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Abelian subvarieties of Drinfeld Jacobians and congruences modulo the characteristic

Mihran Papikian

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Abstract We relate the existence of Frobenius morphisms into the Jacobians of Drinfeld modular curves to the existence of congruences between cusp forms.

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1 Introduction

1.1 Congruence primes

Recall the notion of *congruence primes* in the setting of classical cusp forms. Let $k \geq 2$ and $N \geq 1$ be integers, and let $S(\mathbb{C})$ be the $\mathbb{C}$-vector space of weight-k cusp forms on $\Gamma_0(N)$. This vector space has a natural rational (resp. integral) structure given by the space $S(\mathbb{Q})$ (resp. the lattice $S(\mathbb{Z})$) of cusp forms whose Fourier coefficients are rational numbers (resp. are integers), i.e., $S(\mathbb{C}) = S(\mathbb{Z}) \otimes \mathbb{C}$. Suppose we are given a direct sum decomposition

$$S(\mathbb{Q}) = X \oplus Y. \tag{1}$$

Denote $X(\mathbb{Z}) := X \cap S(\mathbb{Z})$ and $Y(\mathbb{Z}) := Y \cap S(\mathbb{Z})$, so that $X(\mathbb{Z}) \oplus Y(\mathbb{Z}) \subset S(\mathbb{Z})$ is a full lattice in $S(\mathbb{C})$. The primes dividing the order of the finite abelian group $S(\mathbb{Z})/(X(\mathbb{Z}) \oplus Y(\mathbb{Z}))$ are called the *congruence primes* for the decomposition (1), cf. [20,21]. The congruence primes for different decompositions have been extensively studied by Hida, Ribet and others. An especially important case of

M. Papikian (✉)
Department of Mathematics, Stanford University, Stanford, CA 94305, USA
e-mail: papikian@math.stanford.edu

(1) is when one takes X to be the space of ℓ-old forms for some $\ell \parallel N$, and Y to be the orthogonal complement of X with respect to the Petersson inner product. These congruence primes play a key role in Mazur–Ribet level lowering theorem [23], whose famous application is the fact that Fermat's Last Theorem follows from the Shimura-Taniyama conjecture.

1.2 Main result

The aim of this paper is to relate the existence of Frobenius morphisms into Jacobian varieties of Drinfeld modular curves to the existence of mod-p congruences between Drinfeld's cusp forms, where p is the characteristic of the field. Our main result can be interpreted as a certain function field analogue of Mazur–Ribet level lowering theorem mentioned above. Before stating this result, we need to introduce some terminology and notation.

The situation which most closely resembles the classical one is when our function field $F = \mathbb{F}_q(t)$ is the field of rational functions on $\mathbb{P}^1_{\mathbb{F}_q}$. Here $\mathbb{F}_q$ is the finite field of q elements, where q is a power of a prime number p. To get the analogue of $\mathbb{Z}$ one has to fix a closed point on $\mathbb{P}^1_{\mathbb{F}_q}$, suggestively denoted by ∞, and consider the subring A of F consisting of functions regular away from ∞. We will choose ∞ to be rational, $\deg(\infty) = 1$. Without loss of generality, $\infty = \frac{1}{t}$ and $A = \mathbb{F}_q[t]$ is the polynomial ring in one variable over $\mathbb{F}_q$. Let $\mathfrak{n}$ be an ideal of A and consider the Drinfeld modular curve $X_0(\mathfrak{n})_F$ of level $\mathfrak{n}$. This is a compactified coarse moduli scheme for pairs $(D, Z_\mathfrak{n})$ consisting of a Drinfeld A-module D of rank-2 over F and a $\mathfrak{n}$-cyclic subgroup $Z_\mathfrak{n}$ of D. It is a proper, smooth, geometrically connected curve over F. Denote by $J_0(\mathfrak{n})$ the Jacobian variety of $X_0(\mathfrak{n})_F$. This abelian variety has bad reduction exactly at the places in the support of $\mathfrak{n}$ and at ∞.

Let $S(\mathfrak{n}, \mathbb{C})$ be the vector space of $\mathbb{C}$-valued cuspidal harmonic cochains invariant under the action of the Hecke congruence group $\Gamma_0(\mathfrak{n})$. We refer to Sect.3.2 for the precise definition of this space; for now it suffices to say that $S(\mathfrak{n}, \mathbb{C})$ has an interpretation as a space of automorphic cusp forms on $\mathrm{GL}_2(\mathbb{A}_F)$, which plays a role similar to that of weight-2, level-N cusp forms in the classical theory. The space $S(\mathfrak{n}, \mathbb{C})$ has a canonical integral structure given by the $\mathbb{Z}$-valued harmonic cochains $S(\mathfrak{n}, \mathbb{Z})$. It is known that $S(\mathfrak{n}, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank equal to the genus of $X_0(\mathfrak{n})_F$, and $S(\mathfrak{n}, \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{C} = S(\mathfrak{n}, \mathbb{C})$. The lattice $S(\mathfrak{n}, \mathbb{Z})$ is the analogue of weight-2 cusp forms having integral Fourier expansion. Suppose $\mathfrak{n} = \mathfrak{p}\mathfrak{m}$, where $\mathfrak{p}$ is a prime ideal coprime to the ideal $\mathfrak{m}$. There are two natural injective homomorphisms $S(\mathfrak{m}, \mathbb{C}) \rightrightarrows S(\mathfrak{n}, \mathbb{C})$. Denote the subspace generated by the images by $S(\mathfrak{n}, \mathbb{C})^{\mathfrak{p}-\mathrm{old}}$, and denote the intersection $S(\mathfrak{n}, \mathbb{Z}) \cap S(\mathfrak{n}, \mathbb{C})^{\mathfrak{p}-\mathrm{old}}$ by $S(\mathfrak{n}, \mathbb{Z})^{\mathfrak{p}-\mathrm{old}}$. Denote by $S(\mathfrak{n}, \mathbb{Z})^{\mathfrak{p}-\mathrm{new}}$ the orthogonal complement in $S(\mathfrak{n}, \mathbb{Z})$ of $S(\mathfrak{n}, \mathbb{Z})^{\mathfrak{p}-\mathrm{old}}$ with respect to the Petersson inner product. Finally, denote by $C(\mathfrak{p})$ the set of primes which divide the order of the finite abelian group

$$S(\mathfrak{n}, \mathbb{Z})/(S(\mathfrak{n}, \mathbb{Z})^{\mathfrak{p}-\mathrm{old}} \oplus S(\mathfrak{n}, \mathbb{Z})^{\mathfrak{p}-\mathrm{new}}).$$

For any F-scheme X there is a F-scheme $X^{(p)}$, called the (absolute) *Frobenius conjugate* of X, which by definition is the reciprocal image of X under the base change $\mathrm{Spec}(F) \to \mathrm{Spec}(F)$ induced by $x \mapsto x^p$ on F. If X is a commutative group scheme, then there are two canonical homomorphisms $\mathrm{Frob} : X \to X^{(p)}$ and $\mathrm{Ver} : X^{(p)} \to X$ defined over F, whose composition is the multiplication by p on X; see [8, Sect. 4]. The main result of this paper is the following:

Theorem 1.1 *Suppose $\mathfrak{n}$ is square-free. If there is an abelian variety B over F with bad reduction at $\mathfrak{p}$ and such that $B^{(p)}$ is isomorphic to a subvariety of $J_0(\mathfrak{n})$ then $p \in C(\mathfrak{p})$.*

This theorem can be interpreted as a level lowering result, analogous to Theorem 1.1 in [23]. Indeed, the assumptions of the theorem imply that there is an abelian subvariety of $J_0(\mathfrak{n})$ which is not completely "old" at $\mathfrak{p}$ and such that its p-torsion extends to a finite flat group scheme over the localization of A at $\mathfrak{p}$. This replaces the assumption on the modular residual Galois representation being finite at $\ell \parallel N$. Therefore, under a certain "finiteness" condition at $\mathfrak{p}$ we find a mod-p congruence between $\mathfrak{p}$-old and $\mathfrak{p}$-new cusp forms. Note that it is not possible to reformulate our Theorem 1.1 in terms of Galois representations since the p-torsion of an abelian variety over a field of characteristic p is not an étale group scheme.

