On Jacquet–Langlands isogeny over function fields

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A B S T R A C T

We propose a conjectural explicit isogeny from the Jacobians of hyperelliptic Drinfeld modular curves to the Jacobians of hyperelliptic modular curves of D-elliptic sheaves. The kernel of the isogeny is a subgroup of the cuspidal divisor group constructed by examining the canonical maps from the cuspidal divisor group into the component groups.

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1. Introduction

Let N be a square-free integer, divisible by an even number of primes. It is well known that the new part of the modular Jacobian $J_0(N)$ is isogenous to the Jacobian of a Shimura curve; see [33]. The existence of this isogeny can be interpreted as a geometric incarnation of the global Jacquet–Langlands correspondence over $\mathbb{Q}$ between the cusp forms on $GL(2)$ and the multiplicative group of a quaternion algebra [24]. Jacquet–Langlands isogeny has important arithmetic applications, for example, to level lowering [35]. In this paper we are interested in the function field analogue of the Jacquet–Langlands isogeny.

Let $\mathbb{F}_q$ be the finite field with $q$ elements, and let $F = \mathbb{F}_q(T)$ be the field of rational functions on $\mathbb{P}^1_{\mathbb{F}_q}$. The set of places of $F$ will be denoted by $|F|$. Let $A := \mathbb{F}_q[T]$. This is the subring of $F$ consisting of functions which are regular away from the place generated by $1/T$ in $\mathbb{F}_q[1/T]$. The place
generated by $1/T$ will be denoted by $\infty$ and called the place at infinity; it will play a role similar to the archimedean place for $\mathbb{Q}$. The places in $|F| - \infty$ are the finite places.

Let $v \in |F|$. We denote by $F_v$, $O_v$ and $\mathbb{F}_v$ the completion of $F$ at $v$, the ring of integers in $F_v$, and the residue field of $F_v$, respectively. We assume that the valuation $\text{ord}_v : F_v \to \mathbb{Z}$ is normalized by $\text{ord}_v(\pi_v) = 1$, where $\pi_v$ is a uniformizer of $O_v$. The degree of $v$ is $\text{deg}(v) = [\mathbb{F}_v : \mathbb{F}_q]$. Let $q_v := q^{\text{deg}(v)} = \#\mathbb{F}_v$. If $v$ is a finite place, then with an abuse of notation we denote the prime ideal of $A$ corresponding to $v$ by the same letter.

Given a field $K$, we denote by $\bar{K}$ an algebraic closure of $K$.

Let $R \subseteq |F| - \infty$ be a nonempty finite set of places of even cardinality. Let $D$ be the quaternion algebra over $F$ ramified exactly at the places in $R$. Let $X^R_F$ be the modular curve of $D$-elliptic sheaves (see Section 2.2). This curve is the function field analogue of a Shimura curve parametrizing abelian surfaces with multiplication by a maximal order in an indefinite division quaternion algebra over $\mathbb{Q}$. Denote the Jacobian of $X^R_F$ by $J^R$. The role of classical modular curves in this context is played by Drinfeld modular curves. With an abuse of notation, let $R$ also denote the square-free ideal of $A$ whose support consists of the places in $R$. Let $X_0(R)_F$ be the Drinfeld modular curve defined in Section 2.1. Let $J_0(R)$ be the Jacobian of $X_0(R)_F$. The same strategy as over $\mathbb{Q}$ shows that $J^R$ is isogenous to the new part of $J_0(R)$ (see Theorem 7.1 and Remark 7.4). The proof relies on Tate’s conjecture, so it provides no information about the isogenies $J^R \to J_0(R)^{\text{new}}$ beyond their existence. In this paper we carefully examine the simplest non-trivial case, namely $R = \{x, y\}$ with $\text{deg}(x) = 1$ and $\text{deg}(y) = 2$. (When $R = \{x, y\}$ and $\text{deg}(x) = \text{deg}(y) = 1$, both $X^R_F$ and $X_0(R)_F$ have genus 0.)

Notation 1.1. Unless indicated otherwise, throughout the paper $x$ and $y$ will be two fixed finite places of degree 1 and 2, respectively. When $R = \{x, y\}$, we write $X^{xy}_F$ for $X^R_F$, $J^{xy}$ for $J^R$, $X_0(xy)_F$ for $X_0(R)_F$, and $J_0(xy)$ for $J_0(R)$.

