

Rigid-analytic geometry and the uniformization of abelian varieties

Mihran Papikian

ABSTRACT. The purpose of these notes is to introduce some basic notions of rigid-analytic geometry, with the aim of discussing the non-archimedean uniformizations of certain abelian varieties.

1. Introduction

The methods of complex-analytic geometry provide a powerful tool in the study of algebraic geometry over \mathbb{C} , especially with the help of Serre's GAGA theorems [S1]. For many arithmetic questions one would like to have a similar theory over other fields which are complete for an absolute value, namely over non-archimedean fields. If one tries to use the metric of a non-archimedean field directly, then such a theory immediately runs into problems because the field is totally disconnected.

It was John Tate who, in early 60's, understood how to build the theory in such a way that one obtains a reasonably well-behaved coherent sheaf theory and its cohomology [T1]. This theory of so-called rigid-analytic spaces has many results which are strikingly similar to algebraic and complex-analytic geometry. For example, one has the GAGA theorems over any non-archimedean ground field.

Over the last few decades, rigid-analytic geometry has developed into a fundamental tool in modern number theory. Among its impressive and diverse applications one could mention the following: the Langlands correspondence relating automorphic and Galois representations [D], [HT], [LRS], [Ca]; Mumford's work on degenerations and its generalizations [M2], [M3], [CF]; Ribet's proof that Shimura-Taniyama conjecture implies Fermat's Last Theorem [Ri]; fundamental groups of curves in positive characteristic [Ra]; the construction of abelian varieties with a given endomorphism algebra [OvdP]; and many others.

The purpose of these notes is to informally introduce the most basic notions of rigid-analytic geometry, and, as an illustration of these ideas, to explain the analytic uniformization of abelian varieties with split purely toric reduction. The construction of such uniformizations is due to Tate and Mumford. In our exposition we follow [Co1] and [FvdP].

Acknowledgements. I thank the organizers for their invitation to participate in the conference, and for creating stimulating atmosphere during the conference.

I also thank Brian Conrad and the anonymous referee for useful comments on the earlier versions of this article.

2. Tate curve

As a motivation, and to have a concrete example in our later discussions, we start with elliptic curves. Here, the non-archimedean uniformization of so-called Tate curves can be constructed very explicitly by using Weierstrass equations. (Of course, such an equation-based approach does not generalize to higher dimensional abelian varieties.) Later on we will return to the example of Tate curves, and will see how Tate’s theorem can be recovered from a more sophisticated point of view.

Let K be a field complete with respect to a non-trivial non-archimedean valuation which we denote by $|\cdot|$. Let $R = \{z \in K \mid |z| \leq 1\}$ be the ring of integers in K , $\mathfrak{m} = \{z \in K \mid |z| < 1\}$ be the maximal ideal, and $k = R/\mathfrak{m}$ be the residue field. For example, we can take $K = \mathbb{Q}_p$, $R = \mathbb{Z}_p$, $\mathfrak{m} = p\mathbb{Z}_p$ and $k = \mathbb{F}_p$, or $K = \mathbb{F}_q((t))$, $R = \mathbb{F}_q[[t]]$, $\mathfrak{m} = t\mathbb{F}_q[[t]]$ and $k = \mathbb{F}_q$. The field K need not necessarily be locally compact or discretely valued, for example, it can be $\mathbb{C}((t))$ or the completion of an algebraic closure of \mathbb{Q}_p . Nevertheless, if it simplifies the exposition, we shall sometimes assume that the underlying field is either algebraically closed or discretely valued.

EXAMPLE 2.1. Recall that explicitly \mathbb{Q}_p is the field of “Laurent series”

$$\left\{ \sum_{i=n}^{\infty} a_i p^i \mid n \in \mathbb{Z}, a_i \in \mathbb{Z}, 0 \leq a_i \leq p-1 \right\}.$$

The norm of $\alpha = \sum_{i=n}^{\infty} a_i p^i$ is given by (without loss of generality we assume $a_n \neq 0$ if $\alpha \neq 0$)

$$|\alpha| = \begin{cases} p^{-n} & \text{if } \alpha \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The name “non-archimedean” comes from the fact that for such fields the usual triangle inequality holds in a stronger form: $|\alpha + \beta| \leq \max(|\alpha|, |\beta|)$.

Let E be an elliptic curve over \mathbb{C} . Classically it is known that

$$(2.1) \quad E(\mathbb{C}) \cong \mathbb{C}/\Lambda \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau),$$

where $\text{Im}(\tau) > 0$, with the isomorphism given in terms of the Weierstrass \wp -function and its derivative; see [Si1, Ch.V]. To motivate the formulae over K , let

$$u = e^{2\pi iz}, \quad q = e^{2\pi i\tau}, \quad q^{\mathbb{Z}} = \{q^m \mid m \in \mathbb{Z}\},$$

and consider the complex-analytic isomorphism

$$(2.2) \quad \begin{aligned} \mathbb{C}/\Lambda &\xrightarrow{\sim} \mathbb{C}^\times/q^{\mathbb{Z}} \\ z &\longmapsto u. \end{aligned}$$

Note that since $\text{Im}(\tau) > 0$, we have $|q| < 1$. We use q -expansion to explicitly describe the isomorphism $E(\mathbb{C}) \cong \mathbb{C}^\times/q^{\mathbb{Z}}$ resulting from (2.1) and (2.2).

THEOREM 2.2. *Define the quantities*

$$s_k(q) = \sum_{n \geq 1} \frac{n^k q^n}{1 - q^n} \in \mathbb{Z}[[q]],$$

$$a_4(q) = -5s_3(q), \quad a_6(q) = -\frac{5s_3(q) + 7s_5(q)}{12},$$

$$X(u, q) = \sum_{n \in \mathbb{Z}} \frac{q^n u}{(1 - q^n u)^2} - 2s_1(q),$$

$$Y(u, q) = \sum_{n \in \mathbb{Z}} \frac{(q^n u)^2}{(1 - q^n u)^3} + s_1(q),$$

$$E_q : y^2 + xy = x^3 + a_4(q)x + a_6(q).$$

Then E_q is an elliptic curve, and X and Y define a complex-analytic isomorphism

$$(2.3) \quad \begin{aligned} \mathbb{C}^\times / q^{\mathbb{Z}} &\longrightarrow E_q(\mathbb{C}) \\ u &\longmapsto \begin{cases} (X(u, q), Y(u, q)) & \text{if } u \notin q^{\mathbb{Z}}, \\ O & \text{if } u \in q^{\mathbb{Z}}. \end{cases} \end{aligned}$$

PROOF. See [Si2, Ch.V]. □

If we replace \mathbb{C} by K and try to parametrize an elliptic curve E over K by a group of the form K/Λ , then we run into a serious problem: K need not have non-trivial discrete subgroups. For example, if $\Lambda \subset \mathbb{Q}_p$ is any non-zero subgroup and $0 \neq \lambda \in \Lambda$, then $p^n \lambda \in \Lambda$ for all $n \geq 0$, so 0 is an accumulation point of Λ (in p -adic topology!). If the characteristic of K is positive then “lattices” do exist. For example, $\Lambda = \mathbb{F}_q[t]$ is an infinite discrete subgroup of the completion \mathbb{C}_∞ of an algebraic closure of $\mathbb{F}_q((\frac{1}{t}))$. However, the quotient $\mathbb{C}_\infty/\Lambda$ is not an abelian variety; in fact, one can show $\mathbb{C}_\infty/\Lambda \cong \mathbb{C}_\infty$. (Such quotients are very interesting since they naturally lead to the theory of Drinfeld modules [D], but this is not what we want.)

Tate’s observation was that the situation can be salvaged if one uses the multiplicative version of the uniformization. Indeed, K^\times has lots of discrete subgroups, as any $q \in K^\times$ with $|q| < 1$ defines the discrete subgroup $q^{\mathbb{Z}} = \{q^m \mid m \in \mathbb{Z}\} \subset K^\times$.