Remark 1.2 It is not clear what is the relationship between p being a congruence prime and p being the residue characteristic of some prime of fusion in the Hecke algebra. This is because the function field analogue of the pairing $S(\mathbb{Z}) \times \mathbb{T} \to \mathbb{Z}$, cf. [21], is known to be perfect only after inverting p; see [12, pp. 43–44].

Remark 1.3 It is known that "multiplicity-one" fails for Drinfeld rigid-analytic cusp forms $H^0(X_0(\mathfrak{n})_{F_\infty}, \Omega^1)$, that is, there might be two distinct Hecke eigenforms having the same eigenvalues, cf. [14, (9.7.4)]. The reason that this happens is exactly the existence of mod-p congruences in $S(\mathfrak{n}, \mathbb{Z})$; see [14, (6.5.1)]. This should be compared with the point of view on level lowering taken in [16], and the main theorem therein.

Remark 1.4 When $\mathfrak{n} = \mathfrak{p}$ is prime, $C(\mathfrak{p}) = \emptyset$ since $S(\mathfrak{p})^{\mathfrak{p}-\mathrm{old}} = 0$. Therefore, Theorem 1.1 implies that no abelian subvariety of $J_0(\mathfrak{p})$ defined over F is a Frobenius conjugate of another variety over F.

Remark 1.5 The requirement on $\mathfrak{n}$ being square-free is necessary in Theorem 1.1. Indeed, for $\mathfrak{n}$ divisible by a square there may exist an abelian variety B over F with bad reduction at some $\mathfrak{p} \parallel \mathfrak{n}$ and such that $B^{(p)} \hookrightarrow J_0(\mathfrak{n})$, but at the same time $p \notin C(\mathfrak{p})$. We give one example.

Let $F = \mathbb{F}_2(t)$ and $\mathfrak{n} = t^2(t+1)$. Then $X_0(\mathfrak{n})_F$ is an elliptic curve given by the Weierstrass equation $E: \ y^2 + txy + ty = x^3$; see Example 4.3 in [12]. The group of rational points is $E(F) \cong \mathbb{Z}/6\mathbb{Z}$, and is generated by $P = (t, t)$. If we let $E' := E/(3P)$ then E' is an elliptic curve defined over F and $E = (E')^{(p)}$.

1.3 Szpiro's bound

Our study of the existence of Frobenius morphisms into Drinfeld Jacobians was motivated by an attempt to give a refinement of the well-known Szpiro's bound. We recall this theorem. Let E be a non-isotrivial semi-stable elliptic curve over F. Let λ_E be the largest non-negative integer such that $E \cong (E')^{(p^{\lambda_E})}$ for some elliptic curve E' over F. Let $\mathcal{D}_E$ be the minimal discriminant of E, and let $\mathfrak{n}_E$ be its conductor. Szpiro's bound is the following inequality (cf. [19])

$$\deg \mathcal{D}_E \leq 6 \cdot p^{\lambda_E} \cdot (\deg \mathfrak{n}_E - 2).$$

This bound is the function field analogue of a famous (still open) conjecture of Szpiro which asserts a certain inequality between the discriminants and the conductors of elliptic curves over $\mathbb{Q}$. It is clear that in any F-isogeny class there are elliptic curves with arbitrarily large λ_E, and the above inequality is false without p^{λ_E} in it. More precisely, $\deg \mathcal{D}_E$ cannot be uniformly bounded only in terms of some fixed power of $\deg \mathfrak{n}_E$. (This easily can be seen by fixing a non-isotrivial elliptic curve E and considering its Frobenius conjugates $E^{(p^n)}$.)

It is an interesting question[1] whether one could refine Szpiro's bound for arithmetically important curves, such as the optimal elliptic curves, by getting rid of p^{λ_E}. Recall that an elliptic curve over F having conductor $\mathfrak{n} \cdot \infty$ and split multiplicative reduction at ∞ is known to be isogenous to a subvariety of $J_0(\mathfrak{n})$ (this follows from a combination of some deep results due to Deligne, Drinfeld and Zarhin), cf. [14]; we call such elliptic curves *modular*. The *optimal* elliptic curve in a F-isogeny class of modular elliptic curves is the unique curve E which embeds into $J_0(\mathfrak{n})$ (The terminology is motivated by analogous notions over $\mathbb{Q}$.)

Theorem 1.1 implies that if E is a semi-stable optimal curve with conductor $\mathfrak{n} \cdot \infty$ and $\lambda_E > 0$, then $p \in C(\mathfrak{p})$ for all $\mathfrak{p}$ dividing $\mathfrak{n}$. This is rather unlikely since $S(1, \mathbb{Z}) = 0$, so we expect that semi-stable optimal curves are not Frobenius conjugates of other curves over F. By Remark 1.4 this is true at least when $\mathfrak{n}$ is prime.

1.4 Outline of the proof of the main theorem

Let $\mathfrak{n} = \mathfrak{p}\mathfrak{m}$, where $\mathfrak{p}$ is coprime to $\mathfrak{m}$. Let D be an abelian subvariety of J. Denote by $\Phi_{D,\mathfrak{p}}$ and $\Phi_{J,\mathfrak{p}}$ the groups of connected components of the fibres at $\mathfrak{p}$ of the Néron models of D and J over $\mathbb{P}^1_{\mathbb{F}_q}$. In Sect. 2 we prove a few general facts about maps between Néron models of abelian varieties. A consequence of these results is that if D has degenerate reduction at $\mathfrak{p}$ then the non-triviality of the p-torsion of $\ker(\Phi_{D,\mathfrak{p}} \to \Phi_{J,\mathfrak{p}})$ implies congruences between the elements of the character group M_J of the connected component of the identity of the mod-∞ fibre of the Néron model of J. In Sect. 3.3 we show that there is a canonical $\mathbb{T}$-equivariant isomorphism $M_J \cong S(\mathfrak{n}, \mathbb{Z})$. Hence congruences in M_J translate

[1] This question was communicated to me by Barry Mazur.

into congruences in $S(\mathfrak{n}, \mathbb{Z})$. The reason why Theorem 1.1 currently fails to work over general function fields is exactly this last isomorphism, whose proof uses at one step a result of Gekeler and Nonnengardt [13]. (A statement similar to this result is conjecturally valid in general, cf. [14, (6.4.5)], but the proof in [13] works only over $\mathbb{F}_q(t)$.) A combination of these results proves the key Proposition 3.5 which relates $\# \ker(\Phi_{D,\mathfrak{p}} \to \Phi_{J,\mathfrak{p}})$ to $C(\mathfrak{p})$. To deduce Theorem 1.1 from Proposition 3.5 one needs to make two additional observations. First, for an abelian variety $B^{(p)}$ over F with bad semi-abelian reduction at $\mathfrak{p}$ the group $\Phi_{B^{(p)},\mathfrak{p}}$ has p-torsion. Second, if $\mathfrak{n}$ is square-free then $\Phi_{J,\mathfrak{p}}$ has no p-torsion. Both of these facts are valid over general function fields.

2 Character groups and component groups

By a finite flat group scheme over the base scheme S we always mean a finite flat *commutative* S-group scheme. When $S = \mathrm{Spec}(L)$ with L a field, we will abbreviate this to "finite L-group scheme". A finite group scheme G over a field L is said to be *étale* if $G \times_{\mathrm{Spec}(L)} \mathrm{Spec}(\bar{L})$ is reduced, where $\bar{L}$ is the algebraic closure of L. A finite flat group scheme G over S is said to be étale if the fibres G_s are étale over the corresponding residue fields for all closed points s of S. We say that a finite flat group scheme G is *multiplicative* if its Cartier dual $G^\vee$ is étale. Given an abelian variety A, its dual abelian variety will be denoted by $\hat{A}$.

Let R be a complete discrete valuation ring, K be its field of fractions, and k be the residue field. We assume that K has characteristic $p > 0$.