The genus of $X^{xy}_F$ is $q$, which is also the genus of $X_0(xy)_F$. Hence $J_0(xy)$ and $J^{xy}$ are $q$-dimensional Jacobian varieties, which are isogenous over $F$. We would like to construct an explicit isogeny $J_0(xy) \to J^{xy}$. A natural place to look for the kernel of an isogeny defined over $F$ is in the cuspidal divisor group $C$ of $J_0(xy)$. To see which subgroup of $C$ could be the kernel, one needs to compute, besides $C$ itself, the component groups of $J_0(xy)$ and $J^{xy}$, and the canonical specialization maps of $C$ into the component groups of $J_0(xy)$. These calculations constitute the bulk of the paper. Based on these calculations, in Section 7 we propose a conjectural explicit isogeny $J_0(xy) \to J^{xy}$, and prove that the conjecture is true for $q = 2$. We note that $X^{xy}_F$ is hyperelliptic, and in fact for odd $q$ these are the only $X^R_F$ which are hyperelliptic [31]. The curve $X_0(xy)_F$ is also hyperelliptic, and for levels which decompose into a product of two prime factors these are the only hyperelliptic Drinfeld modular curves [36]. Hence this paper can also be considered as a study of hyperelliptic modular Jacobians over $F$ which interrelates [31] and [36].

The approach to explicating the Jacquet–Langlands isogeny through the study of component groups and cuspidal divisor groups was initiated in the classical context by Ogg. In [27], Ogg proposed in several cases conjectural explicit isogenies between the modular Jacobians and the Jacobians of Shimura curves (as far as I know, these conjectures are still mostly open, but see [19] and [23] for some advances).

We summarize the main results of the paper.

- The cuspidal divisor group $C \subset J_0(xy)(F)$ is isomorphic to

$$C \cong \mathbb{Z}/(q + 1)\mathbb{Z} \oplus \mathbb{Z}/(q^2 + 1)\mathbb{Z}.$$  

- The component groups of $J_0(xy)$ and $J^{xy}$ at $x$, $y$, and $\infty$ are listed in Table 1. ($J_0(xy)$ and $J^{xy}$ have good reduction away from $x$, $y$ and $\infty$, so the component groups are trivial away from these three places.)

- If we denote the component group of $J_0(xy)$ at $\ast$ by $\Phi_\ast$, and the canonical map $C \to \Phi_\ast$ by $\phi_\ast$, then there are exact sequences
Table 1

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<td>$J_0(xy)$</td>
<td>$\mathbb{Z}/(q^2 + 1)(q + 1)\mathbb{Z}$</td>
<td>$\mathbb{Z}/(q + 1)\mathbb{Z}$</td>
<td>$\mathbb{Z}/(q^2 + 1)(q + 1)\mathbb{Z}$</td>
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<tr>
<td>$J^{xy}$</td>
<td>$\mathbb{Z}/(q + 1)\mathbb{Z}$</td>
<td>$\mathbb{Z}/(q^2 + 1)(q + 1)\mathbb{Z}$</td>
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$$0 \rightarrow \mathbb{Z}/(q + 1)\mathbb{Z} \rightarrow C \xrightarrow{\phi_x} \mathbb{Z}/(q + 1)\mathbb{Z} \rightarrow 0,$$

$$0 \rightarrow \mathbb{Z}/(q^2 + 1)\mathbb{Z} \rightarrow C \xrightarrow{\phi_y} \mathbb{Z}/(q + 1)\mathbb{Z} \rightarrow 0,$$

$$\phi_\infty : C \xrightarrow{\sim} \Phi_\infty \text{ if } q \text{ is even},$$

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow C \xrightarrow{\phi_\infty} \Phi_\infty \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0 \text{ if } q \text{ is odd}.$$

- The kernel $C_0 \cong \mathbb{Z}/(q^2 + 1)\mathbb{Z}$ of $\phi$ maps injectively into $\Phi_x$ and $\Phi_\infty$.

Conjecture 7.3 then states that there is an isogeny $J_0(xy) \rightarrow J^{xy}$ whose kernel is $C_0$. As an evidence for the conjecture, we prove that the quotient abelian variety $J_0(xy)/C_0$ has component groups of the same order as $J^{xy}$. This is a consequence of a general result (Theorem 4.3), which describes how the component groups of abelian varieties with toric reduction change under isogenies. Finally, we prove Conjecture 7.3 for $q = 2$ (Theorem 7.12); the proof relies on the fact that $J_0(xy)$ in this case is isogenous to a product of two elliptic curves. Two other interesting consequences of our results are the following. First, we deduce the genus formula for $X^8_F$ proven in [30] by a different argument (Corollary 6.3). Second, assuming $q$ is even and Conjecture 7.3 is true, we are able to tell how the optimal elliptic curve with conductor $xy\infty$ changes in a given $F$-isogeny class when we change the modular parametrization from $X_0(xy)_F$ to $X^{xy}_F$ (Proposition 7.10).