THEOREM 2.3 (Tate). *Let $q \in K^\times$ with $|q| < 1$. Let $a_4(q), a_6(q)$ be defined as in Theorem 2.2. (This makes sense since $a_4(q), a_6(q) \in \mathbb{Z}[[q]]$.) The series defining a_4 and a_6 converge in K . (Thanks to non-archimedean topology, $\sum_{n \geq 0} \alpha_n$ is convergent if $|\alpha_n| \rightarrow 0$.) Hence, we can define E_q over K in terms of the Weierstrass equation as in Theorem 2.2. The map (2.3) defines a surjective homomorphism*

$$\phi : L^\times \rightarrow E_q(L)$$

with kernel $q^{\mathbb{Z}}$, where L is any algebraic extension of K . Moreover, when L is Galois over K , this homomorphism is compatible with respect to the action of the Galois group $\text{Gal}(L/K)$.

PROOF. See [T2]. □

REMARK 2.4. The absolute Galois groups of local fields, e.g. $\text{Gal}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$, play a prominent role in arithmetic geometry. The Galois equivariance of non-archimedean uniformizations, as in Tate’s theorem, makes such uniformizations quite useful in the study of these Galois groups.

Over the complex numbers we know that every elliptic curve E/\mathbb{C} is isomorphic to E_q for some $q \in \mathbb{C}^\times$. In the non-archimedean case, using q -expansions, we see that $a_4(q), a_6(q) \in \mathfrak{m}$ because $q \in \mathfrak{m}$. Hence, the reduction \overline{E}_q of E_q over the residue field is given by $x^2 + xy = x^3$. In particular, \overline{E}_q has a node and the slopes of the tangent lines at the node are in k . Elliptic curves with this property are said to have *split multiplicative* reduction over K . We conclude that not every elliptic curve over K can be isomorphic to some E_q – a necessary condition is that E must have split multiplicative reduction. This also turns out to be a sufficient condition, i.e., if E has split multiplicative reduction then there is a unique $q \in K^\times$ with $|q| < 1$ such that $E_q \cong E$. This last statement again can be proven by using the explicit nature of the uniformization: First, E has multiplicative reduction over K if and only if $|j(E)| > 1$. Next, one writes down the j -invariant of E_q as an infinite Laurent series in q with integer coefficients $j = q^{-1} + 744 + 196884 \cdot q + \dots$, and, by carefully analyzing these series, shows that $j(q)$ defines a bijection between the two sets $\{q \in K^\times \mid |q| < 1\}$ and $\{z \in K \mid |z| > 1\}$. Finally, one shows that E with $j(E) = \alpha$, $|\alpha| > 1$, is isomorphic over K to E_q with $j(q) = \alpha$ if and only if E has split multiplicative reduction. To prove a similar statement for general abelian varieties one needs to use more sophisticated arguments. We will return to this in §4.5.

We would like to understand the higher-dimensional generalizations of Theorem 2.3; we also would like to have an honest “analytic object” E^{an} with underlying set equal to the set of closed points of E and an isomorphism of “analytic spaces” $E^{\text{an}} \cong \mathbb{G}_{m,K}^{\text{an}}/q^{\mathbb{Z}}$ in an appropriate geometric category. To do all of this we need rigid-analytic spaces – the subject of the next section.

3. Rigid-analytic geometry

As in §2, let K be a non-archimedean field. We would like to define analytic geometry over K . The attempt to construct such a theory in a straightforward manner, by using the topology on K , does not work since K is totally disconnected. To see why this is so, let $a \in K$ and let $D(a, r) = \{x \in K \mid |x - a| \leq r\}$ be the closed disc around a of radius r , where $r \in \mathbb{R}$ is a value of $|\cdot|$ on K^\times . Let $D(a, r^-) = \{x \in K \mid |x - a| < r\}$ be the open disc, and $C(a, t) = \{x \in K \mid |x - a| = t\}$ be the circle of radius t around a . Then a counterintuitive thing happens, namely $C(a, t)$ is an open subset of K ! Indeed, for $y \in C(a, t)$, the non-archimedean nature of the norm implies $D(y, t^-) \subset C(a, t)$. Since $D(a, r) = \bigcup_{t \leq r} C(a, t)$, the disc $D(a, r)$ is both open and closed. (It is worth mentioning that for some other questions, such as the theory of Lie groups over non-archimedean fields, the total disconnectedness of K does not cause serious problems; see [S2] where the theory of Lie groups is developed uniformly for any ground field K , complete with respect to a non-trivial absolute value.)

Tate’s idea for constructing a theory of analytic spaces over K was to imitate the construction of algebraic varieties: an algebraic variety over a field K is obtained by gluing affine varieties over K with respect to Zariski topology. Rigid-analytic spaces over K are formed in a similar way, by gluing “affinoids” with respect to a certain Grothendieck topology.

We start by introducing the Tate algebra $T_n(K)$, which plays a role similar to the role of finitely generated polynomial algebra $K[Z_1, \dots, Z_n]$ in algebraic geometry. Then we define the notion of an affinoid, which is the analogue of a finitely

generated algebra over K . Finally, we explain how to “glue” the affinoids to construct rigid-analytic spaces. The standard references for this section are [T1], [BGR] and [FvdP].

3.1. Tate algebra. The *Tate algebra* $T_n(K)$ over K is the algebra of formal power series in Z_1, Z_2, \dots, Z_n with coefficients in K satisfying the following condition:

$$T_n(K) = K\langle Z_1, \dots, Z_n \rangle = \left\{ \sum_{\mathbf{s}=(s_1, \dots, s_n) \in \mathbb{N}^n} a_{\mathbf{s}} Z_1^{s_1} \cdots Z_n^{s_n} \mid \lim_{s_1 + \dots + s_n \rightarrow \infty} |a_{\mathbf{s}}| = 0 \right\}.$$

One can think of T_n as the algebra of holomorphic functions on the polydisc

$$\mathbb{D}^n(K) := \{(z_1, \dots, z_n) \in K^n \mid |z_i| \leq 1 \text{ for } i = 1, \dots, n\}.$$

The norm on K can be extended to a norm on T_n by:

$$(3.1) \quad \left\| \sum_{\mathbf{s} \in \mathbb{N}^n} a_{\mathbf{s}} Z_1^{s_1} \cdots Z_n^{s_n} \right\| = \max_{\mathbf{s} \in \mathbb{N}^n} |a_{\mathbf{s}}|.$$

PROPOSITION 3.1. *$T_n(K)$ is a Banach algebra with respect to $\|\cdot\|$. That is, for any $f, g \in T_n$ and any $a \in K$ we have*

- (1) $\|f\| \geq 0$ with equality if and only if $f = 0$,
- (2) $\|f + g\| \leq \max(\|f\|, \|g\|)$,
- (3) $\|a \cdot f\| = |a| \cdot \|f\|$,
- (4) $\|1\| = 1$,
- (5) $\|f \cdot g\| \leq \|f\| \cdot \|g\|$,
- (6) T_n is complete with respect to $\|\cdot\|$.

It is easy to see that the norm on T_n is in fact multiplicative: $\|f \cdot g\| = \|f\| \cdot \|g\|$. Also note that T_n is the completion of the polynomial algebra $K[Z_1, \dots, Z_n]$ with respect to this norm. Some of the statements in the next proposition can be proven by using an analogue of the Weierstrass preparation theorem, cf. Theorem 3.2.1 in [FvdP].

PROPOSITION 3.2.

- (1) T_n is a regular noetherian unique factorization domain whose maximal ideals all have height n .
- (2) If $I \subset T_n$ is a proper ideal then there exists a finite injective morphism $T_d \rightarrow T_n/I$ with d the Krull dimension of T_n/I .
- (3) For any maximal ideal $M \subset T_n$, the field T_n/M is a finite extension of K and the valuation on K uniquely extends to T_n/M .
- (4) Every ideal in T_n is closed with respect to $\|\cdot\|$.
- (5) T_n is a Jacobson ring, i.e., the radical of an ideal I is equal to the intersection of all the maximal ideals containing I .

3.2. Affinoid algebras. Let $I \subset T_n$ be an ideal. The algebra $A = T_n/I$ is called a *K -affinoid algebra*. Since I is closed in T_n , the algebra A is noetherian and is a Banach algebra with respect to the quotient norm on A , i.e., $\|\bar{f}\| = \inf\{\|f + g\| \mid g \in I\}$. It can be shown that the K -Banach topology on A

is unique and that all K -algebra maps between K -affinoid algebras are continuous. For simplicity of the exposition we will always assume that A is reduced.