2.1 Component groups of abelian varieties

Let A be an abelian variety over K. Denote by $\mathcal{A}$ its Néron model over R and denote by $\mathcal{A}_k^0$ the connected component of the identity of the closed fibre $\mathcal{A}_k$ of $\mathcal{A}$. We have an exact sequence

$$0 \to \mathcal{A}_k^0 \to \mathcal{A}_k \to \Phi_A \to 0,$$

where Φ_A is a finite étale group scheme over k called the *component group of A*. We say that A has *semi-abelian reduction* if the identity component is an extension of an abelian variety A'_k by an affine algebraic torus T_k over k

$$0 \to T_k \to \mathcal{A}_k^0 \to A'_k \to 0.$$

We say that A has *(split) toric reduction* if $\mathcal{A}_k^0 = T_k$ is a (split) torus. Assume A has semi-abelian reduction. The *character group*

$$M_A = \mathrm{Hom}_{\bar{k}}(T_{\bar{k}}, \mathbb{G}_{m,\bar{k}})$$

is a free abelian group contravariantly associated to A and its $\mathbb{Z}$-rank is equal to the dimension of T_k. Let $M_{\hat{A}}$ be the analogous group associated to the dual abelian variety $\hat{A}$. Grothendieck [15] defined a functorial bilinear $\mathrm{Gal}(\bar{k}/k)$-equivariant pairing,

$$u_A : M_A \times M_{\hat{A}} \to \mathbb{Z},$$

which he called the *monodromy pairing*. This pairing is uniquely characterized by the property that its extension of scalars $u_A \otimes \mathbb{Z}_\ell$, for a prime $\ell \neq \mathrm{char}(k)$, can be expressed in terms of the ℓ-adic Weil pairing on $T_\ell(A) \times T_\ell(\hat{A})$ via a formula given in [15, Sect. 9]. We have the following key fact (see [15, Sect. 11]):

Theorem 2.1 (Grothendieck) *There is a* $\mathrm{Gal}(\bar{k}/k)$*-equivariant exact sequence*

$$0 \longrightarrow M_{\hat{A}} \xrightarrow{u_A} \mathrm{Hom}(M_A, \mathbb{Z}) \longrightarrow \Phi_A \longrightarrow 0.$$

Lemma 2.2 *Suppose A has semi-abelian reduction. Let $d := \mathrm{rank}_{\mathbb{Z}}(M_A)$. Then*

$$\#\Phi_{A^{(p)}} = p^d \#\Phi_A.$$

Proof Denote the torus in $\mathcal{A}_k^0$ by T_A. Frob induces the raising to p-th power map on the tori, so there is an exact sequence

$$0 \to T_A[p] \to T_A \xrightarrow{\mathrm{Frob}} T_{A^{(p)}} \to 0.$$

Hence $\mathrm{Frob}^*(M_{A^{(p)}}) = pM_A$. On the other hand, Ver induces an isomorphism on the tori, so $\mathrm{Ver}^*(M_{\hat{A}}) = M_{\widehat{A^{(p)}}}$. Theorem 2.1 and the functoriality of the monodromy pairing imply

$$\begin{aligned}
\#\Phi_{A^{(p)}} &= u_{A^{(p)}}(\wedge^d M_{A^{(p)}}, \wedge^d M_{\widehat{A^{(p)}}}) \\
&= u_{A^{(p)}}(\wedge^d M_{A^{(p)}}, \wedge^d \mathrm{Ver}^*(M_{\hat{A}})) \\
&= u_A(\wedge^d \mathrm{Frob}^*(M_{A^{(p)}}), \wedge^d M_{\hat{A}}) \\
&= p^d \cdot u_A(\wedge^d M_A, \wedge^d M_{\hat{A}}) \\
&= p^d \#\Phi_A.
\end{aligned}$$

□

2.2 Maps between component groups

Let B be an abelian variety over K with semi-abelian reduction, and let A be an abelian subvariety of B also defined over K. Denote the quotient abelian variety B/A by C, so that there is an exact sequence of abelian varieties over K

$$0 \to A \xrightarrow{f_K} B \xrightarrow{h_K} C \to 0. \tag{2}$$

Here f_K is a closed immersion. By the Néron mapping property, f_K extends to a canonical homomorphism $f : \mathcal{A} \to \mathcal{B}$ of the Néron models. Consider the induced homomorphism $f_k : \mathcal{A}_k \to \mathcal{B}_k$ on the closed fibres, and the resulting commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}_k^0 & \longrightarrow & \mathcal{A}_k & \longrightarrow & \Phi_A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_k^0} & & \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_\Phi} & & \\ 0 & \longrightarrow & \mathcal{B}_k^0 & \longrightarrow & \mathcal{B}_k & \longrightarrow & \Phi_B & \longrightarrow & 0. \end{array}$$

Lemma 2.3 *If B has toric reduction, then f_k^0 is a closed immersion and* $\ker(f_\Phi)$ *has trivial p-power torsion.*

Proof For a proof of a somewhat stronger statement we refer to [17, Prop. 2.3]. Below we give a different argument, which uses rigid-analytic geometry.

By slightly modifying the argument in [2, Lem.7.4/2], one can show that A also has toric reduction. Since a monomorphism between smooth finite type group schemes over a field is necessarily a closed immersion, to shaw that f_k^0 is a closed immersion it suffices to show that $G := \ker f_k^0$ is trivial. G is a finite multiplicative k-group scheme, being the kernel of an isogeny between tori over k. It is easy to see that there is a unique multiplicative finite flat R-group scheme $\widetilde{G}$ with closed fibre G, and moreover $\widetilde{G}_K \hookrightarrow \ker(f_K)$; cf. [4, Thm. 8.6]. Since f_K is a closed immersion, $\ker(f_K)$ is trivial, which implies that G also has to be trivial.

Next, let $H := \ker(f_\Phi)[p^\infty]$. We want to show that $H = 1$. Since the base change to an unramified extension of K commutes with the formation of Néron models, we can assume that A and B have split toric reduction. By the Mumford–Raynaud theory, A and B have rigid-analytic uniformization, and there is a functorial commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda_A & \longrightarrow & X_A & \longrightarrow & A^{\mathrm{an}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_\Lambda} & & \downarrow{\scriptstyle f_X} & & \downarrow{\scriptstyle f_K^{\mathrm{an}}} & & \\ 0 & \longrightarrow & \Lambda_B & \longrightarrow & X_B & \longrightarrow & B^{\mathrm{an}} & \longrightarrow & 0, \end{array} \tag{3}$$

where X_A and X_B are split analytic tori over K, $\Lambda_A \subset X_A(K)$ and $\Lambda_B \subset X_B(K)$ are lattices; see [7, Ch. 6] or [1]. Moreover, there are natural isomorphisms $\Lambda_A \cong M_{\hat{A}}$, $\Lambda_B \cong M_{\hat{B}}$, and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda_A & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Lambda_{\hat{A}}, \mathbb{Z}) & \longrightarrow & \Phi_A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_\Lambda} & & \downarrow & & \downarrow{\scriptstyle f_\Phi} & & \\ 0 & \longrightarrow & \Lambda_B & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Lambda_{\hat{B}}, \mathbb{Z}) & \longrightarrow & \Phi_B & \longrightarrow & 0, \end{array} \tag{4}$$

where the rows are the rigid-analytic realizations of Theorem 2.1; see [3, Prop. 5.2]. Using the fact that f_K^{an} in (3) is a closed immersion, it is easy to check that f_Λ is an injection, so $\Lambda_B/f_\Lambda(\Lambda_A) \hookrightarrow (X_B/f_X(X_A))(K) \cong (K^\times)^m$, where $m = \dim(B) - \dim(A)$. Next, it is clear that the middle vertical map in (4) is injective, so we have an injection $H \hookrightarrow \Lambda_B/f_\Lambda(\Lambda_A)$. Combining these maps, we get

$$H \hookrightarrow \Lambda_B/f_\Lambda(\Lambda_A) \hookrightarrow (K^\times)^m.$$

Since K has characteristic p, the p-power torsion of $(K^\times)^m$ is connected. On the other hand, H is a finite étale p-group, hence it has to be trivial. □

From now on we assume that B is principally polarized, $\theta : \hat{B} \xrightarrow{\sim} B$. Taking the dual of h_K in (2) and using θ to identify $\hat{B}$ with B, we can identify $\hat{C}$ with a closed subvariety of B; see [4, Prop.3.3]. Since $\hat{C}$ is isogenous to C under $B \to C$, it is clear that $\hat{C} \cap A$ is a finite group scheme, where the scheme-theoretic intersection is taken inside of B.