2. Preliminaries

2.1. Drinfeld modular curves

Let $K$ be an $A$-field, i.e., $K$ is a field equipped with a homomorphism $\gamma : A \rightarrow K$. In particular, $K$ contains $\mathbb{F}_q$ as a subfield. The $A$-characteristic of $K$ is the ideal $\ker(\gamma) \triangleleft A$. Let $K(\tau)$ be the twisted polynomial ring with commutation rule $\tau s = s^2 \tau$, $s \in K$. A rank-2 Drinfeld $A$-module over $K$ is a ring homomorphism $\phi : A \rightarrow K(\tau)$, $a \mapsto \phi_a$ such that $\deg_{\mathbb{Z}} \phi_a = -2\operatorname{ord}_a(a)$ and the constant term of $\phi_a$ is $\gamma(a)$. A homomorphism of two Drinfeld modules $u : \phi \rightarrow \psi$ is $u \in K(\tau)$ such that $\phi_a u = u \psi_a$ for all $a$ in $A$; $u$ is an isomorphism if $u \in K^\times$. Note that $\phi$ is uniquely determined by the image of $T$:

$$\phi_T = \gamma(T) + gT + \Delta T^2,$$

where $g \in K$ and $\Delta \in K^\times$. The $j$-invariant of $\phi$ is $j(\phi) = g^{q+1}/\Delta$. It is easy to check that if $K$ is algebraically closed, then $\phi \cong \psi$ if and only if $j(\phi) = j(\psi)$.

Treating $\tau$ as the automorphism of $K$ given by $k \mapsto k^2$, the field $K$ acquires a new $A$-module structure via $\phi$. Let $a \triangleleft A$ be an ideal. Since $A$ is a principal ideal domain, we can choose a generator $a \in A$ of $a$. The $A$-module $\phi[a] = \ker \phi_a(\bar{K})$ does not depend on the choice of $a$ and is called the $a$-torsion of $\phi$. If $a$ is coprime to the $A$-characteristic of $K$, then $\phi[a] \cong (A/a)^2$. On the other hand, if $p = \ker(\gamma) \neq 0$, then $\phi[p] \cong (A/p)$ or $0$; when $\phi[p] = 0$, $\phi$ is called supersingular.

Lemma 2.1. Up to isomorphism, there is a unique supersingular rank-2 Drinfeld $A$-module over $\mathbb{F}_q$; it is the Drinfeld module with $j$-invariant equal to 0. Up to isomorphism, there is a unique supersingular rank-2 Drinfeld $A$-module over $\mathbb{F}_q$, and its $j$-invariant is non-zero.

Proof. This follows from [9, (5.9)] since $\deg(x) = 1$ and $\deg(y) = 2$. □
Let \( \text{End}(\phi) \) denote the centralizer of \( \phi(A) \) in \( \mathcal{K}\{\tau\} \), i.e., the ring of all homomorphisms \( \phi \to \phi \) over \( \mathcal{K} \). The automorphism group \( \text{Aut}(\phi) \) is the group of units \( \text{End}(\phi)^\times \).

**Lemma 2.2.** If \( j(\phi) \neq 0 \), then \( \text{Aut}(\phi) \cong \mathbb{F}_q^\times \). If \( j(\phi) = 0 \), then \( \text{Aut}(\phi) \cong \mathbb{F}_{q^2}^\times \).

**Proof.** If \( u \in \mathcal{K}^\times \) commutes with \( \phi_\tau = \gamma(T) + g \tau + \Delta \tau^2 \), then \( u^{q^2-1} = 1 \) and \( u^{q-1} = 1 \) if \( g \neq 0 \). This implies that \( u \in \mathbb{F}_q^\times \) if \( j(\phi) \neq 0 \), and \( u \in \mathbb{F}_{q^2}^\times \) if \( j(\phi) = 0 \). On the other hand, we clearly have the inclusions \( \mathbb{F}_q^\times \subset \text{Aut}(\phi) \) and, if \( j(\phi) = 0 \), \( \mathbb{F}_{q^2}^\times \subset \text{Aut}(\phi) \). This finishes the proof. \( \square \)

**Lemma 2.3.** Let \( p \triangleleft A \) be a prime ideal and \( \mathbb{F}_p := A/p \). Let \( \phi \) be a rank-2 Drinfeld \( A \)-module over \( \mathbb{F}_p \). Let \( n \triangleleft A \) be an ideal coprime to \( p \). Let \( C_n \) be an \( A \)-submodule of \( \phi[n] \) isomorphic to \( A/n \). Denote by \( \text{Aut}(\phi, C_n) \) the subgroup of automorphisms of \( \phi \) which map \( C_n \) to itself. Then \( \text{Aut}(\phi, C_n) \cong \mathbb{F}_q^\times \) or \( \mathbb{F}_{q^2}^\times \). The second case is possible only if \( j(\phi) = 0 \).