Let $X = Z(I) \subset \text{Max}(T_n)$ be the zero set of I . By restriction, the elements of T_n can be regarded as functions on X . Since T_n is Jacobson and A is assumed to be reduced, the element $f \in T_n$ is 0 on X if and only if $f \in I$. In particular, the elements of A can then be regarded as (holomorphic) functions on X . The canonical surjection $T_n \rightarrow A$ induces a map between the sets of maximal ideals $\text{Max}(A) \rightarrow \text{Max}(T_n)$. This map identifies X and $\text{Max}(A)$, so the elements of A can be regarded as functions on $\text{Max}(A)$. As with algebraic varieties over an abstract field, the value $f(x)$ of $f \in A$ at $x \in X$ is the image of f in the residue field A/\mathfrak{m}_x at x , where \mathfrak{m}_x denotes the maximal ideal of A corresponding to x .

DEFINITION 3.3. A subset $U \subseteq X$ is an *affinoid subset* of X if there is a homomorphism of K -affinoid algebras $\varphi : A \rightarrow B$ with $U = \varphi^*(\text{Max}(B))$, and moreover, for every homomorphism $\psi : A \rightarrow C$ such that $\psi^*(\text{Max}(C)) \subseteq U$ there is a unique homomorphism of K -affinoid algebras $\phi : B \rightarrow C$ with $\psi = \phi \circ \varphi$. (Note that the analogous condition on finite type maps between noetherian affine schemes corresponds to open immersions.)

It is not hard to show that we can provide the set $X = \text{Max}(A)$ with a Grothendieck topology, where the admissible opens are the affinoid subsets and the admissible coverings are the coverings by affinoid subsets which have *finite* sub-coverings. One essentially needs to check that: (1) The intersection of two affinoid subsets in X is again an affinoid subset of X ; (2) If $U \subseteq X$ is an affinoid subset of X and $W \subseteq U$ is an affinoid subset of U then W is an affinoid subset of X . For the convenience of the reader, we recall the definition of Grothendieck topology; see [BGR, §9.1] for more details.

DEFINITION 3.4. Let X be a set. To endow X with a *Grothendieck topology* means to specify a family \mathcal{G} of subsets of X , and for each $U \in \mathcal{G}$ a set $\text{Cov}(U)$ of coverings of U by elements of \mathcal{G} , satisfying the following conditions:

- (1) $\emptyset, X \in \mathcal{G}$;
- (2) If $U, V \in \mathcal{G}$ then $U \cap V \in \mathcal{G}$;
- (3) $\{U\} \in \text{Cov}(U)$;
- (4) If $U, V \in \mathcal{G}$ with $V \subset U$, and $\mathcal{U} \in \text{Cov}(U)$, then $\mathcal{U} \cap V \in \text{Cov}(V)$;
- (5) If $U \in \mathcal{G}$, $\{U_i\}_{i \in I} \in \text{Cov}(U)$ and $\mathcal{U}_i \in \text{Cov}(U_i)$, then $\bigcup_{i \in I} \mathcal{U}_i \in \text{Cov}(U)$;
- (6) If $\{U_i\}_{i \in I} \in \text{Cov}(U)$ then $\bigcup_{i \in I} U_i = U$.

The elements of \mathcal{G} are called *admissible opens*, and the elements of $\text{Cov}(U)$ are called *admissible coverings*. One defines presheaves (sheaves, Čech cohomology) for a Grothendieck topological space in a usual manner.

Given an admissible open $U \subseteq X$ with $U = \text{Max}(B)$, let $\mathcal{O}_X(U) = B$. The map $U \mapsto \mathcal{O}_X(U)$ from admissible opens of X to affinoid K -algebras defines a presheaf \mathcal{O} on X . It is a consequence of rather non-trivial results of Gerritzen, Grauert and Tate that \mathcal{O} is in fact a sheaf.

DEFINITION 3.5. The Grothendieck topological space $\text{Max}(A)$ with the structure sheaf \mathcal{O} is called a *K -affinoid space* and is denoted $\text{Sp}(A)$.

REMARK 3.6. Strictly speaking, for the purposes of the next definition we should have defined a more general notion of “admissible opens” in an affinoid space,

called *rational subsets*. The structure sheaf canonically extends to this “stronger” topology, and in what follows it is understood that admissibles in affinoid spaces are to be taken in this more general sense. Nevertheless, for practical purposes it suffices to think with affinoid subsets, and in fact affinoid subsets of $\mathrm{Sp}(A)$ serve as a “base for the topology”, quite similar to open balls in a manifold.

By gluing the affinoids along admissible opens, we obtain more general spaces:

DEFINITION 3.7. A *rigid-analytic space* is a pair (X, \mathcal{O}) , where X is a Grothendieck topological space and \mathcal{O} is a sheaf of K -algebras on X , such that there exists an admissible covering $\{X_i\}$ of X with $(X_i, \mathcal{O}|_{X_i})$ being an affinoid space.

In all examples in this section we assume that K is algebraically closed (this is done for expository simplicity and for no other reason). The same examples can be pushed through for general K once one has a suitable language for working with rigid spaces on which there are non-rational points.

EXAMPLE 3.8. For any affinoid space $\mathrm{Sp}(A)$, the set $\mathrm{Max}(A)$ can be identified with the set of K -algebra homomorphisms $A \rightarrow K$. For example, the underlying topological space of $\mathrm{Sp}(T_n)$ is the polydisc $\mathbb{D}_n(K)$. Indeed, any K -homomorphism $K\langle Z_1, \dots, Z_n \rangle \rightarrow K$ is uniquely determined by the images of Z_i 's, which we have to (and can) send to some point in the polydisc (note that for $\mathbf{z} \in K^n$ the sums $f(\mathbf{z})$ are convergent for all $f \in T_n$ exactly when $\mathbf{z} \in \mathbb{D}_n(K)$).

EXAMPLE 3.9. Let $a, b \in K$ satisfy $0 < |a| \leq |b| \leq 1$. It is not hard to check that the set $\{z \in K \mid |a| \leq |z| \leq |b|\}$ is an affinoid subset of $\mathbb{B} = \mathrm{Sp} K\langle Z \rangle$ corresponding to the affinoid algebra $K\langle Z, U, W \rangle / (ZW - a, Z - bU)$. The covering $\mathbb{B} = U_1 \cup U_2$, where $U_1 = \{z \in K \mid |z| \leq |a|\}$ and $U_2 = \{z \in K \mid |a| \leq |z| \leq 1\}$, is an admissible covering.

To emphasize the importance of the requirement that admissible coverings admit finite subcoverings by affinoid subsets, consider the following covering of \mathbb{B} . Let $r_1 < r_2 < \dots < 1$ be real numbers in the image of the norm on K such that r_i converge to 1. Let $U_j = \{z \in K \mid |z| \leq r_j\}$, and let $\partial\mathbb{B} = \{z \in K \mid |z| = 1\}$, so $\mathbb{B} = (\bigcup_j U_j) \cup \partial\mathbb{B}$. If we had allowed such a covering, then \mathbb{B} would be disconnected (something we do not want). Fortunately, since this covering obviously has no finite subcovering, it is not an admissible covering of the affinoid space \mathbb{B} . The notion of admissible coverings is introduced to rule out such unpleasant phenomena as the unit disc being disconnected, and in fact if A is a K -affinoid algebra with no idempotents then $\mathrm{Sp}(A)$ is connected (in the sense that for any admissible covering $\{U_1, U_2\}$, with U_1 and U_2 non-empty, the intersection $U_1 \cap U_2$ is non-empty).

REMARK 3.10. As an instructive analogue of the idea behind the definition of a K -affinoid space, the reader is invited to try to make the totally disconnected space $\{q \in \mathbb{Q} \mid 0 \leq q \leq 1\}$ acquire the compactness and connectedness properties of $[0, 1]$ by restricting the class of “opens” and “covers”.

EXAMPLE 3.11. We can “analytify” algebraic K -schemes, much as is done for algebraic varieties over \mathbb{C} . Let $X = \mathrm{Spec} K[Z_1, \dots, Z_n] / (f_1, \dots, f_s)$ be a reduced affine scheme of finite type over K . The topological space X can be identified with a closed (in Zariski topology) subset of K^n :

$$X = \{\mathbf{z} = (z_1, \dots, z_n) \in K^n \mid 0 = f_1(\mathbf{z}) = \dots = f_s(\mathbf{z})\}.$$

Choose $\pi \in K^\times$ with $|\pi| < 1$. For $m \geq 1$ let

$$X_m = \{\mathbf{z} \in X \mid |z_i| \leq |\pi|^{-m} \text{ for } 1 \leq i \leq n\},$$

so X_m has a natural structure of an affinoid (that is in fact reduced):

$$X_m = \text{Sp } K\langle \pi^m Z_1, \dots, \pi^m Z_n \rangle / (f_1, \dots, f_s).$$

The affinoids X_m can be glued together in an obvious manner, where X_m is an admissible open in X_{m+1} . One can check that this gives a well-defined rigid-analytic space X^{an} .