Definition 2.4 *We say that the abelian subvariety A of B is* maximal toric *if for any abelian subvariety D of B having toric reduction the canonical closed immersion $D \hookrightarrow B$ factors through $D \hookrightarrow A \hookrightarrow B$.*

Lemma 2.5 *If A has toric reduction and C has good reduction (i.e., C can be extended to an abelian scheme over R) then A is maximal toric.*

Proof This is clear. □

Let D be an abelian subvariety of B. Let $H_D := \ker(\Phi_D \to \Phi_B)[p^\infty]$, where the homomorphism between the component groups is induced by the Néron mapping property from the closed immersion $D \hookrightarrow B$.

Proposition 2.6 *Assume D has toric reduction and A is maximal toric. If H_D is non-trivial then $(A \cap \hat{C})$ has a non-trivial connected subgroup.*

Proof We have the exact sequence

$$0 \to \ker(\Phi_D \to \Phi_A) \to \ker(\Phi_D \to \Phi_B) \to \ker(\Phi_A \to \Phi_B)$$

functorially arising from $D \hookrightarrow A \hookrightarrow B$. By Lemma 2.3 the map $\Phi_D \to \Phi_A$ is injective on p-torsion, so $H_D \hookrightarrow H_A$. Therefore, without loss of generality, we can assume $D = A$. Consider the polarization $\varphi_K : A \to \hat{A}$ obtained as the composition

$$\varphi_K : A \xrightarrow{f_K} B \xrightarrow{\hat{\theta}} \hat{B} \xrightarrow{\hat{f}_K} \hat{A}.$$

Using Theorem 2.1, we get a commutative diagram (cf. [22])

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{\hat{A}} & \xrightarrow{u_A} & \mathrm{Hom}(M_A,\mathbb{Z}) & \longrightarrow & \Phi_A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \mathrm{Hom}(\hat{\varphi}^*,\mathbb{Z})} & & \downarrow{\scriptstyle \varphi_\Phi} & & \\ 0 & \longrightarrow & M_A & \xrightarrow{u_{\hat{A}}} & \mathrm{Hom}(M_{\hat{A}},\mathbb{Z}) & \longrightarrow & \Phi_{\hat{A}} & \longrightarrow & 0. \end{array}$$

The middle vertical arrow is injective, so $\ker(\varphi_\Phi) \hookrightarrow \operatorname{coker}(\varphi^*)$. By functoriality the map φ_Φ factors through f_Φ, and we get the injective homomorphisms

$$H_A \hookrightarrow \ker(\varphi_\Phi)[p^\infty] \hookrightarrow \operatorname{coker}(\varphi^*)[p^\infty].$$

Denote $\delta_k := \ker(\mathcal{A}_k^0 \to \hat{\mathcal{A}}_k^0)[p^\infty]$. This is a finite connected multiplicative group scheme over k, whose Cartier dual $(\delta_k)^\vee$ is isomorphic to $\operatorname{coker}(\varphi^*)[p^\infty]$. The preceding discussion implies that δ_k is non-trivial. As we already mentioned, for any finite multiplicative k-group scheme G_k there is a unique multiplicative finite flat R-group scheme $\widetilde{G}$ with closed fibre G_k. It is easy to see that there is a natural closed immersion (cf. [4, p. 762]) $\widetilde{\delta}_K \hookrightarrow \ker(\varphi_K) = A \cap \hat{C}$. Since K has characteristic p, the multiplicative p-group $\widetilde{\delta}_K$ is connected, and we conclude that $(A \cap \hat{C})^0$ is non-trivial. □

2.3 Maps between character groups

In addition to the assumptions of Sect. 2.2, in this subsection we assume that B has toric reduction. This implies, in particular, that any subvariety or a quotient of B also has toric reduction. To simplify the notation, denote $\mathcal{A}_k^0$ by T_A, $\mathcal{B}_k^0$ by T_B and etc.

The closed immersion f_K induces a functorial homomorphism

$$\hat{f}^* : M_{\hat{A}} \to M_{\hat{B}} \overset{\theta^*}{=} M_B.$$

The image of $M_{\hat{A}}$ under $\hat{f}^*$ need not be saturated, i.e., the quotient group $M_B/(\hat{f}^* M_{\hat{A}})$ might have torsion. We denote by

$$\overline{M}_{\hat{A}} = ((\hat{f}^* M_{\hat{A}}) \otimes \mathbb{Q}) \cap M_B \tag{5}$$

the *saturation* of $\hat{f}^* M_{\hat{A}}$ inside M_B, and likewise by $\overline{M}_C$ the saturation of $h^* M_C$ inside M_B. It is easy to see that $\overline{M}_C \cap \overline{M}_{\hat{A}} = 0$.

Proposition 2.7 *If $(A \cap \hat{C})^0$ is non-trivial, then $M_B/(\overline{M}_C \oplus \overline{M}_{\hat{A}})$ has non-trivial p-power torsion.*

Proof Consider the polarization $\phi_K : \hat{C} \to C$, where $\phi := h_K \circ \theta \circ \hat{h}_K$. Let $\varphi_K : A \to \hat{A}$ be the similar polarization on A. Since $(A \cap \hat{C})$ is the kernel of both ϕ_K and φ_K, the assumption of the proposition implies that $\ker(\phi_K)^0 = \ker(\varphi_K)^0$ is non-trivial. Let $\phi_t : T_{\hat{C}} \to T_C$ be the isogeny functorially induced by ϕ_K. The k-group scheme $\ker(\phi_t)$ is finite multiplicative, and Theorem 8.6 in [4] implies that there is a canonical isomorphism $\ker(\phi_t)^0 \cong \ker(\phi_K)^0$. Hence the finite multiplicative group scheme $T_{\hat{C}} \cap T_A$ has a non-trivial connected subgroup. (Note that we implicitly use Lemma 2.3 to identify T_A and $T_{\hat{C}}$ with subtori of T_B.) This implies that the kernel of the natural surjective homomorphism $T_B \to T_C \times T_{\hat{A}}$ is geometrically non-reduced. Taking the duals, i.e., passing to the character groups, we get that $M_B/(M_C \oplus M_{\hat{A}})$ has non-trivial p-power torsion. (Here we omit h^* and etc. from notation.) It remains to show that the quotient $M_B/(\overline{M}_C \oplus \overline{M}_{\hat{A}})$ of this last finite abelian group also has non-trivial p-power torsion. In fact we will show that these two groups have isomorphic p-power torsion. There is a short-exact sequence

$$0 \to \frac{\overline{M}_{\hat{A}}}{M_{\hat{A}}} \oplus \frac{\overline{M}_C}{M_C} \to \frac{M_B}{M_{\hat{A}} \oplus M_C} \to \frac{M_B}{\overline{M}_{\hat{A}} \oplus \overline{M}_C} \to 0.$$

We claim that the group on the left-hand side has no p-torsion. Consider the commutative diagram arising from Theorem 2.1:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_B & \xrightarrow{u_B} & \mathrm{Hom}(M_B,\mathbb{Z}) & \longrightarrow & \Phi_B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \hat{h}^*} & & \downarrow{\scriptstyle \mathrm{Hom}(h^*,\mathbb{Z})} & & \downarrow{\scriptstyle h_\Phi} & & \\
0 & \longrightarrow & M_{\hat{C}} & \xrightarrow{u_C} & \mathrm{Hom}(M_C,\mathbb{Z}) & \longrightarrow & \Phi_C & \longrightarrow & 0.
\end{array}$$

By Lemma 2.3 $T_{\hat{C}} \to T_B$ is a closed immersion, so the left vertical arrow is surjective. Thus, from the snake lemma

$$\#\mathrm{coker}(h_\Phi) = \#\mathrm{Ext}^1_{\mathbb{Z}}(M_B/M_C, \mathbb{Z}) = \#(M_B/M_C)_{\mathrm{tor}} = \#(\overline{M}_C/M_C).$$

By [15, pp. 5–6], $\mathrm{coker}(h_\Phi) \cong \mathrm{Hom}_{\mathbb{Z}}(\ker(\hat{h}_\Phi), \mathbb{Q}/\mathbb{Z})$. Therefore, by Lemma 2.3, $\overline{M}_C/M_C$ has no p-torsion. A similar argument shows that $\overline{M}_{\hat{A}}/M_{\hat{A}}$ also has no p-torsion. □

3 Drinfeld modular curves and congruences

Let $F = \mathbb{F}_q(t)$ and $A = \mathbb{F}_q[t]$ be as in the introduction. For a prime ideal $\mathfrak{p}$ of the Dedekind domain A we denote the completion of A at $\mathfrak{p}$ by $A_\mathfrak{p}$, the fraction field of $A_\mathfrak{p}$ by $F_\mathfrak{p}$, and the residue field $A_\mathfrak{p}/\mathfrak{p}$ by $\mathbb{F}_\mathfrak{p}$. Denote the completion of F at $\infty = 1/t$ by K, the ring of integers in K by R, and the residue field at ∞ by k.