**Proof.** The action of \( \mathbb{F}_q^\times \) obviously stabilizes \( C_n \), hence, using Lemma 2.2, it is enough to show that if \( \text{Aut}(\phi, C_n) \neq \mathbb{F}_q^\times \), then \( \text{Aut}(\phi, C_n) \cong \mathbb{F}_{q^2}^\times \). Let \( u \in \text{Aut}(\phi, C_n) \) be an element which is not in \( \mathbb{F}_q \). Then \( \text{Aut}(\phi) = \mathbb{F}_q[u]^\times \cong \mathbb{F}_{q^2}^\times \), where \( \mathbb{F}_q[u] \) is considered as a finite subring of \( \text{End}(\phi) \). It remains to show that \( \alpha + u\beta \) stabilizes \( C_n \) for any \( \alpha, \beta \in \mathbb{F}_q \) not both equal to zero. But this is obvious since \( \alpha \) and \( u\beta \) stabilize \( C_n \) and \( C_n \cong A/n \) is cyclic. \( \square \)

One can generalize the notion of Drinfeld modules over an \( A \)-field to the notion of Drinfeld modules over an arbitrary \( A \)-scheme \( S \) [8]. The functor which associates to an \( A \)-scheme \( S \) the set of isomorphism classes of pairs \( (\phi, C_n) \), where \( \phi \) is a Drinfeld \( A \)-module of rank 2 over \( S \) and \( C_n \cong A/n \) is an \( A \)-submodule of \( \phi[n] \), possesses a coarse moduli scheme \( Y_0(n) \) that is affine, flat and of finite type over \( A \) of pure relative dimension 1. There is a canonical compactification \( X_0(n) \) of \( Y_0(n) \) over \( \text{Spec}(A) \); see [8, §9] or [41]. The finitely many points \( X_0(n)(\bar{F}) - Y_0(n)(\bar{F}) \) are called the cusps of \( X_0(n) \).

Denote by \( \mathbb{C}_\infty \) the completion of an algebraic closure of \( F_\infty \). Let \( \Omega = \mathbb{C}_\infty - F_\infty \) be the Drinfeld upper half-plane; \( \Omega \) has a natural structure of a smooth connected rigid-analytic space over \( F_\infty \). Denote by \( \Gamma_0(n) \) the Hecke congruence subgroup of level \( n \):

\[
\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(A) \mid c \in n \right\}.
\]

The group \( \Gamma_0(n) \) naturally acts on \( \Omega \) via linear fractional transformations, and the action is discrete in the sense of [8, p. 582]. Hence we may construct the quotient \( \Gamma_0(n) \backslash \Omega \) as a 1-dimensional connected smooth analytic space over \( F_\infty \).

The following theorem can be deduced from the results in [8]:

**Theorem 2.4.** \( X_0(n) \) is a proper flat scheme of pure relative dimension 1 over \( \text{Spec}(A) \), which is smooth away from the support of \( n \). There is an isomorphism of rigid-analytic spaces \( \Gamma_0(n) \backslash \Omega \cong Y_0(n)_{F_\infty}^{\text{ph}} \).

There is a genus formula for \( X_0(n)_{F} \) which depends on the prime decomposition of \( n \); see [16, Thm. 2.17]. By this formula, the genera of \( X_0(x)_{F}, X_0(y)_{F} \) and \( X_0(xy)_{F} \) are 0, 0 and \( q \), respectively.

### 2.2. Modular curves of \( \mathcal{D} \)-elliptic sheaves

Let \( D \) be a quaternion algebra over \( F \). Let \( R \subseteq |F| \) be the set of places which ramify in \( D \), i.e., \( D \otimes F_v \) is a division algebra for \( v \in R \). It is known that \( R \) is finite of even cardinality, and, up to isomorphism, this set uniquely determines \( D \); see [42]. Assume \( R \neq \emptyset \) and \( \infty \notin R \). In particular, \( D \) is
Let $\mathbb{P}^1_{\mathbb{F}_q}$ be a division algebra. Let $C := \mathbb{P}^1_{\mathbb{F}_q}$. Fix a locally free sheaf $D$ of $O_C$-algebras with stalk at the generic point equal to $D$ and such that $D_v := D \otimes_{O_C} O_v$ is a maximal order in $D_v := D \otimes_F F_v$.

Let $S$ be an $\mathbb{F}_q$-scheme. Denote by Frob$_S$ its Frobenius endomorphism, which is the identity on the points and the $q$th power map on the functions. Denote by $C \times S$ the fibered product $C \times_{\text{Spec}(\mathbb{F}_q)} S$. Let $z : S \to C$ be a morphism of $\mathbb{F}_q$-schemes. A $D$-elliptic sheaf over $S$, with pole $\infty$ and zero $z$, is an irreducible polynomial in $O_S$.

Proof. Let $\chi_0$ be an $\mathbb{F}_q$-scheme. Denote by Frob$_S$ its Frobenius endomorphism, which is the identity on the points and the $q$th power map on the functions. Denote by $C \times S$ the fibered product $C \times_{\text{Spec}(\mathbb{F}_q)} S$. Let $z : S \to C$ be a morphism of $\mathbb{F}_q$-schemes. A $D$-elliptic sheaf over $S$, with pole $\infty$ and zero $z$, is a sequence $E = (E_i, j_i, t_i)_{i \in \mathbb{Z}}$, where each $E_i$ is a locally free sheaf of $O_{C \times S}$-modules of rank 4 equipped with a right action of $D$ compatible with the $O_C$-action, and where