As a more concrete example, let's consider $(\mathbb{A}_K^1)^{\text{an}}$. Pick $c \in K^\times$ with $|c| > 1$. Let $\mathbb{B}_r = \text{Sp } K\langle \frac{Z}{c^r} \rangle := \{z \in K \mid |z| \leq |c|^r\}$ for $r = 0, 1, 2, \dots$. Then

$$\begin{aligned} \mathbb{B}_0 \hookrightarrow \mathbb{B}_1 \hookrightarrow \dots \hookrightarrow \mathbb{B}_r \hookrightarrow \mathbb{B}_{r+1} \hookrightarrow \dots \\ z_r/c \leftarrow z_{r+1} \end{aligned}$$

are glued together as indicated to give $\mathbb{A}_K^{1,\text{an}}$.

Now given a general separated (reduced) scheme X over K , we can associate to it a (reduced) rigid-analytic space X^{an} by considering an open affine cover of X . Since the intersection of two open affines in X is again an open affine, we can analytify each open affine and glue them along the intersections.

Of course, one can show that $X \mapsto X^{\text{an}}$ is a functor from the category of separated schemes of finite type over K into the category of rigid-analytic spaces. The space X^{an} represents the functor which to a rigid-analytic space \mathcal{X} associates the set of morphisms $\text{Hom}_K(\mathcal{X}, X)$ of Grothendieck topological spaces over K . That this functor is representable can be proven along the lines of Theorem 1.1 in **SGA 1**, Exposé XII. One eventually reduces the question to the representability of $\mathcal{X} \mapsto \text{Hom}_K(\mathcal{X}, \mathbb{A}_K^1)$. This last functor is representable by a rigid space with a distinguished global function, and the space $\mathbb{A}_K^{1,\text{an}}$, which we constructed above, has this universal property. In view of the construction of $\mathbb{A}_K^{1,\text{an}}$ via gluing of balls, to prove its universal property one has to show that on any affinoid space every global function is bounded. Such boundedness is a fundamental property of affinoid algebras which is not hard to verify; cf. [**FvdP**, §3.3]. Most of the properties of morphisms and spaces with respect to analytification over \mathbb{C} , as proved in the above mentioned Exposé, carry over to the rigid-analytic setting. For example, $X \mapsto X^{\text{an}}$ is fully faithful on proper schemes; cf. [**FvdP**, Ch.4].

EXAMPLE 3.12. Now let's return to the Tate curve example and consider it from the rigid-analytic point of view. Let $\mathbb{G}_{m,K} = \text{Spec } K[Z, W]/(ZW - 1)$ be the algebraic torus, and let $\mathbb{G}_{m,K}^{\text{an}}$ be its analytification. Let $q \in K^\times$ with $|q| < 1$. We can identify q with the automorphism of the rigid space $\mathbb{G}_{m,K}^{\text{an}}$, given by $z \mapsto zq$. The action of the group $\Gamma = q^{\mathbb{Z}}$ of the rigid space $\mathbb{G}_{m,K}^{\text{an}}$ is *discontinuous*. This means that $\mathbb{G}_{m,K}^{\text{an}}$ has an admissible covering $\{X_i\}$ such that for each i the set $\{\gamma \in \Gamma \mid \gamma X_i \cap X_i \neq \emptyset\}$ is finite. An example of such a covering is $\{X_i\}_{i \in \mathbb{Z}}$, where the affinoid subset X_n of $\mathbb{G}_{m,K}^{\text{an}}$ is the annulus

$$X_n = \{z \in K^\times \mid |q|^{(n+1)/2} \leq |z| \leq |q|^{n/2}\}.$$

Note that $X_n \cap X_{n-1} = \{z \in K^\times \mid |z| = |q|^{n/2}\}$ is also an affinoid subset, and q acts on this covering by $X_n \mapsto X_{n+2}$. Now we can form the quotient $\mathcal{E} = \mathbb{G}_{m,K}^{\text{an}}/\Gamma$ in the usual manner, i.e., by gluing annuli appropriately. The Grothendieck topology on \mathcal{E} is defined as follows: $U \subset \mathcal{E}$ is admissible if there is an admissible open $V \subset \mathbb{G}_{m,K}^{\text{an}}$

such that under the canonical projection $\mathbb{G}_{m,K}^{\text{an}} \rightarrow \mathcal{E}$ the set V maps bijectively to U . For example, X_0 and X_1 map bijectively into \mathcal{E} and form a covering $U_0 \cup U_1$ of this space. The intersection $U_0 \cap U_1$ is the image of $(X_{-1} \cap X_0) \cup (X_0 \cap X_1)$. The structure sheaf \mathcal{O} on \mathcal{E} is defined by $\mathcal{O}(\emptyset) = \{0\}$, $\mathcal{O}(\mathcal{E}) = K$, $\mathcal{O}(U_i) = \mathcal{O}(X_i)$ and

$$\mathcal{O}(U_0 \cap U_1) = \mathcal{O}(X_{-1} \cap X_0) \oplus \mathcal{O}(X_0 \cap X_1).$$

One checks that this indeed gives a well-defined structure of a rigid-analytic space on \mathcal{E} . Moreover, it is possible to show that $\mathcal{E} \cong E_q^{\text{an}}$. This is done by carefully analyzing the field of meromorphic functions $\mathcal{M}(\mathcal{E})$ on \mathcal{E} and showing $\mathcal{M}(\mathcal{E}) = K(X(z, q), Y(z, q))$. For the details we refer to [FvdP, §5.1].

4. Uniformization of abelian varieties

In this section we show that the classical theory of analytic uniformization of abelian varieties over \mathbb{C} has a good analogue in the category of rigid-analytic spaces. We study the quotients of $(\mathbb{G}_{m,K}^{\text{an}})^g$ by discrete rank- g lattices, and we consider questions of algebraicity of such quotients. As a motivation, we first recall the well-known corresponding theorem over \mathbb{C} .

4.1. Complex case. Let $V \cong \mathbb{C}^g$ be a finite-dimensional complex vector space over \mathbb{C} . A map $H : V \times V \rightarrow \mathbb{C}$ is a *Hermitian form* if it is linear in the first variable, anti-linear in the second variable, and $H(u, v) = \overline{H(v, u)}$. The Hermitian form H is *positive-definite* if $H(u, u) > 0$ for all $u \in V \setminus \{0\}$. By a *lattice* Λ in V we mean a discrete subgroup isomorphic to \mathbb{Z}^{2g} . A *Riemann form* on $G = V/\Lambda$ is a positive-definite Hermitian form on V such that $\text{Im}(H)$ is \mathbb{Z} -valued on $\Lambda \times \Lambda$.

THEOREM 4.1. *The quotient $G = V/\Lambda$ is the analytification of a complex abelian variety if and only if G possesses a Riemann form.*

G is an abelian variety if and only if it can be embedded into some projective space $\mathbb{P}_{\mathbb{C}}^N$. (The image of G is an algebraic group variety according to Serre's GAGA.) This translates into a question of existence of line bundles on \mathbb{C}^g with certain properties, which then can be shown to be equivalent to the existence of a Riemann form. Relatively few G have such a form, hence not all of them are algebraic. For the details we refer to [M1, Ch.1].

4.2. Non-archimedean case. As with Tate curves, we consider the multiplicative version of the construction in §4.1.

Let $T = \mathbb{G}_{m,K}^g = \text{Spec } K[Z_1, Z_1^{-1}, \dots, Z_g, Z_g^{-1}]$ be the algebraic g -dimensional torus. Let T^{an} be the rigid-analytic space corresponding to T . There is a natural group homomorphism $\ell : T^{\text{an}} \rightarrow \mathbb{R}^g$ given by $\ell(\mathbf{z}) = (-\log |z_1|, \dots, -\log |z_g|)$. A *lattice* Λ in T^{an} is a torsion-free, \mathbb{Z} -rank g subgroup of $T(K)$ which is discrete in T^{an} , i.e., the intersection of each affinoid with Λ is finite. Equivalently, $\ell(\Lambda)$ is a rank- g lattice in \mathbb{R}^g . We would like to give $G = T^{\text{an}}/\Lambda$ a structure of a rigid-analytic space.

