2 u_{xx} = u_t, u(0,t) = 0, u(L,t) = 0, u(x,0) = f(x)$$

Step 1: Apply the method of separation of variables to separate the equation into two ODEs

$$X''(x) + \lambda X(x) = 0, T'(t) + \alpha^2\lambda T(t) = 0$$

Find the boundary conditions for the ODE: $X(0) = 0, X(L) = 0$
Step 2: Solve the two-point boundary value problem and get the eigenvalues $\lambda_n = (n\pi/L)^2$ and the corresponding eigenfunction $X_n(x) = \sin\frac{n\pi x}{L}$.
Step 3: Plug in the eigenvalues and solve the other ODE, $T_n(t) = c_n e^{-n^2\pi^2\alpha^2 t/L^2}$. For each eigenvalue λ_n, the product $u_n(x,t) = X_n(x)T_n(t)$ is a solution to the PDE satisfying the homogeneous boundary conditions.
Step 4: Find the solution as the linear combination of the solutions obtained in Step 3 which also satisfies the initial condition $u(x,0) = f(x)$. (Plug in the linear combination you will see the sine series for $f(x)$).
Solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\frac{n\pi x}{L}$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

10.3 Heat Conduction Problem with Nonhomogeneous Boundary Conditions

$$\alpha^2 u_{xx} = u_t, u(0,t) = T_1, u(L,t) = T_2, u(x,0) = f(x)$$

10.3.1 Steady-state Solution

The steady-state solution is the solution to the equation satisfying the nonhomogeneous boundary conditions that has the form $v(x,t) = v(x)$ (independent of t). For the above problem, the steady-state solution is $v(x) = \frac{T_2 - T_1}{L}x + T_1$.

10.3.2 solution to the problem

The solution is the sum of the steady-state solution and the solution to the problem with homogeneous boundary condition.

$$u(x,t) = v(x) + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\frac{n\pi x}{L}$$

where

$$c_n = \frac{2}{L}\int_0^L (f(x) - v(x))\sin\frac{n\pi x}{L}dx$$

10.4 Heat Conduction with Insulated Ends

$$\alpha^2 u_{xx} = u_t, u_x(0,t) = 0, u_x(L,t) = 0, u(x,0) = f(x)$$

Follow the steps in 10.2. Note in this case the boundary condition for the ODE is $X'(0) = 0, X'(L) = 0$. Hence the eigenvalues and eigenfunctions for

the two-point boundary value problem are $\lambda_n = (n\pi/L)^2$ and $X_n(x) = \cos\frac{n\pi x}{L}$. Also note here n can take the value zero.
Solution:

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \cos\frac{n\pi x}{L}$$

where

$$c_0 = \frac{2}{L}\int_0^L f(x)dx, c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

10.5 The Wave Equations

10.5.1 1st Type

$$a^2 u_{xx} = u_{tt}, u(0,t) = 0, u(L,t) = 0, u(x,0) = f(x), u_t(x,0) = 0$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

10.5.2 2nd Type

$$a^2 u_{xx} = u_{tt}, u(0,t) = 0, u(L,t) = 0, u(x,0) = 0, u_t(x,0) = g(x)$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L}$$

where

$$\frac{n\pi a}{L}c_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

11 Highly Recommended Exercises

Note the textbook does not contain all necessary exercises for the exam. You may need to work on the previous exams.
2.5: 3, 12
2.6: 3, 8
3.3: 2, 3
3.5: 3, 5, 6
3.8: 5, 17
3.9: 7
4.2: 20, 22
6.2: 5, 16
6.3: 14, 16

6.4: 4, 6
9.1: 3, 5, 8, 9, 10, 11
9.3: 7, 9
10.1: 1, 3, 14, 17, 18
10.2: 16, 17
10.4: 7, 16, 20, 23
10.5: 5, 6, 7, 8
10.6: 2, 4, 8

This study guide may contain mistakes or typos. If you find any, please tell me as soon as possible.