$$j_i : E_i \to E_{i+1},$$

$$t_i : E_i := (\text{Id}_C \times \text{Frob}_S)^* E_i \to E_{i+1}$$

are injective $O_{C \times S}$-linear homomorphisms compatible with the $D$-action. The maps $j_i$ and $t_i$ are sheaf modifications at $\infty$ and $z$, respectively, which satisfy certain conditions, and it is assumed that for each closed point $w$ of $S$, the Euler–Poincaré characteristic $\chi(E_0|_{C \times w})$ is in the interval $[0, 2)$; we refer to [26, §2] and [22, §1] for the precise definition. Moreover, to obtain moduli schemes with good properties at the closed points $w$ of $S$ such that $z(w) \in \mathbb{R}$ one imposes an extra condition on $E$ to be “special” [22, p. 1305]. Note that, unlike the original definition in [26], $\infty$ is allowed to be in the image of $S$; here we refer to [1, §4.4] for the details. Denote by $E\ell D(S)$ the set of isomorphism classes of $D$-elliptic sheaves over $S$. The following theorem can be deduced from some of the main results in [26] and [22]:

**Theorem 2.5.** The functor $S \mapsto E\ell D(S)$ has a coarse moduli scheme $X^R$, which is proper and flat of pure relative dimension 1 over $C$ and is smooth over $C - R - \infty$.

**Remark 2.6.** Theorems 2.4 and 2.5 imply that $J_0(R)$ and $J^R$ have good reduction at any place $v \in |F| - R - \infty$; cf. [2, Ch. 9].

### 3. Cuspidal divisor group

For a field $K$, we represent the elements of $\mathbb{P}^1(K)$ as column vectors $\left( \begin{array}{c} u \\ v \end{array} \right)$ where $u, v \in K$ are not both zero and $\left( \begin{array}{c} u \\ v \end{array} \right)$ is identified with $\left( \begin{array}{c} au \\ av \end{array} \right)$ if $\alpha \in K^\times$. We assume that $GL_2(K)$ acts on $\mathbb{P}^1(K)$ on the left by

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} au + bv \\ cu + dv \end{array} \right).$$

Let $n \triangleleft A$ be an ideal. The cusps of $X_0(n)_F$ are in natural bijection with the orbits of $\Gamma_0(n)$ acting from the left on $\mathbb{P}^1(F)$.

**Lemma 3.1.** If $n$ is square-free, then there are $2^s$ cusps on $X_0(n)_F$, where $s$ is the number of prime divisors of $n$. All the cusps are $F$-rational.

**Proof.** See Proposition 3.3 and Corollary 3.4 in [11].

For every $m|n$ with $(m, n/m) = 1$ there is an Atkin–Lehner involution $W_m$ on $X_0(n)_F$, cf. [36]. Its action is given by multiplication from the left with any matrix $\left( \begin{array}{cc} ma & b \\ n & m \end{array} \right)$ whose determinant generates $m$, and where $a, b, m, n \in A$, $(n) = n, (m) = m$.

From now on assume $n = xy$. Recall that we denote by $x$ and $y$ the prime ideals of $A$ corresponding to the places $x$ and $y$, respectively. With an abuse of notation, we will denote by $x$ also the monic irreducible polynomial in $A$ generating the ideal $x$, and similarly for $y$. It should be clear from the
context in which capacity $x$ and $y$ are being used. With this notation, $X_0(xy)_F$ has 4 cusps, which can be represented by

$$[\infty] := \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad [0] := \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad [x] := \left( \begin{array}{c} 1 \\ x \end{array} \right), \quad [y] := \left( \begin{array}{c} 1 \\ y \end{array} \right).$$

cf. [36, p. 333] and [15, p. 196].

There are 3 non-trivial Atkin–Lehner involutions $W_x, W_y, W_{xy}$ which generate a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$: these involutions commute with each other and satisfy

$$W_x W_y = W_{xy}, \quad W_x^2 = W_y^2 = W_{xy}^2 = 1.$$ 

By [36, Prop. 9], none of these involutions fixes a cusp. In fact, a simple direct calculation shows that

$$W_{xy}([\infty]) = [0], \quad W_{xy}([x]) = [y];$$
$$W_x([\infty]) = [y], \quad W_x([0]) = [x];$$
$$W_y([\infty]) = [x], \quad W_y([0]) = [y].$$  \hfill (3.1)

Let $\Delta(z), z \in \Omega$, denote the Drinfeld discriminant function; see [11] or [15] for the definition. This is a holomorphic and nowhere vanishing function on $\Omega$. In fact, $\Delta(z)$ is a type-0 and weight-$(q^2 - 1)$ cusp form for $\text{GL}_2(A)$. Its order of vanishing at the cusps of $X_0(n)_F$ can be calculated using [15]. When $n = xy$, [15, (3.10)] implies

$$\text{ord}_{[\infty]} \Delta = 1, \quad \text{ord}_{[0]} \Delta = q_x q_y, \quad \text{ord}_{[x]} \Delta = q_y, \quad \text{ord}_{[y]} \Delta = q_x.$$  \hfill (3.2)

The functions

$$\Delta_x(z) := \Delta(xz), \quad \Delta_y(z) := \Delta(yz), \quad \Delta_{xy}(z) := \Delta(xy z)$$

are type-0 and weight-$(q^2 - 1)$ cusp forms for $\Gamma_0(xy)$. Hence the fractions $\Delta/\Delta_x, \Delta/\Delta_y, \Delta/\Delta_{xy}$ define rational functions on $X_0(xy)_C$. We compute the divisors of these functions.