3.1 Drinfeld modular curves

Let $\mathfrak{n}$ be an ideal in A. The functor which associates to an A-scheme S the set of isomorphism classes of pairs $(D, Z_{\mathfrak{n}})$, where D is a Drinfeld module of rank 2 over S and $Z_{\mathfrak{n}}$ is a $\mathfrak{n}$-cyclic subgroup of D, possesses a coarse moduli scheme $M_0(\mathfrak{n})/A$ that is affine of finite type over A, and is A-flat with pure relative dimension 1. There is a canonical compactification $X_0(\mathfrak{n})$ of $M_0(\mathfrak{n})$ over $\mathrm{Spec}(A)$; see [6, Sect. 9].

Let $\Omega = \mathbb{P}^1_K - \mathbb{P}^1_K(K)$ be the *Drinfeld upper half-plane*. Ω has a natural structure of a smooth connected rigid-analytic space. Denote by $\Gamma_0(\mathfrak{n})$ the *Hecke congruence subgroup* of level $\mathfrak{n}$:

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \mid c \in \mathfrak{n} \right\}.$$

In this section we assume that $\mathfrak{n}$ is fixed and, to simplify the notation, let $\Gamma = \Gamma_0(\mathfrak{n})$. The group Γ naturally acts on Ω via linear fractional transformations, and the action is *discrete* in the sense of [6, p. 582]. Hence we may construct the quotient $\Gamma \setminus \Omega$ as a 1-dimensional connected smooth analytic space over K.

The following theorem can be deduced from the results in [6]:

Theorem 3.1

(a) $X_0(\mathfrak{n})$ *is a proper, normal, flat, irreducible scheme of pure relative dimension* 1 *over* $\mathrm{Spec}(A)$.
(b) $X_0(\mathfrak{n}) \to \mathrm{Spec} A\left[\mathfrak{n}^{-1}\right]$ *is smooth.*
(c) $X_0(\mathfrak{n})_F$ *is a smooth, proper, geometrically connected curve over* F.
(d) *There is an isomorphism of rigid-analytic spaces* $\Gamma \setminus \Omega \cong M_0(\mathfrak{n})^{\mathrm{an}}_K$.

3.2 Cuspidal harmonic cochains

Let $\mathcal{T}$ be the *Bruhat-Tits tree* of $\mathrm{PGL}_2(K)$; see [14, Sect. 1] for the definition. We denote by $X(\mathcal{T})$ and $Y(\mathcal{T})$ the vertices and the oriented edges of $\mathcal{T}$, respectively. There is a natural action of Γ on $\mathcal{T}$ as a coset space of $\mathrm{GL}_2(K)$. This action preserves the simplicial structure of $\mathcal{T}$. For an edge $e \in Y(\mathcal{T})$ we denote by $\bar{e}$, $t(e)$, $o(e)$ the inversely oriented edge, the terminus of e, and the origin of e, respectively. For any abelian group B, let $S(\Gamma, B)$ be the group of maps $Y(\mathcal{T}) \to B$ subject to

(i) $\varphi(\bar{e}) = -\varphi(e)$ for any $e \in Y(\mathcal{T})$;
(ii) $\sum_{t(e)=v} \varphi(e) = 0$ for any $v \in X(\mathcal{T})$;
(iii) $\varphi(\gamma e) = \varphi(e)$ for any $\gamma \in \Gamma$;
(iv) φ has compact (= finite) support modulo Γ.

We call this group the *group of B-valued cuspidal harmonic cochains* for Γ, cf. [14, Sect. 3]. Using the *strong approximation theorem* for function fields, it can be shown that $S(\Gamma, \mathbb{C})$ may be interpreted as a space of automorphic cusp forms on $\mathrm{GL}_2(\mathbb{A}_F)$ which are special at ∞; see [14, Sect. 4]. This space plays a role similar to the role of weight-2 cusp forms of fixed level in the classical theory. From the definition it is easy to see that for any ring $\mathcal{O}$ contained in $\mathbb{C}$, in particular for $\mathbb{C}$ itself, there is the canonical isomorphism $S(\Gamma, \mathcal{O}) = S(\Gamma, \mathbb{Z}) \otimes \mathcal{O}$. This is the analogue of the fact that the space of weight-2 cusp forms has a basis consisting of cusp forms with integral Fourier coefficients. Thus, $S(\Gamma, \mathbb{C})$ has a canonical integral structure given by $S(\Gamma, \mathbb{Z})$.

3.3 Hecke algebra

Following [12, (1.10)], for any ideal $\mathfrak{m}$ of A we define a Hecke operator $T_\mathfrak{m}$ acting on $S(\Gamma, \mathbb{C})$. This is derived from a correspondence on

$$Y(\Gamma \setminus \mathcal{T}) = \Gamma \setminus \mathrm{GL}_2(K)/\mathcal{I} \cdot Z(K),$$

where $\mathcal{I}$ is the Iwahori group at ∞ and Z is the center of GL_2. Any function $\varphi \in S(\Gamma, \mathbb{C})$ can be considered as a function on $\mathrm{GL}_2(K)$. Define

$$T_\mathfrak{m}\varphi(g) = \sum \varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right),$$

where the sum is over $a, b, d \in A$ such that a, d are monic, the ideal (ad) is $\mathfrak{m}$, a is coprime to $\mathfrak{n}$, and $\deg b < \deg d$. We call $T_\mathfrak{m}$ the $\mathfrak{m}$-th *Hecke operator*. The Hecke operators commute with each other, and satisfy recursive relationships which allow to express each $T_\mathfrak{m}$ in terms of a polynomial with integral coefficients in $T_\mathfrak{p}$'s, where the $\mathfrak{p}$'s are the prime divisors of $\mathfrak{m}$. Let $\mathbb{T} := \mathbb{Z}[\dots, T_\mathfrak{m}, \dots]$ be the commutative $\mathbb{Z}$-algebra generated by the Hecke operators acting on $S(\Gamma, \mathbb{C})$. It is clear from the definition that $\mathbb{T}$ preserves the integral structure $S(\Gamma, \mathbb{Z})$, and since it is known that $S(\Gamma, \mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module, $\mathbb{T}$ is a finitely generated free $\mathbb{Z}$-module.

There is an equivalent "modular" definition of the $T_\mathfrak{m}$ as a correspondence on $X_0(\mathfrak{n})$ given by

$$(D, Z_\mathfrak{n}) \mapsto \sum_{Z_\mathfrak{m} \cap Z_\mathfrak{n} = 0} (D/Z_\mathfrak{m}, (Z_\mathfrak{n} + Z_\mathfrak{m})/Z_\mathfrak{m}).$$

Let $J = J_0(\mathfrak{n})$ be the Jacobian variety of $X_0(\mathfrak{n})_F$. $T_\mathfrak{m}$ induces an endomorphism of J, and we can consider the commutative subalgebra $\mathbb{T} \subseteq \mathrm{End}_F(J)$ generated by the Hecke operators. Let $V_\ell(J)$ be the ℓ-adic Tate vector space of J, $\ell \neq p$. As a consequence of a theorem of Zarhin, there is a canonical isomorphism

$$\mathrm{End}_F(J) \otimes \mathbb{Q}_\ell = \mathrm{End}_{\mathrm{Gal}(F^{\mathrm{sep}}/F)}(V_\ell(J))$$

Hence $\mathbb{T}$ is naturally a subalgebra of $\mathrm{End}(V_\ell(J))$. The fundamental theorem of Drinfeld [6, Thm. 2], among other things, relates the two Hecke algebras we defined:

Theorem 3.2 (Drinfeld) *There is a canonical isomorphism between* $V_\ell(J)^*$ *and* $S(\Gamma, \mathbb{Q}_\ell) \otimes \mathrm{sp}(2)$ *compatible with the action of* $\mathbb{T}$*, where* $\mathrm{sp}(2)$ *is the two-dimensional special* ℓ*-adic representation of* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$*, and* $V_\ell(J)^* = \mathrm{Hom}(V_\ell(J), \mathbb{Q}_\ell)$.