For simplicity assume the valuation on K is discrete, and choose a basis for T^{an} and \mathbb{R}^g such that $\ell(\Lambda) = \mathbb{Z}^g$. Consider the unit cube

$$S := \{(x_1, \dots, x_g) \in \mathbb{R}^g \mid |x_i| \leq 1/2 \text{ for all } i\},$$

so \mathbb{R}^g is covered by $\{a + S\}$ with $a \in \mathbb{Z}^g$. The set $U := \ell^{-1}(S)$ is an affinoid subspace of T^{an} . The translates of U by elements of Λ , $\lambda U := \ell^{-1}(\ell(\lambda) + S)$, are again affinoid subsets and give an admissible covering $T^{\text{an}} = \bigcup_{\lambda \in \Lambda} \lambda U$. We use

the covering of G by the images $V_\lambda := \text{pr}(\lambda U)$ to endow it with a structure of a rigid-analytic space. There is a notion of *proper* rigid-analytic space, and it is not hard to show that the group space G is proper. The Riemann form condition in rigid setting translates into the following:

THEOREM 4.2. *G is an abelian variety if and only if there is a homomorphism*

$$H : \Lambda \rightarrow \Lambda^\vee := \text{Hom}(T^{\text{an}}, \mathbb{G}_{m,K}^{\text{an}})$$

such that $H(\lambda)(\lambda') = H(\lambda')(\lambda)$, and the symmetric bilinear form on $\Lambda \times \Lambda$

$$\begin{aligned} \Lambda \times \Lambda &\rightarrow \mathbb{Z} \\ \lambda, \lambda' &\mapsto \langle \lambda, \lambda' \rangle := -\log |H(\lambda')(\lambda)| \end{aligned}$$

is positive definite.

The strategy of the proof of this theorem is very similar to the one over \mathbb{C} . First, we have the following analogue of the *Appell-Humbert theorem*.

PROPOSITION 4.3. *There is a functorial isomorphism of groups*

$$\text{Pic}(G) \cong \text{H}^1(\Lambda, \mathcal{O}(T^{\text{an}})^\times),$$

where $\mathcal{O}(T^{\text{an}})^\times = \{\beta \cdot Z_1^{\alpha_1} \cdots Z_g^{\alpha_g} \mid \beta \in K^\times \text{ and } (\alpha_1, \dots, \alpha_g) \in \mathbb{Z}^g\}$ is the multiplicative group of nowhere-vanishing holomorphic functions on T^{an} , and $\Lambda \subset T^{\text{an}}(K)$ acts through its translation action on T^{an} . Moreover, every element in $\text{H}^1(\Lambda, \mathcal{O}(T^{\text{an}})^\times)$ can be uniquely represented by a 1-cocycle $Z_\lambda(\mathbf{z}) = d(\lambda)H(\lambda)(\mathbf{z})$, where

$$H : \Lambda \rightarrow \Lambda^\vee = \{Z_1^{\alpha_1} \cdots Z_g^{\alpha_g} \mid (\alpha_1, \dots, \alpha_g) \in \mathbb{Z}^g\}$$

is a group homomorphism and $d : \Lambda \rightarrow K^\times$ is a map satisfying

$$d(\lambda_1 \lambda_2) d(\lambda_1)^{-1} d(\lambda_2)^{-1} = H(\lambda_2)(\lambda_1).$$

PROOF. The proof is essentially the same as over \mathbb{C} [**M1**, Ch.1], using the fact [**FvdP**, Ch.VI] that the line bundles on T^{an} are trivial. \square

Thus, every line bundle L on G corresponds to a cocycle Z_λ and every such cocycle is given by a pair (H, d) . That is, every line bundle corresponds to a pair (H, d) , and we will denote this line bundle by $L(H, d)$. One easily checks that $L(H_1, d_1) \cong L(H_2, d_2)$ if and only if $H_1 = H_2$ and $d_1(\lambda)/d_2(\lambda) = \lambda_1^{\alpha_1} \cdots \lambda_g^{\alpha_g}$ for some fixed $(\alpha_1, \dots, \alpha_g) \in \mathbb{Z}^g$.

The global sections of a line bundle L on G can be explicitly described using theta series and their Fourier expansions (this relies on the explicit description of L given above). It turns out that $L(H, d)$ is ample if and only if $-\log |H|$ is positive definite. The positive definiteness is used to show that enough explicit formal Fourier series are convergent, and hence give honest sections, so that $L^{\otimes 3}$ is very ample. Finally, one uses rigid-analytic GAGA theorems, proved by Kiehl, to conclude that G is algebraic (every closed analytic subspace of $\mathbb{P}_K^{n,\text{an}}$ is the analytification of a unique closed subscheme of \mathbb{P}_K^n).

Given an abelian variety A of dimension g over K it is natural to ask whether there is an isomorphism $A^{\text{an}} \cong T^{\text{an}}/\Lambda$ for some lattice Λ . If there is such an isomorphism then we say that A *admits analytic uniformization*. The corresponding question over \mathbb{C} has an affirmative answer, i.e., all abelian varieties over \mathbb{C} have an

analytic uniformization. Indeed, let $V_A = H^0(A, \Omega^1)^\vee$ be the tangent space to A at the identity, and let $\Delta_A = H_1(A, \mathbb{Z})$. There is a natural injective homomorphism

$$\begin{aligned} \Delta_A &\rightarrow V_A \\ \gamma &\mapsto (\omega \mapsto \int_\gamma \omega), \end{aligned}$$

where $\omega \in H^0(A, \Omega^1)$. The quotient V_A/Δ_A , which is isomorphic to $(\mathbb{C}^\times)^g/\Lambda$ via \exp , can be identified with $A(\mathbb{C})$; see [M1, Ch.1]. We already saw on the example of the Tate curve that the same question over non-archimedean fields has a more complicated answer. To characterize the abelian varieties which admit analytic uniformization one needs three fundamental concepts: Néron models, formal schemes with their “generic fibre” functor, and the notion of analytic reduction of a rigid-analytic space. We will only outline the ideas involved.

4.3. Analytic reductions. Given a rigid-analytic space X and an admissible affinoid covering \mathcal{U} having certain good properties, one can associate to it a locally finite type scheme \bar{X} over k , called the *analytic reduction* of X with respect to \mathcal{U} .

First, let $X = \mathrm{Sp}(A)$. Denote by A° the R -algebra $\{f \in A \mid \|f\| \leq 1\}$. There is a natural ideal $A^{\circ\circ} = \{f \in A \mid \|f\| < 1\}$. The canonical analytic reduction of X is the k -scheme $\bar{X} = \mathrm{Spec}(\bar{A})$, where $\bar{A} = A^\circ/A^{\circ\circ}$. This can be proved to be a reduced scheme of finite type over the residue field. Moreover, there is a canonical surjective map of sets $\mathrm{Max}(A) \rightarrow \mathrm{Max}(\bar{A})$. This last map is constructed as follows (for simplicity assume K is algebraically closed). Using an analogue of the *maximum modulus principle*, it is possible to show that for $f \in A$, $\|f\| = \max_{x \in X} |f(x)|$. Hence under the quotient map $A \rightarrow A/\mathfrak{m}_x = K$ given by evaluation at $x \in X$, A° maps to R and $A^{\circ\circ}$ maps to \mathfrak{m} . This induces a k -algebra homomorphism $\bar{A} \rightarrow k$. Since any such homomorphism uniquely corresponds to some $\bar{x} \in \mathrm{Max}(\bar{A})$, we get the map $\mathrm{Max}(A) \rightarrow \mathrm{Max}(\bar{A})$, $x \mapsto \bar{x}$.

Next, if X is an analytic space and $X = \bigcup U_i$ is an admissible covering such that the maps $\overline{U_i \cap U_j} \rightrightarrows \overline{U_i}, \overline{U_j}$ are open immersions for all i and j , then the canonical reductions $\overline{U_i}$ can be glued together along $\overline{U_i \cap U_j}$'s to give a k -scheme \bar{X} (depending on the affinoid covering $\bigcup U_i$) with a surjective map of sets $X \rightarrow \bar{X}$.