The matrix $W_{xy} = \begin{pmatrix} 0 & 1 \\ xy & 0 \end{pmatrix}$ normalizes $\Gamma_0(xy)$ and interchanges $\Delta(z)$ and $\Delta_{xy}(z)$. Thus by (3.1) and (3.2)

$$\text{ord}_{[\infty]} \Delta_{xy} = q_x q_y, \quad \text{ord}_{[0]} \Delta_{xy} = 1, \quad \text{ord}_{[x]} \Delta_{xy} = q_x, \quad \text{ord}_{[y]} \Delta_{xy} = q_y.$$ 

A similar argument involving the actions of $W_x$ and $W_y$ gives

$$\text{ord}_{[\infty]} \Delta_x = q_x, \quad \text{ord}_{[0]} \Delta_x = q_y, \quad \text{ord}_{[x]} \Delta_x = q_x q_y, \quad \text{ord}_{[y]} \Delta_x = 1;$$
$$\text{ord}_{[\infty]} \Delta_y = q_y, \quad \text{ord}_{[0]} \Delta_y = q_x, \quad \text{ord}_{[x]} \Delta_y = 1, \quad \text{ord}_{[y]} \Delta_y = q_x q_y.$$ 

From these calculations we obtain

$$\text{div}(\Delta/\Delta_{xy}) = (1 - q_x q_y)[\infty] + (q_x q_y - 1)[0] + (q_y - q_x)[x] + (q_x - q_y)[y]$$
$$= (q^3 - 1)([0] - [\infty]) + (q^2 - q)([x] - [y]),$$

and similarly,
\[ \text{div}(\Delta/\Delta_x) = (q - 1)([y] - [\infty]) + (q^3 - q^2)([0] - [x]), \]
\[ \text{div}(\Delta/\Delta_y) = (q^2 - 1)([x] - [\infty]) + (q^3 - q)([0] - [y]). \]

Next, by [15, p. 200], the largest positive integer \( k \) such that \( \Delta/\Delta_{xy} \) has a \( k \)th root in the field of modular functions for \( \Gamma_0(xy) \) is \( (q - 1)^2/(q - 1) = (q - 1) \). We can apply the same argument to \( \Delta/\Delta_x \) as a modular function for \( \Gamma_0(x) \) to deduce that \( \Delta/\Delta_x \) has \( (q - 1)^2/(q - 1) \)th root. Similarly, \( \Delta/\Delta_y \) has \( (q - 1)(q^2 - 1)/(q - 1) \)th root. Therefore, the following relations hold in \( \text{Pic}^0(X_0(xy)_{/F}) \):

\[
(q^2 + q + 1)([0] - [\infty]) + q([x] - [y]) = 0, \\
([y] - [\infty]) + q^2([0] - [x]) = 0, \\
([x] - [\infty]) + q([0] - [y]) = 0. \tag{3.3}
\]

There is one more relation between the cuspidal divisors which comes from the fact that \( X_0(xy)_{/F} \) is hyperelliptic. By a theorem of Schweizer [36, Thm. 20], \( X_0(xy)_{/F} \) is hyperelliptic, and \( W_{xy} \) is the hyperelliptic involution. Consider the degree-2 covering

\[
\pi : X_0(xy)_{/F} \to X_0(xy)_{/F}/W_{xy} \cong \mathbb{P}^1_F.
\]

Denote \( P := \pi([\infty]), Q := \pi([x]). \) Since \( W_{xy}([\infty]) \neq [x], \ P \neq Q \). There is a function \( f \) on \( \mathbb{P}^1_F \) with divisor \( P - Q \). Now

\[
\text{div}(\pi^*f) = \pi^*(\text{div}(f)) = \pi^*(P - Q) \\
= ([\infty] + W_{xy}([\infty])) - ([x] + W_{xy}([x])) = [\infty] + [0] - [x] - [y].
\]

This gives the relation in \( \text{Pic}^0(X_0(xy)_{/F}) \)

\[
[\infty] + [0] - [x] - [y] = 0. \tag{3.4}
\]

Fixing \([\infty] \in X_0(xy)(F)\) as an \( F \)-rational point, we have the Abel–Jacobi map \( X_0(xy)_{/F} \to J_0(xy) \) which sends a point \( P \in X_0(xy)_{/F} \) to the linear equivalence class of the degree-0 divisor \( P - [\infty] \).

