Exam II Study Guide

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1 Spring-Mass Systems

1.1 Modeling

$$mu'' + \gamma u' + ku = F(t)$$

where u is the displacement from the equilibrium position, m the mass, γ the damping constant and F(t) the external force. k is the spring constant which can be determined from the equilibrium condition $mg = kL$, where mg is the weight and L is the elongation of the spring at equilibrium position.

1.2 Undamped Free Vibrations

$$mu'' + ku = 0$$

General solution: $u = A\cos\omega_0 t + B\sin\omega_0 t = R\cos(\omega_0 t - \delta)$
Amplitude: $R = \sqrt{A^2 + B^2}$
Natural Frequency: $\omega_0 = \sqrt{k/m}$
Period: $T = 2\pi/\omega_0$
Phase: δ, such that $R\cos\delta = A$ and $R\sin\delta = B$

1.3 Damped Free Vibrations

$$mu'' + \gamma u' + ku = 0$$

The system is losing its energy so every solution asymptotically approaches zero as $t \to \infty$, regardless of the type of damping.

1.3.1 Over-damping and Critical damping

The characteristic equation may have two (negative) real roots, repeated (negative) real roots or two complex roots (with negative real parts).
Over-damping: $\gamma^2 - 4km > 0$, two real roots, solution does not oscillate
Critical damping: $\gamma^2 - 4km = 0$, repeated roots, solution does not oscillate
In both cases, the solution does not oscillate and can cross the equilibrium position at most once.

1.3.2 Under-damping

$\gamma^2 - 4km < 0$, two complex roots, solution oscillates with decreasing amplitude and crosses the equilibrium position infinitely many times.
General solution: $u = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t) = Re^{-\gamma t/2m}\cos(\mu t - \delta)$,
Quasi frequency: $\mu = \sqrt{4km - \gamma^2}/2m$
Quasi period: $T_d = 2\pi/\mu$

1.4 Undamped Forced Vibration

$$mu'' + ku = F_0 \cos\omega t$$

Resonance: $\omega_0 = \omega$, where $\omega_0 = \sqrt{k/m}$ is the natural frequency

2 Laplace Transform

2.1 Definition

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Additivity: $\mathcal{L}\{af(t) \pm bg(t)\} = a\mathcal{L}\{f(t)\} \pm b\mathcal{L}\{g(t)\}$.

2.2 Solving Initial Value Problems

$$y'' + py' + qy = g(t), y(0) = y_0, y'(0) = y_0'$$

Step 1: Laplace transform:

$$s^2Y(s) - sy(0) - y'(0) + p(sY(s) - y(0)) + qY(s) = \mathcal{L}\{g(t)\}$$

where $Y(s) = \mathcal{L}\{y\}$
Step 2: Plug in initial conditions and solve for $Y(s)$ from the equation above.
Step 3: Take inverse transform and find $y = \mathcal{L}^{-1}\{Y(s)\}$

2.3 Step Functions

2.3.1 Unit Step Functions

$$u_c(t) = u(t-c) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

2.3.2 Piecewise Defined Functions as Step Functions

$$f(t) = \begin{cases} f_1(t), & t < t_1 \\ f_2(t), & t_1 \leq t < t_2 \\ \vdots \\ f_n(t), & t_{n-1} \leq t \end{cases} = f_1(t)+u(t-t_1)(f_2(t)-f_1(t))+\cdots+u(t-t_{n-1})(f_n(t)-f_{n-1}(t))$$

Also need to know: Evaluation of a step function.

2.3.3 Laplace Transform of Step Functions

$$\mathcal{L}\{u(t-c)\} = e^{-cs}/s$$
$$\mathcal{L}\{u(t-c)f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}$$

2.3.4 Inverse Transform

To find the inverse transform for $e^{-cs}F(s)$, first find $f(t) = \mathcal{L}^{-1}\{F(s)\}$. Then

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c)$$

2.4 Impulse Functions

2.4.1 Definition

Unit impulse function $\delta(t)$ is an idealized function with the following properties:

$$\delta(t) = 0 \text{ if } t \neq 0; \int_{-\infty}^{\infty} \delta(t)dt = 1$$

More properties:

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0) = f(t_0) \quad \text{and} \quad f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$$

2.4.2 Laplace Transform

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$
$$\mathcal{L}\{\delta(t-t_0)f(t)\} = f(t_0)e^{-st_0}$$

3 Homogeneous Linear Systems with Constant Coefficients

3.1 Solving the Equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$

Step 1: Look for eigenvalues: set $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and solve for λ.
Step 2: Look for eigenvectors: Plug in the λ you get in step 1, find a nonzero vector ξ such that $(\mathbf{A} - \lambda\mathbf{I})\xi = 0$
Step 3: 3 cases here depending on the roots of the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

3.1.1 Different Real Roots

General solution: $\mathbf{x} = c_1 e^{\lambda_1 t}\xi_1 + c_2 e^{\lambda_2 t}\xi_2$
Phase portrait: Two straight lines; Critical point (origin) is a node or a saddle point.

3.1.2 Complex Roots

Take one complex root $a+bi$ and its eigenvector ξ. Find real valued vectors ξ_1 and ξ_2 such that

$$e^{(a+bi)t}\xi = e^{at}(\cos bt + i\sin bt)\xi = \xi_1 + i\xi_2$$

Then the general solution is $\mathbf{x} = c_1\xi_1 + c_2\xi_2$
Phase portrait: Spiral or center (if the complex roots have zero real parts)

3.1.3 Repeated Roots

Find a vector η such that $(\mathbf{A} - \lambda\mathbf{I})\eta = \xi$
The general solution is $\mathbf{x} = c_1 e^{\lambda t}\xi + c_2 e^{\lambda t}(t\xi + \eta)$
Phase portrait: Improper node (one eigenvector) or proper node (every vector is an eigenvector)

3.2 Phase Portrait and Stability

A detailed summary with pictures is provided in Section 9.1. You may find Table 9.1.1 on page 494 and Figure 9.1.9 on page 497 helpful.

3.3 Transform between Higher Order Equations and Systems of First Order Equations

An arbitrary nth order equation $y^{(n)} = F(t, y, y', \ldots, y^{(n-1)})$ can be transformed into a system of n first order equations. Set $x_1 = y$, $x_2 = y'$, $\ldots$, $x_n = y^{(n-1)}$. The system is

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = F(t, x_1, x_2, \ldots, x_n) \end{cases}$$

4 Nonlinear Systems $\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$

Critical points: can be solved from $\begin{cases} F(x,y) = 0 \\ G(x,y) = 0 \end{cases}$.

Type of the critical point (x_0, y_0): determined by the linear system with the matrix $\begin{pmatrix} \partial F/\partial x(x_0,y_0) & \partial F/\partial y(x_0,y_0) \\ \partial G/\partial x(x_0,y_0) & \partial G/\partial y(x_0,y_0) \end{pmatrix}$. For correspondence between linear and nonlinear systems, see Table 9.3.1 on Page 513.

This study guide may contain mistakes or typos. If you find any, please tell me as soon as possible.