In the examples of this subsection we assume that K is algebraically closed. As in §3.2, this is done for expository purposes only, and all conclusions in the examples are valid for general K .

EXAMPLE 4.4. Let $A = T_1 = K\langle Z \rangle$ be the 1-dimensional Tate algebra. From the definition of the norm (3.1) on T_n it is clear that $A^\circ = R\langle Z \rangle$ and $A^{\circ\circ} = \mathfrak{m}\langle Z \rangle$. Since any power series in A° , up to a finite number of summands, belongs to $A^{\circ\circ}$, we get $\bar{A} = k[Z]$. The topological space of $\mathrm{Sp}(A)$ is the unit disc $\mathbb{D} = \{z \in K \mid |z| \leq 1\}$. It is easy to see that the evaluation at z of the elements of A induces the evaluation at $\bar{z} = z \pmod{\mathfrak{m}}$ of the elements of $k[Z]$. Hence under $\mathrm{Max}(A) \rightarrow \mathrm{Max}(\bar{A})$ the interior $\mathbb{D}^0 = \{z \in K \mid |z| < 1\}$ of \mathbb{D} maps to the origin 0 of \mathbb{A}_k^1 , and the boundary $\partial\mathbb{D} = \{z \in K \mid |z| = 1\}$ maps onto $\mathbb{A}_k^1 - \{0\}$. A similar argument shows that $\bar{T}_n = k[Z_1, \dots, Z_n]$.

EXAMPLE 4.5. Let $A = K\langle Z, U \rangle / (ZU - 1) = K\langle Z, Z^{-1} \rangle$. As a power series ring A is given by $\left\{ \sum_{n \in \mathbb{Z}} a_n Z^n \mid \lim_{n \rightarrow \pm\infty} |a_n| = 0 \right\}$, with the norm $\left\| \sum_{n \in \mathbb{Z}} a_n Z^n \right\| =$

$\max_{n \in \mathbb{Z}} |a_n|$. The underlying topological space is $\{z \in K \mid |z| = 1\}$. As in Example 4.4 we see that $\overline{A} = k[Z, Z^{-1}]$, so $\overline{X} = \mathbb{G}_{m,k}$.

EXAMPLE 4.6. Let $X = \{z \in K \mid |q| \leq |z| \leq 1\}$ with $q \in K^\times$ and $|q| < 1$. We have

$$A = K\langle Z, U \rangle / (ZU - q) = K\langle Z, \frac{q}{Z} \rangle$$

$$= \left\{ f = \sum_{n \geq 0} a_n Z^n + \sum_{n > 0} b_n \left(\frac{q}{Z}\right)^n \mid \lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} |b_n| = 0 \right\},$$

and $\|f\| = \max_{n_1, n_2} \{|a_{n_1}|, |b_{n_2}|\}$. Hence $\overline{A} = k[Z', U'] / Z'U'$, where Z' is the image of Z and U' is the image of q/Z . The analytic reduction \overline{X} consists of two affine lines ℓ_1 and ℓ_2 intersecting (transversally) at one point P . The points in X with $|z| = 1$ are mapped onto $\ell_1 - \{P\}$, and the points with $|z| = |q|$ are mapped onto $\ell_2 - \{P\}$. Finally, the points which satisfy $|q| < |z| < 1$ are mapped to P . This is an analytic incarnation of a deformation of a node.

EXAMPLE 4.7. Let $X = \mathbb{G}_{m,K}^{\text{an}}$. Take $q \in K^\times$ with $|q| < 1$. For the affinoid covering $\bigcup_{n \in \mathbb{Z}} X_n$ in Example 3.12, \overline{X} is an infinite chain of \mathbb{P}_k^1 's. This follows from the previous example. Indeed, \overline{X}_n consists of two affine lines $\ell_{1,n}$ and $\ell_{2,n}$ intersecting at a point P_n . The reduction \overline{X}_{n+1} is glued to \overline{X}_n by identifying $\ell_{2,n} - \{P_n\}$ and $\ell_{1,n+1} - \{P_{n+1}\}$ so that there results \mathbb{P}_k^1 with $\mathbb{P}_k^1 - \{0\} = \ell_{2,n}$ and $\mathbb{P}_k^1 - \{\infty\} = \ell_{1,n+1}$. To see this last part note that the affinoid algebra corresponding to X_n is $K\langle q^{-n/2}Z, q^{(n+1)/2}Z^{-1} \rangle$, and X_n is glued to X_{n+1} via $q^{(n+1)/2}Z^{-1} \mapsto q^{-(n+1)/2}Z$. On the other hand, $q^{(n+1)/2}Z^{-1}$ gives the parameter u on $\ell_{2,n} = \text{Spec}(k[u])$, whereas $q^{-(n+1)/2}Z$ gives the parameter z on $\ell_{1,n+1} = \text{Spec}(k[z])$, cf. Example 4.6. Hence $\ell_{1,n+1}$ is glued to $\ell_{2,n}$ via $u \mapsto z = 1/u$.

EXAMPLE 4.8. The analytic reduction of $T^{\text{an}} = (\mathbb{G}_{m,K}^{\text{an}})^g$ is the product situation of the previous example: $\overline{T^{\text{an}}} = \overline{X}^g$ with $X = \mathbb{G}_{m,K}^{\text{an}}$. Each irreducible component of $\overline{T^{\text{an}}}$ is isomorphic to the product of g copies of \mathbb{P}_k^1 's. The irreducible components are glued together along unions of coordinate hyperplanes in k^g . In particular, every singular point in $\overline{T^{\text{an}}}$ is locally isomorphic to a singularity in the intersection of d coordinate hyperplanes in k^g with some (varying) $d \leq g$. If we remove the singular locus from $\overline{T^{\text{an}}}$ then we get $\bigcup_{\alpha \in \mathbb{Z}^g} (\mathbb{G}_{m,k}^g)_\alpha$, i.e., a disjoint union of copies of $\mathbb{G}_{m,k}^g$ labelled by g -tuples of integers.

EXAMPLE 4.9. Let $G = T^{\text{an}}/\Lambda$. We use the affinoid covering of T^{an} in §4.2. By construction, this affinoid covering is preserved under the action of Λ . Hence the group Λ acts equivariantly on the analytic reduction $T^{\text{an}} \rightarrow \overline{T^{\text{an}}}$. The analytic reduction of T^{an}/Λ is a finite union of copies of g -tuple products $(\mathbb{P}_k^1) \times \cdots \times (\mathbb{P}_k^1)$ glued together along coordinate hyperplanes, and the smooth locus of \overline{G} is an extension of a finite abelian group by $\mathbb{G}_{m,k}^g$.

As a more concrete example of this, let's consider the analytic reduction of the Tate curve $\mathcal{E} = \mathbb{G}_{m,K}^{\text{an}}/q^{\mathbb{Z}}$. Refine the affinoid covering $\{X_i\}_{i \in \mathbb{Z}}$ of $\mathbb{G}_{m,K}^{\text{an}}$ given in Example 3.12, so that q acts by $X_n \mapsto X_{n+2m}$ for some fixed $m \in \mathbb{Z}^+$ (this is easily done by appropriately decreasing the radii of X_i). Then q acts on the infinite chain $\{\mathbb{P}_k^1\}_{i \in \mathbb{Z}}$ forming the reduction of $\mathbb{G}_{m,K}^{\text{an}}$ by m -shifts, i.e., by mapping

$(\mathbb{P}_k^1)_i$ identically onto $(\mathbb{P}_k^1)_{i+m}$. The analytic reduction $\bar{\mathcal{E}}$ is a union of m copies of \mathbb{P}_k^1 , arranged as the sides of a polygon, and intersecting each other transversally (if $m = 1$ then there is one \mathbb{P}_k^1 forming a node); see Figure 1.