**Definition 3.2.** Let \( c_0, c_x, c_y \in J_0(xy)(F) \) be the classes of \([0] - [\infty], [x] - [\infty] \), and \([y] - [\infty] \), respectively. These give \( F \)-rational points on the Jacobian since the cusps are \( F \)-rational. The **cuspidal divisor group** is the subgroup \( C \subset J_0(xy) \) generated by \( c_0, c_x, \) and \( c_y \).

From (3.3) and (3.4) we obtain the following relations:

\[
(q^2 + q + 1)c_0 + qc_x - qc_y = 0, \\
q^2c_0 - q^2c_x + c_y = 0, \\
qc_0 + c_x - qc_y = 0, \\
c_0 - c_x - c_y. \]

**Lemma 3.3.** The cuspidal divisor group \( C \) is generated by \( c_x \) and \( c_y \), which have orders dividing \( q + 1 \) and \( q^2 + 1 \), respectively.
Proof. Substituting \( c_0 = c_x + c_y \) into the first three equations above, we see that \( C \) is generated by \( c_x \) and \( c_y \) subject to relations:

\[
(q + 1)c_x = 0,
\]
\[
(q^2 + 1)c_y = 0.
\]

The following simple lemma, which will be used later on, shows that the factors \( (q^2 + 1) \) and 
\( (q + 1) \) appearing in Lemma 3.3 are almost coprime.

**Lemma 3.4.** Let \( n \) be a positive integer. Then

\[
\gcd(n^2 + 1, n + 1) = \begin{cases} 
1, & \text{if } n \text{ is even;} \\
2, & \text{if } n \text{ is odd.}
\end{cases}
\]

**Proof.** Let \( d = \gcd(n^2 + 1, n + 1) \). Then \( d \) divides \((n^2 + 1) - (n + 1) = n(n - 1)\). Since \( n \) is coprime to \( n + 1 \), \( d \) must divide \( n - 1 \), hence also must divide \((n + 1) - (n - 1) = 2\). For \( n \) even, \( d \) is obviously odd, so \( d = 1 \). For \( n \) odd, \( n + 1 \) and \( n^2 + 1 \) are both even, so \( d = 2 \).

4. Néron models and component groups

4.1. Terminology and notation

The notation in this section will be somewhat different from the rest of the paper. Let \( R \) be a complete discrete valuation ring, with fraction field \( K \) and algebraically closed residue field \( k \).

Let \( A_K \) be an abelian variety over \( K \). Denote by \( A \) its Néron model over \( R \) and denote by \( A^0_k \) the connected component of the identity of the special fiber \( A_k \) of \( A \). There is an exact sequence

\[
0 \to A^0_k \to A_k \to \Phi_A \to 0,
\]

where \( \Phi_A \) is a finite (abelian) group called the component group of \( A_K \). We say that \( A_K \) has semi-abelian reduction if \( A^0_k \) is an extension of an abelian variety \( A'_k \) by an affine algebraic torus \( T_A \) over \( k \) (cf. [2, p. 181]):

\[
0 \to T_A \to A^0_k \to A'_k \to 0.
\]

We say that \( A_K \) has toric reduction if \( A^0_k = T_A \). The character group

\[
M_A := \text{Hom}(T_A, \mathbb{G}_m, k)
\]

is a free abelian group contravariantly associated to \( A \).

Let \( X_K \) be a smooth, proper, geometrically connected curve over \( K \). We say that \( X \) is a semi-stable model of \( X_K \) over \( R \) if (cf. [2, p. 245]):

(i) \( X \) is a proper flat \( R \)-scheme.

(ii) The generic fiber of \( X \) is \( X_K \).

(iii) The special fiber \( X_k \) is reduced, connected, and has only ordinary double points as singularities.

We will denote the set of irreducible components of \( X_k \) by \( C(X) \) and the set of singular points of \( X_k \) by \( S(X) \). Let \( G(X) \) be the dual graph of \( X \): The set of vertices of \( G(X) \) is the set \( C(X) \), the set of edges is the set \( S(X) \), the end points of an edge \( x \) are the two components containing \( x \). Locally at \( x \in S(X) \) for the étale topology, \( X \) is given by the equation \( uv = \pi^{m(x)} \), where \( \pi \) is a uniformizer of \( R \). The integer \( m(x) \geq 1 \) is well defined, and will be called the thickness of \( x \). One obtains from \( G(X) \) a graph with length by assigning to each edge \( x \in S(X) \) the length \( m(x) \).
4.2. Raynaud’s theorem

Let $X_K$ be a curve over $K$ with semi-stable model $X$ over $R$. Let $J_K$ be the Jacobian of $X_K$, let $J$ be the Néron model of $J_K$ over $R$, and $\Phi := J_K/J_0^0$. Let $\tilde{X} \to X$ be the minimal resolution of $X$. Let $B(\tilde{X})$ be the free abelian group generated by the elements of $C(\tilde{X})$. Let $B^0(\tilde{X})$ be the kernel of the homomorphism

$$B(\tilde{X}) \to \mathbb{Z}, \quad \sum_{C_1 \in C(\tilde{X})} n_i C_1 \mapsto \sum n_i.$$