One can conclude from this theorem that the abelian subvarieties of J which are stable under the action of $\mathbb{T}$ as a subalgebra of $\mathrm{End}_F(J)$ are in one-to-one correspondence with the $\mathbb{T}$-stable subspaces of $S(\Gamma, \mathbb{Q})$.

It is well-known that the existence of the rigid-analytic uniformization in Theorem 3.1(d) implies that J has split toric reduction over K. Denote by $\mathcal{J}$ the Néron model of J over R, and let M_J be the character group of $\mathcal{J}_k^0$. By the Néron mapping property, any endomorphism of J uniquely extends to an endomorphism of $\mathcal{J}$. Specializing to the closed fibre, we get a natural homomorphism $\mathrm{End}_K(J) \to \mathrm{End}_k(\mathcal{J}_k^0)$. Since J has toric reduction, this homomorphism is injective. On the other hand, since $\mathcal{J}_k^0$ is a split torus, there is a canonical isomorphism $\mathrm{End}_k(\mathcal{J}_k^0) \cong \mathrm{End}_{\mathbb{Z}}(M_J)$. Thus, we find a faithful representation $\mathbb{T} \to \mathrm{End}_{\mathbb{Z}}(M_J)$.

Proposition 3.3 *There is a canonical isomorphism of* $\mathbb{T}$*-modules* $M_J \xrightarrow{\cong} S(\Gamma, \mathbb{Z})$.

Proof Let $\overline{\Gamma} := \Gamma^{\mathrm{ab}}/(\Gamma^{\mathrm{ab}})_{\mathrm{tor}}$. One can show, cf. [14, (3.2)], that $\overline{\Gamma}$ is a free abelian group of rank g =genus($X_0(\mathfrak{n})$). Moreover, by [13] there is a canonical isomorphism $j : \overline{\Gamma} \xrightarrow{\sim} S(\Gamma, \mathbb{Z})$. Gekeler and Reversat [14, (9.3)] define a natural action of $\mathbb{T}$ on $\overline{\Gamma}$ and show that j is an isomorphism of $\mathbb{T}$-modules. Next, recall one of the principal results of [14], which says that there is an exact sequence

$$1 \to \overline{\Gamma} \xrightarrow{\bar{c}} \mathrm{Hom}(\overline{\Gamma}, \mathbb{G}_{m,K}^{\mathrm{an}}) \to J^{\mathrm{an}} \to 1, \tag{6}$$

where $\bar{c}$ is an explicit rigid-analytic period. Moreover, (6) is $\mathbb{T}$-equivariant in the following sense, cf. [14, (9.4)]: $T^{\mathrm{an}} := \mathrm{Hom}(\overline{\Gamma}, \mathbb{G}_{m,K}^{\mathrm{an}})$ is the universal covering space of J^{an} in the rigid-analytic category, and hence every endomorphism of J^{an} uniquely lifts to an endomorphism of the torus T^{an}. Since the endomorphisms of a split analytic torus are algebraic, $\mathrm{End}_K(T^{\mathrm{an}}) \cong \mathrm{End}_{\mathbb{Z}}(\overline{\Gamma})$. So we get two actions of $\mathbb{T}$ on $\overline{\Gamma}$, and the equivariance is expressed by the fact that these two actions are the same.

Now we recall a more abstract description of the analytic uniformization of J which brings into the picture the character group M_J. Since J has split toric reduction over K, it has an analytic uniformization [1, Thm. 1.2]

$$0 \to \Lambda \to T^{\mathrm{an}} \to J^{\mathrm{an}} \to 0,$$

where $\Lambda \subset T^{\mathrm{an}}(K)$ is a lattice of rank g. Moreover, $\Lambda \cong \mathrm{Hom}(T^{\mathrm{an}}, \mathbb{G}_{m,K}^{\mathrm{an}})$. (Here we implicitly use the canonical principal polarization of J.) Comparing with (6), we see that there is a canonical isomorphism $\Lambda \cong \overline{\Gamma}$, compatible with the action of $\mathrm{End}_K(J)$.

Let $\mathcal{J}^0$ be the relative connected component of the identity of $\mathcal{J}$, i.e., the largest open subscheme of $\mathcal{J}$ containing the identity section which has connected fibres. The formal completion $\widehat{\mathcal{J}^0}$ of $\mathcal{J}^0$ along its closed fibre is uniquely isomorphic to a formal split torus over $\mathrm{Spf}(R)$ respecting a choice of isomorphism $\mathcal{J}^0_k \cong \mathbb{G}^g_{m,k}$. Raynaud's "generic fibre" functor produces the functorial isomorphisms, cf. [1]

$$\mathrm{Hom}(\widehat{\mathcal{J}^0}, \widehat{\mathbb{G}}_{m,R}) \cong \mathrm{Hom}(T^{\mathrm{an}}, \mathbb{G}^{\mathrm{an}}_{m,K}) \cong \Lambda.$$

On the other hand, we have the functorial isomorphisms

$$\mathrm{Hom}(\widehat{\mathcal{J}^0}, \widehat{\mathbb{G}}_{m,R}) \cong \mathrm{Hom}((\widehat{\mathcal{J}^0})_k, \mathbb{G}_{m,k}) = \mathrm{Hom}(\mathcal{J}^0_k, \mathbb{G}_{m,k}) = M_J.$$

We conclude that M_J and Λ are canonically isomorphic $\mathrm{End}_K(J)$-modules. Composing this isomorphism with the isomorphisms j and $\Lambda \cong \overline{\Gamma}$ gives the claim. □

The upshot of this proposition is that the $\mathbb{T}$-stable subvarieties of J are in one-to-one correspondence with the saturated $\mathbb{T}$-stable subgroups of M_J. Moreover, the congruences between cusp forms can be deduced from the congruences between the elements of M_J. The dictionary between $S(\Gamma, \mathbb{Z})$ and M_J goes even further. Since $S(\Gamma, \mathbb{Z})$ is a lattice in a space of automorphic forms, there is a positive definite (a priori) $\mathbb{C}$-valued pairing on this group, which comes from the Petersson inner product on the space of cusp forms. For a particular choice of the Haar measure on $Y(\Gamma \setminus \mathcal{T})$ this pairing is $\mathbb{Z}$-valued and turns out to be equal to the monodromy pairing on M_J; see [18, Prop. 4.5].