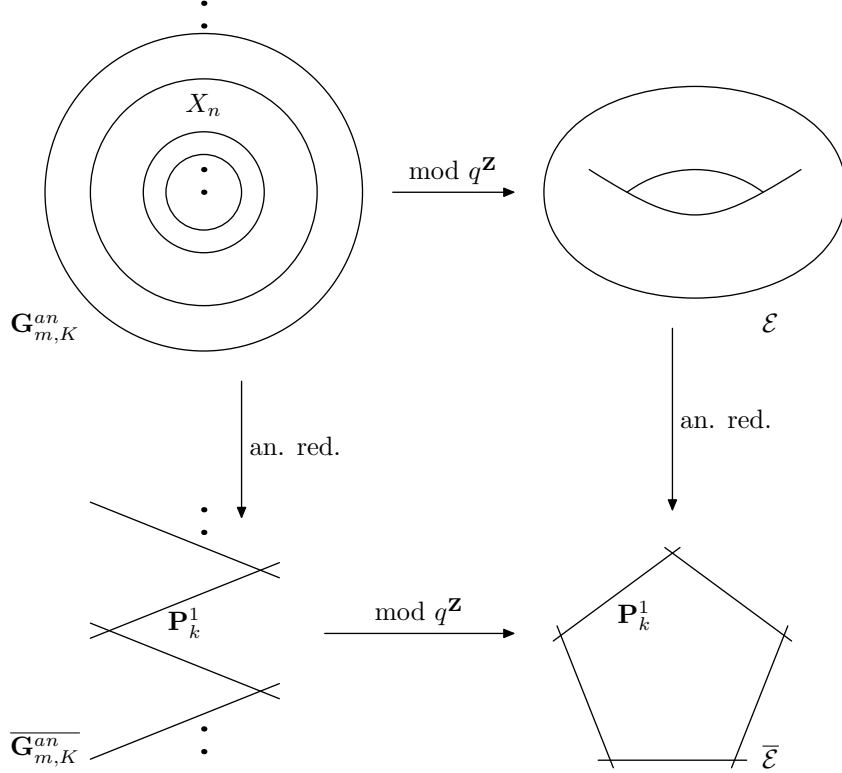


FIGURE 1. Analytic reductions of $\mathbb{G}_{m,K}^{an}$ and \mathcal{E}

4.4. Formal schemes and their generic fibres. There is a close and very useful relationship (discovered by Raynaud) between formal schemes over R and rigid spaces over K . We will indicate this relationship first making it explicit for the affinoid space $\mathrm{Sp}(T_n)$. Consider the polynomial ring $P = R[Z_1, \dots, Z_n]$. The formal completion of P with respect to the ideal $\mathfrak{m}P$ is, by definition, the projective limit $\varprojlim (P/\mathfrak{m}^m P)$. This R -algebra is denoted by $\mathfrak{T}_n = R\langle Z_1, \dots, Z_n \rangle$. It consists of the power series $\sum_{\mathbf{s} \in \mathbb{N}^n} c_{\mathbf{s}} Z_1^{s_1} \cdots Z_n^{s_n}$ with all $c_{\mathbf{s}} \in R$ and such that for every $m > 0$ there are only finitely many \mathbf{s} 's with $c_{\mathbf{s}} \notin \mathfrak{m}^m R$. That is, this is the set of elements in T_n with coefficients in R . The affine formal scheme $\mathrm{Spf}(\mathfrak{T}_n)$ is the ringed space with underlying topological space $\mathrm{Spec} k[Z_1, \dots, Z_n]$ and the structure sheaf $R\langle Z_1, \dots, Z_n \rangle$; this is the “formal n -ball”. The connection to $\mathrm{Sp}(T_n)$ results from the isomorphism $K \otimes_R \mathfrak{T}_n = T_n$. Now we extend this construction to a functor from (certain) formal schemes over R to rigid spaces over K .

Let A be a reduced affinoid algebra. The R -algebra A° can be proved to be \mathfrak{m} -adically separated and complete, and in fact a quotient of $R\langle Z_1, \dots, Z_n \rangle$ for some n , so the formal scheme $\mathrm{Spf}(A^\circ)$ may be realized as a closed formal subscheme

of a “formal n -ball” over $\mathrm{Spf}(R)$ for some $n \geq 0$. The underlying topological space of $\mathrm{Spf}(A^\circ)$ is the analytic reduction $\mathrm{Spec}(\overline{A})$, and $\mathrm{Sp}(K \otimes_R A^\circ) = \mathrm{Sp}(A)$ is (by definition) the rigid space associated to $\mathrm{Spf}(A^\circ)$. This construction can be extended to formal schemes which are separated and locally of finite type over R . One covers the formal scheme \mathfrak{X} by formal affines $\mathrm{Spf}(\mathfrak{A})$, and glues $\mathrm{Sp}(\mathfrak{A} \otimes_R K)$'s. The resulting rigid-analytic space is called the *generic fibre* of \mathfrak{X} , and is denoted by $\mathfrak{X}^{\mathrm{rig}}$. One can check that $\mathfrak{X} \rightarrow \mathfrak{X}^{\mathrm{rig}}$ is functorial; see [BL2].

Given a separated scheme X over R which admits a locally finite affine covering, we now have two ways to associate a rigid-analytic space to it. First, we could consider the generic fibre $X_K := X \times_R K$ of X , which is a locally finite type K -scheme, and take its analytification X_K^{an} . Second, we could consider the formal completion \mathfrak{X} of X along its closed fibre (i.e., the formal completion of X with respect to an ideal of definition \mathfrak{m} of R), and then take its generic fibre $\mathfrak{X}^{\mathrm{rig}}$. In general, there is a morphism

$$(4.1) \quad i_X : \mathfrak{X}^{\mathrm{rig}} \rightarrow X_K^{\mathrm{an}}$$

which is an open immersion, and it is an isomorphism for proper X over R (essentially because of the valuative criterion for properness); see Theorem 5.3.1 in [Co2].

EXAMPLE 4.10. Let $X = \mathbb{A}_R^n$. In this case $X \times_R K = \mathbb{A}_K^n$, hence $(X_K)^{\mathrm{an}} = \mathbb{A}_K^{n, \mathrm{an}}$ is the analytification of the n -dimensional affine space over K . The completion along the closed fibre is $\mathrm{Spf}(\mathfrak{T}_n)$, so $\mathfrak{X}^{\mathrm{rig}}$ is the unit polydisc $\mathrm{Sp}(T_n)$. It is easy to give an explicit admissible covering of X^{an} such that $\mathfrak{X}^{\mathrm{rig}}$ is one of the affinoids in that covering; see Example 3.11 for the case $n = 1$.

EXAMPLE 4.11. The most important example for us is when $X = \mathbb{G}_{m, R}$, a split torus over R , so $(X \times_R K)^{\mathrm{an}} = \mathbb{G}_{m, K}^{\mathrm{an}}$ is the analytic one-dimensional torus over K and $\mathfrak{X} = \mathrm{Spf}(R\langle Z, Z^{-1} \rangle)$. Hence $\mathfrak{X}^{\mathrm{rig}} = \mathrm{Sp}(K\langle Z, Z^{-1} \rangle)$. In this case i_X is the open immersion of the “unit circle” into the “punctured plane”.

Finally, we recall Grothendieck’s formal GAGA theorem from Chapter 5 of EGA III, which we will need in §4.5. Let X be a proper scheme over $\mathrm{Spec}(R)$, and let \widehat{X} denote the formal completion of X along \mathfrak{m} . Grothendieck’s theorem is two-fold: first of all, $X \rightarrow \widehat{X}$ is a fully faithful functor. Moreover, if \mathfrak{X} is proper over $\mathrm{Spf}(R)$ and \mathcal{L} is an invertible $\mathcal{O}_{\mathfrak{X}}$ -module such that $\mathcal{L}_0 := \mathcal{L}/\mathfrak{m}\mathcal{L}$ is ample on $\mathfrak{X}_0 = \mathfrak{X} \times_{\mathrm{Spf}(R)} \mathrm{Spec}(k)$ then there is a proper scheme X over $\mathrm{Spec}(R)$ such that $\mathfrak{X} \cong \widehat{X}$. In this latter case we say \mathfrak{X} is *algebraic*.

4.5. Néron models. Assume K is a complete, discretely valued field. We return to our initial goal of characterizing the abelian varieties representable as quotients of analytic tori by lattices. First we recall the notion of Néron model, which plays an important role in our characterization.

DEFINITION 4.12. Let A be an abelian variety over K . A *Néron model* of A is a smooth, separated, finite type scheme \mathcal{A} over R such that $\mathcal{A}_K \cong A$ and which satisfies the following universal property, called *Néron mapping property*: for each smooth R -scheme \mathcal{B} and each K -morphism $u_K : \mathcal{B}_K \rightarrow A$ there is a unique R -morphism $u : \mathcal{B} \rightarrow \mathcal{A}$ extending u_K .