3.4 Congruences

Let $\mathfrak{p}$ be a prime ideal of degree d, and let $\mathfrak{n} = \mathfrak{p}\mathfrak{m}$, with $\mathfrak{m}$ coprime to $\mathfrak{p}$. Recall the two natural degeneracy maps $\alpha, \beta : M_0(\mathfrak{n}) \rightrightarrows M_0(\mathfrak{m})$, where α, β are induced by the maps defined in terms of the moduli problem

$$\alpha : (D, Z_{\mathfrak{p}} Z_{\mathfrak{m}}) \mapsto (D, Z_{\mathfrak{m}})$$
$$\beta : (D, Z_{\mathfrak{p}} Z_{\mathfrak{m}}) \mapsto (D/Z_{\mathfrak{p}}, Z_{\mathfrak{p}} Z_{\mathfrak{m}}/Z_{\mathfrak{p}}).$$

These morphisms uniquely extend to $X_0(\mathfrak{n})$ and $X_0(\mathfrak{m})$. By Picard functoriality we get two homomorphisms $J_0(\mathfrak{m}) \rightrightarrows J_0(\mathfrak{n})$. The subvariety of $J_0(\mathfrak{n})$ generated by the images of these homomorphisms is called the $\mathfrak{p}$-*old subvariety*, and is denoted by $J_0(\mathfrak{n})^{\mathfrak{p}-\mathrm{old}}$. The quotient abelian variety $J_0(\mathfrak{n})/J_0(\mathfrak{n})^{\mathfrak{p}-\mathrm{old}}$ is called the $\mathfrak{p}$-*new quotient* of $J_0(\mathfrak{n})$ and is denoted $J_0(\mathfrak{n})_{\mathfrak{p}-\mathrm{new}}$. By taking the dual of the quotient map $J_0(\mathfrak{n}) \to J_0(\mathfrak{n})_{\mathfrak{p}-\mathrm{new}}$ and using the canonical principal polarization on the Jacobian $J_0(\mathfrak{n})$, we obtain a subvariety $J_0(\mathfrak{n})^{\mathfrak{p}-\mathrm{new}}$ of $J_0(\mathfrak{n})$, called the $\mathfrak{p}$-*new subvariety*. We will assume that $\mathfrak{p}$ and $\mathfrak{n}$ are fixed, and to simplify the notation we let $J, J^{\mathrm{old}}, J^{\mathrm{new}}$ denote the Drinfeld Jacobian $J_0(\mathfrak{n})$ and its corresponding $\mathfrak{p}$-old and $\mathfrak{p}$-new subvarieties. $J^{\text{new}} \cap J^{\text{old}}$ is finite, and $\mathbb{T}$ preserves both J^{old} and J^{new}. We can identify the corresponding $\mathbb{T}$-stable subgroups of $S(\Gamma_0(\mathfrak{n}), \mathbb{Z})$ as follows. There are two natural injections $S(\Gamma_0(\mathfrak{m}), \mathbb{Q}) \rightrightarrows S(\Gamma_0(\mathfrak{n}), \mathbb{Q})$. The $\mathbb{Q}$-linear subspace $S(\mathbb{Q})^{\mathfrak{p}-\text{old}}$ generated by the images is the $\mathfrak{p}$-*old subspace*. The intersection $S(\mathbb{Z})^{\text{old}} := S(\Gamma_0(\mathfrak{n}), \mathbb{Z}) \cap S(\mathbb{Q})^{\mathfrak{p}-\text{old}}$ is the $\mathbb{T}$-stable saturated subgroup corresponding to J^{old}. The subgroup $S(\mathbb{Z})^{\text{new}}$ corresponding to J^{new} is the orthogonal complement in $S(\Gamma, \mathbb{Z})$ of $S(\mathbb{Z})^{\text{old}}$ with respect to the Petersson norm (equiv. with respect to the monodromy pairing on M_J, cf. Sect. 3.3). As in the introduction, denote by $C(\mathfrak{p})$ the set of primes dividing $\#(S(\Gamma, \mathbb{Z})/(S(\mathbb{Z})^{\text{old}} \oplus S(\mathbb{Z})^{\text{new}}))$.

Next, we claim that J^{new} is the maximal toric subvariety of J over $F_\mathfrak{p}$ in the sense of Sect. 2.2. To see this we need to recall the structure of $X_0(\mathfrak{n})$ over $A_\mathfrak{p}$.

A Drinfeld module D over an extension of $\mathbb{F}_\mathfrak{p}$ is called *supersingular* if its $\mathfrak{p}$-torsion is connected, cf. [10]. Consider $X_0(\mathfrak{m})$ over $\text{Spec}(A_\mathfrak{p})$. It is smooth by Theorem 3.1. We will call the points of $X_0(\mathfrak{m})_{\overline{\mathbb{F}}_\mathfrak{p}}$ represented by pairs $(D, Z_\mathfrak{m})$, with D supersingular, the *supersingular points*. There are only finitely many of these. The automorphism group $\text{Aut}(D, Z_\mathfrak{m})$ of a pair $(D, Z_\mathfrak{m})$ is defined in an obvious manner. It is known that there are inclusions of groups $\mathbb{F}_q^\times \subseteq \text{Aut}(D, Z_\mathfrak{m}) \subseteq \text{Aut}(D) \subseteq \mathbb{F}_{q^2}^\times$. Let τ be the Frobenius endomorphism relative to $\mathbb{F}_q$, i.e., the map $x \mapsto x^q$. The endomorphism τ^d can be canonically identified with an involution of the set of supersingular points of $X_0(\mathfrak{m})_{\overline{\mathbb{F}}_\mathfrak{p}}$. The following theorem, which is the analogue of [5, Thm. VI.6.9], describes the structure of the special fibre $X_0(\mathfrak{n})_{\overline{\mathbb{F}}_\mathfrak{p}}$. The proof of the theorem in case $\mathfrak{n} = \mathfrak{p}$ is carefully discussed in [10].

Theorem 3.4

(a) *The special fibre $X_0(\mathfrak{n})_{\overline{\mathbb{F}}_\mathfrak{p}}$ is reduced and is a union of two copies of the smooth curve $X_0(\mathfrak{m})_{\overline{\mathbb{F}}_\mathfrak{p}}$, intersecting transversally at the supersingular points. The supersingular point x on the first copy of $X_0(\mathfrak{m})_{\overline{\mathbb{F}}_\mathfrak{p}}$ is glued to $\tau^d(x)$ on the second copy.*

(b) *Let x be a supersingular point of $X_0(\mathfrak{m})_{\overline{\mathbb{F}}_\mathfrak{p}}$ defined by a pair $(D, Z_\mathfrak{m})$, and let $n := \frac{1}{q-1}\#\text{Aut}(D, Z_\mathfrak{m})$. Then*

$$\widehat{\mathcal{O}^{\text{sh}}_{X_0(\mathfrak{n}),x}} \cong \widehat{A^{\text{sh}}_\mathfrak{p}}[\![v, w]\!]/(v \cdot w - \mathfrak{p}^n).$$

Using Theorem 3.4 and Raynaud's theorem of specializations of the Picard functor, cf. [2, Ch.9], one obtains an exact sequence

$$0 \to T \to \mathcal{J}^0_{\mathbb{F}_\mathfrak{p}} \to J_0(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}} \times J_0(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}} \to 0, \tag{7}$$

where $\mathcal{J}$ is the Néron model of J over $A_\mathfrak{p}$ and T is a torus. Since J/J^{new} is isogenous to $J_0(\mathfrak{m}) \times J_0(\mathfrak{m})$, it has good reduction over $F_\mathfrak{p}$. Next, comparing the dimensions in (7) with the dimensions in $0 \to J^{\text{new}} \to J \to J/J^{\text{new}} \to 0$, we conclude that J^{new} has toric reduction. Therefore, by Lemma 2.5, J^{new} is the maximal toric subvariety of J over $F_{\mathfrak{p}}$.

Proposition 3.5 *Let B be an abelian subvariety of J. Denote by $\Phi_{J,\mathfrak{p}}$ and $\Phi_{B,\mathfrak{p}}$ the component groups of the Néron models of J and B over $A_{\mathfrak{p}}$. Assume B has toric reduction at $\mathfrak{p}$. If the kernel of the homomorphism $\Phi_{B,\mathfrak{p}} \to \Phi_{J,\mathfrak{p}}$, functorially induced from the closed immersion $B \hookrightarrow J$, has non-trivial p-power torsion, then $p \in C(\mathfrak{p})$.*

Proof First, note that a finite F-group scheme G_F is étale if and only if $G_{F_v} := G_F \otimes_F F_v$ is étale over F_v, where F_v is the completion of F at some place v. The "only if" part is a consequence of the base change property of étale morphisms, and the "if" part is true because $F \subset F_v$. We apply this to $\mathfrak{G}_F := (J^{\text{new}} \cap J^{\text{old}})$ as a group scheme over F. If $\ker(\Phi_{B,\mathfrak{p}} \to \Phi_{J,\mathfrak{p}})$ has p-torsion, then $\mathfrak{G}^0_{F_{\mathfrak{p}}}$ is non-trivial by Proposition 2.6. Hence $\mathfrak{G}^0_F$ is non-trivial, and so $\mathfrak{G}^0_K$ is also non-trivial.