It is clear from the Néron mapping property that if \mathcal{A} exists then it is unique, and has a unique R -group scheme structure which extends the K -group scheme

structure on A . It is a non-trivial theorem (due to Néron) that \mathcal{A} always exists; see [BLR].

Let $G = T^{\text{an}}/\Lambda$ and let \mathcal{U} be the explicit affinoid covering used to give G its analytic structure as in Example 4.9. As we discussed in §4.4, we can construct a formal scheme \mathfrak{G} , using the covering \mathcal{U} , such that $\mathfrak{G}^{\text{rig}} \cong G$ (recall that G is proper). Assume G is algebraic. Using Grothendieck’s GAGA, one can deduce that \mathfrak{G} is also algebraic. Let \mathcal{G} be the proper scheme over $\text{Spec}(R)$ such that $\widehat{\mathcal{G}} \cong \mathfrak{G}$. There is an isomorphism $\mathcal{G}_K^{\text{an}} \cong G$ via (4.1). Let A be the abelian variety for which $A^{\text{an}} \cong G$. Then \mathcal{G} is a model of A over R . Let \mathcal{G}' be the scheme over R which is obtained by removing the singular locus of \mathcal{G}_k from \mathcal{G} . The R -scheme \mathcal{G}' is smooth, separated, of finite type, and the identity section (extending that on A) lies in \mathcal{G}' . Its closed fibre is an extension of a finite abelian group by $\mathbb{G}_{m,k}^g$; c.f. Example 4.9. Let \mathcal{A} be the Néron model of A over R . By the Néron mapping property the isomorphism $\mathcal{G}'_K \xrightarrow{\sim} \mathcal{A}_K$ uniquely extends to an R -morphism $\mathcal{G}' \rightarrow \mathcal{A}$. The image of \mathcal{G}' is open in the fibres of \mathcal{A} . Hence the connected component of the identity \mathcal{A}_k^0 of \mathcal{A}_k is isomorphic to the split algebraic torus $\mathbb{G}_{m,k}^g$. Abelian varieties whose Néron models have this property are called *totally degenerate*. We conclude that abelian varieties which admit analytic uniformization must be totally degenerate. (Being totally degenerate is a rather special property. For example, the abelian varieties which extend to abelian schemes over $\text{Spec}(R)$ do not have this property.)

The property of being “totally degenerate” is also sufficient for the existence of an analytic uniformization in the sense discussed above. To see how this goes, let A be a totally degenerate abelian variety of dimension g over K . We apply (4.1) to the relative connected component of the identity \mathcal{A}^0 of \mathcal{A} , that is, to the largest open subscheme of \mathcal{A} containing the identity section which has connected fibres. Since $\mathcal{A}_K^0 = A$, on the right side of (4.1) we have the analytification A^{an} of A . On the other hand, $\mathcal{A}_k^0 \cong \mathbb{G}_{m,k}^g$. The rigidity of tori (cf. Theorem 3.6 in Exposé IX of **SGA 3**) implies that the formal completion of \mathcal{A}^0 along its closed fibre is uniquely isomorphic to the formal split torus $\widehat{\mathbb{G}}_m^g = (\text{Spf } R\langle Z, Z^{-1} \rangle)^g$ respecting a choice of isomorphism $\mathcal{A}_k^0 \cong \mathbb{G}_{m,k}^g$. Thus, by Example 4.11 we get an open immersion of analytic groups $i_{\mathcal{A}^0} : (\widehat{\mathbb{G}}_m^g)^{\text{rig}} \hookrightarrow A^{\text{an}}$. We also have the analytic torus $T^{\text{an}} = (\mathbb{G}_{m,K}^{\text{an}})^g$ associated to $(\widehat{\mathbb{G}}_m^g)^{\text{rig}}$ and an open immersion $(\widehat{\mathbb{G}}_m^g)^{\text{rig}} \hookrightarrow T^{\text{an}}$. The key fact is that $i_{\mathcal{A}^0}$ extends uniquely to a rigid-analytic group morphism $T^{\text{an}} \rightarrow A^{\text{an}}$, and its kernel is a lattice $\Lambda \subset (K^\times)^g$ of rank g . This induces an isomorphism of analytic groups $T^{\text{an}}/\Lambda \cong A^{\text{an}}$; see Theorem 1.2 in [BL1].

$$\begin{array}{ccc}
 & T^{\text{an}} & \\
 \swarrow & & \searrow \\
 (\widehat{\mathbb{G}}_m^g)^{\text{rig}} & \xrightarrow{i_{\mathcal{A}^0}} & A^{\text{an}}
 \end{array}$$

We record all of this into a theorem:

THEOREM 4.13. *An abelian variety over K admits analytic uniformization (in the sense of §4.2) if and only if it is totally degenerate.*

Note that totally degenerate abelian varieties are essentially the ones considered in Mumford’s fundamental paper [M3], except that Mumford allowed a normal complete base with dimension possibly greater than 1 and so he had to use formal

geometry throughout. For a 1-dimensional base, his work can be motivated by the above picture from rigid-analytic geometry; cf. [BL1].

References

- [BGR] S. Bosch, U. Güntzer, and R. Remmert, *Non-archimedean analysis. A systematic approach to rigid analytic geometry*, Springer, 1984.
- [BL1] S. Bosch and W. Lütkebohmert, *Degenerating abelian varieties*, *Topology* **30** (1991), 653–698.
- [BL2] S. Bosch and W. Lütkebohmert, *Formal and rigid geometry I*, *Math. Ann.* **295** (1993), 291–317.
- [BLR] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Springer, 1990.
- [Ca] H. Carayol, *Non-abelian Lubin-Tate theory*, in: *Automorphic forms, Shimura varieties, and L-functions* (1990), 15–39.
- [CF] C.-L. Chai and G. Faltings, *Degenerations of abelian varieties*, Springer, 1990.
- [Co1] B. Conrad, *Rigid-analytic geometry*, Course at the University of Michigan, Spring 2003.
- [Co2] B. Conrad, *Irreducible components of rigid spaces*, *Ann. Inst. Fourier* **49** (1999), 473–541.
- [D] V. Drinfeld, *Elliptic modules*, *Math. Sbornik* **94** (1974), 594–627.
- [FvdP] J. Fresnel and M. van der Put, *Rigid analytic geometry and its applications*, Birkhäuser, 2004.
- [HT] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, *Ann. Math. Studies* 151, Princeton Univ. Press, 2001.
- [LRS] G. Laumon, M. Rapoport, and U. Stuhler, *\mathcal{D} -elliptic sheaves and the Langlands correspondence*, *Invent. Math.* **113** (1993), 217–338.
- [M1] D. Mumford, *Abelian varieties*, Oxford University Press, 1970.
- [M2] D. Mumford, *An analytic construction of degenerating curves over complete local rings*, *Compositio Math.* **24** (1972), 129–174.
- [M3] D. Mumford, *An analytic construction of degenerating abelian varieties over complete local rings*, *Compositio Math.* **24** (1972), 239–272.
- [OvdP] F. Oort and M. van der Put, *A construction of an abelian variety with a given endomorphism algebra*, *Compositio Math.* **67** (1988), 103–120.
- [Ra] M. Raynaud, *Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d’Abhyankar*, *Invent. Math.* **116** (1994), 425–462.
- [Ri] K. Ribet, *On the modular representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms*, *Invent. Math.* **100** (1990), 431–476.
- [S1] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, *Ann. Inst. Fourier* **6** (1956), 1–42.
- [S2] J.-P. Serre, *Lie algebras and Lie groups*, Benjamin, 1965.
- [Si1] J. Silverman, *The arithmetic of elliptic curves*, GTM 106, Springer, 1986.
- [Si2] J. Silverman, *Advanced topics in the arithmetic of elliptic curves*, GTM 151, Springer, 1994.
- [T1] J. Tate, *Rigid analytic spaces*, *Invent. Math.* **12** (1971), 257–289.
- [T2] J. Tate, *A review of non-Archimedean elliptic functions*, in: *Elliptic curves, modular forms and Fermat’s last theorem*, J. Coates and S.-T. Yau (eds.) (1995), 162–184.

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA 94305
E-mail address: papikian@math.stanford.edu