Denote by $\pi : J \to J_{\text{new}}$ the quotient map constructed at the beginning of this subsection. Let $M_{J_{\text{new}}}$ be the character group of the Néron model of J_{new} over R. Let $\overline{M}_{J_{\text{new}}}$ be the saturation of $\pi^*(M_{J_{\text{new}}})$ in M_J, cf. (5). Denote by $\overline{M}_{J_{\text{old}}}$ the similar group for the old quotient $J_{\text{old}} := J/J^{\text{new}}$. Under the isomorphism of Proposition 3.3, there is a canonical isomorphism

$$S(\Gamma, \mathbb{Z})/(S(\mathbb{Z})^{\text{new}} \oplus S(\mathbb{Z})^{\text{old}}) = M_J/(\overline{M}_{J_{\text{new}}} \oplus \overline{M}_{J_{\text{old}}}),$$

so the claim follows from Proposition 2.7. □

Lemma 3.6 *If $\mathfrak{n}$ is square-free then for any prime $\mathfrak{p}$ dividing $\mathfrak{n}$ there is the congruence $\#\Phi_{J,\mathfrak{p}} \equiv 1 \pmod{p}$. In particular, $\Phi_{J,\mathfrak{p}}$ has no p-power torsion.*

Proof The main ingredient of the proof is a result of Raynaud reproduced in a convenient form in [2, 9.6/10, 9.6/11]. Using *loc.cit.* and Theorem 3.4, one can actually determine the structure of $\Phi_{J,\mathfrak{p}}$. Since we are only interested in the order of this group modulo p, we will take a more direct route.

The formation of the Néron model of J over $A_{\mathfrak{p}}$ is compatible with the unramified extensions of $A_{\mathfrak{p}}$; see [2, Cor. 7.2/2]. Therefore we can pass to the strict henselization of $A_{\mathfrak{p}}$ to put ourselves in the set-up of [2, Sect. 9.6]. Let the number of supersingular points on $X_0(\mathfrak{m})_{\overline{\mathbb{F}}_{\mathfrak{p}}}$ be s. Denote these points by $x_1, \dots, x_s$. Let $(D, Z_{\mathfrak{m}})_i$ be the pair corresponding to x_i, and let $n_i = \frac{1}{q-1}\#\text{Aut}(D, Z_{\mathfrak{m}})_i$. Then Raynaud's result implies that

$$\#\Phi_{J,\mathfrak{p}} = \sum_{i=1}^{s} \prod_{j \neq i} n_j.$$

As we mentioned, $\mathbb{F}_q^\times \subseteq \text{Aut}(D, Z_{\mathfrak{m}})_i \subseteq \mathbb{F}_{q^2}^\times$. Using this, one easily shows that $\text{Aut}(D, Z_{\mathfrak{m}})_i$ is either $\mathbb{F}_q^\times$ or $\mathbb{F}_{q^2}^\times$. Thus each n_i is either 1 or $q + 1$, so

$$\#\Phi_{J,\mathfrak{p}} \equiv \sum_{i=1}^{s} 1 \equiv s \pmod{p}. \tag{8}$$

By blowing-up $X_0(\mathfrak{n})_{\overline{\mathbb{F}}_\mathfrak{p}}$ at the supersingular points, one obtains two disjoint copies of the smooth curve $X_0(\mathfrak{m})_{\overline{\mathbb{F}}_\mathfrak{p}}$. A standard argument gives the formula

$$g(X_0(\mathfrak{n})_{\overline{\mathbb{F}}_\mathfrak{p}}) = s - 1 + 2g(X_0(\mathfrak{m})_{\overline{\mathbb{F}}_\mathfrak{p}}), \tag{9}$$

where g denotes the arithmetic genus of the corresponding curve. On the other hand, since $X_0(\mathfrak{n})$ is flat, the arithmetic genus of $X_0(\mathfrak{n})_F$ is equal to the arithmetic genus of $X_0(\mathfrak{n})_{\overline{\mathbb{F}}_\mathfrak{p}}$. Note that so far we have not used the assumption on $\mathfrak{n}$ being square-free. The assumption enters as the fact that $g(X_0(\mathfrak{n})_F) \equiv 0 \pmod{p}$ when $\mathfrak{n}$ is square-free. This can be easily deduced from the formula for $g(X_0(\mathfrak{n})_F)$ in [13, Cor.2.19] (in general the congruence is false if $\mathfrak{n}$ is not square-free). Since $\mathfrak{m}$ is also square-free, we also have $g(X_0(\mathfrak{m})_F) \equiv 0 \pmod{p}$. Therefore, from (9) we get $s \equiv 1$, which combined with (8) gives the proposition. □

Remark 3.7 The statement of Lemma 3.6 is true over general function fields. Let C be a proper smooth curve over $\mathbb{F}_q$ and let ∞ be a fixed closed point on C. Let F be the field of rational functions on C, and $A := H^0(C - \infty, \mathcal{O})$. Let $\mathfrak{n}$ be an arbitrary ideal of A, and $\mathfrak{p} \lhd A$ be a prime ideal. Let L be the finite unramified extension of $F_\mathfrak{p}$ corresponding to the maximal unramified abelian extension of F contained in F_∞. Then $M_0(\mathfrak{n})_L$ is a disjoint union of $\mathrm{Pic}(A)$ copies of an affine smooth geometrically connected curve defined over L. We denote by X_L the compactification of this curve, cf. [14] or [9]. Let $J = \mathrm{Pic}^0_{X_L/L}$ and let $\mathcal{J}$ be the Néron model of J over the ring of integers $\mathcal{L}$ of L. Let $\Phi_J := \mathcal{J}_{\bar{k}}/\mathcal{J}^0_{\bar{k}}$, where k is the residue field of $\mathcal{L}$. We claim that if $\mathfrak{n}$ is square-free then $\#\Phi_J \equiv 1 \pmod{p}$. It is easy to see that the argument in the proof Lemma 3.6 applies if $g(X_L) \equiv 0 \pmod{p}$ for $\mathfrak{n}$ square-free (for a discussion of Theorem 3.4 over general function fields we refer to [11]). We leave it to the reader to verify that by the methods outlined in [9, p. 90] one obtains for $\mathfrak{n} = \mathfrak{p}_1 \dots \mathfrak{p}_s$ ($\mathfrak{p}_i$'s are distinct prime ideals)

$$g(X_L) = 1 + (q^2-1)^{-1}\left(\prod_{i=1}^{s}(q^{d_i}+1)\frac{q^\delta - 1}{q-1}P(q) - 2^{s-1}q(\theta+\eta)\right),$$

where $d_i = \deg(\mathfrak{p}_i)$, $\delta = \deg(\infty)$, $\theta = \delta(q+1)P(1)$, $P(X)$ is the numerator of the zeta function of C, $Z_C(X) = \frac{P(X)}{(1-X)(1-qX)}$, and $\eta = (q-1)P(-1)$ if all d_i are even and δ is odd, and $\eta = 0$ otherwise. Since the constant term of $P(X)$ is 1, $g(X_L)$ is indeed divisible by p.

Finally we are ready to prove the main result of the paper.

Proof of Theorem 1.1 Let B be an abelian variety over F with bad reduction at $\mathfrak{p}$. Assume $B^{(p)}$ is isomorphic over F to an abelian subvariety of J. From now on we think of $B^{(p)}$ as being embedded into J. Consider the intersection $W' = J^{\text{new}} \cap B^{(p)}$. This is necessarily a positive dimensional abelian variety, as otherwise $B^{(p)}$, and therefore also B, has good reduction. Let W be the preimage of W' in B. Then $W^{(p)} = W'$, so without loss of generality we can assume that B (and hence also $B^{(p)}$) has toric reduction at $\mathfrak{p}$.

By Lemma 2.2, $\Phi_{B^{(p)},\mathfrak{p}}$ has non-trivial p-torsion. On the other hand, Lemma 3.6 says that $\Phi_{J,\mathfrak{p}}$ has no p-torsion. Therefore the conditions in Proposition 3.5 are satisfied and the theorem follows. □

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