The elements of $C(\tilde{X})$ are Cartier divisors on $\tilde{X}$, hence for any two of them, say $C$ and $C'$, we have an intersection number $(C \cdot C')$. The image of the homomorphism

$$\alpha : B(\tilde{X}) \to B(\tilde{X}), \quad C \mapsto \sum_{C' \in C(\tilde{X})} (C \cdot C')C'$$

lies in $B^0(\tilde{X})$. A theorem of Raynaud [2, Thm. 9.6/1] says that $\Phi$ is canonically isomorphic to $B^0(\tilde{X})/\alpha(B(\tilde{X}))$.

The homomorphism $\phi : J_K(K) \to \Phi$ obtained from the composition

$$J_K(K) = J(R) \to J_K(k) \to \Phi$$

will be called the canonical specialization map. Let $D = \sum_Q n_Q Q$ be a degree-0 divisor on $X_K$ whose support is in the set of $K$-rational points. Let $P \in J_K(K)$ be the linear equivalence class of $D$. The image $\phi(P)$ can be explicitly described as follows. Since $X$ and $\tilde{X}$ are proper, $X(K) = X(R) = \tilde{X}(R)$. Since $\tilde{X}$ is regular, each point $Q \in X(K)$ specializes to a unique element $c(Q)$ of $C(\tilde{X})$. With this notation, $\phi(P)$ is the image of $\sum_Q n_Q c(Q) \in B^0(\tilde{X})$ in $\Phi$.

We apply Raynaud’s theorem to compute $\Phi$ explicitly for a special type of $X$. Assume that $X_k$ consists of two components $Z$ and $Z'$ crossing transversally at $n \geq 2$ points $x_1, \ldots, x_n$. Denote $m_i := m(x_i)$. Let $r : \tilde{X} \to X$ denote the resolution morphism; it is a composition of blow-ups at the singular points. It is well known that $r^{-1}(x_i)$ is a chain of $m_i - 1$ projective lines. More precisely, the special fiber $\tilde{X}_k$ consists of $Z$ and $Z'$ but now, instead of intersecting at $x_i$, these components are joined by a chain $E_1, \ldots, E_{m_i-1}$ of projective lines, where $E_i$ intersects $E_{i+1}$, $E_1$ intersects $Z$ at $x_i$ and $E_{m_i-1}$ intersects $Z'$ at $x_i$. All the singularities are ordinary double points.

Assume $m_1 = m_2 = \cdots = m_n = 1$ if $n \geq 3$.

If $m = 1$, then $X = \tilde{X}$, so $B^0(\tilde{X})$ is freely generated by $z := Z - Z'$. In this case Raynaud’s theorem implies that $\Phi$ is isomorphic to $B^0(\tilde{X})$ modulo the relation $nz = 0$.

If $m \geq 2$, let $E_1, \ldots, E_{m-1}$ be the chain of projective lines at $x_1$ and $G_1, \ldots, G_{m-1}$ be the chain of projective lines at $x_n$, with the convention that $Z$ in $\tilde{X}$ intersects $E_1$ and $G_1$, cf. Fig. 1. The elements $z := Z - Z'$, $e_i := E_i - Z'$, $g_i := G_i - Z'$, $1 \leq i \leq m - 1$ form a $\mathbb{Z}$-basis of $B^0(\tilde{X})$. By Raynaud’s theorem, $\Phi$ is isomorphic to $B^0(\tilde{X})$ modulo the following relations:

if $m = 2$,

$$-nz + e_1 + g_1 = 0, \quad z - 2e_1 = 0, \quad z - 2g_1 = 0;$$
if \( m = 3 \),

\[-nz + e_1 + g_1 = 0, \quad z - 2e_1 + e_2 = 0, \quad z - 2g_1 + g_2 = 0,\]
\[e_1 - 2e_2 = 0, \quad g_1 - 2g_2 = 0;\]

if \( m \geq 4 \)

\[-nz + e_1 + g_1 = 0, \quad z - 2e_1 + e_2 = 0, \quad z - 2g_1 + g_2 = 0,\]
\[e_i - 2e_{i+1} + e_{i+2} = 0, \quad g_i - 2g_{i+1} + g_{i+2} = 0, \quad 1 \leq i \leq m - 3,\]
\[e_{m-2} - 2e_{m-1} = 0, \quad g_{m-2} - 2g_{m-1} = 0.\]

**Theorem 4.1.** Denote the images of \( z, e_i, g_i \) in \( \Phi \) by the same letters, and let \( \langle z \rangle \) be the cyclic subgroup generated by \( z \) in \( \Phi \). Then for any \( n \geq 2 \) and \( m \geq 1 \):

(i) \( \Phi \cong \mathbb{Z}/m(m(n - 2) + 2)\mathbb{Z} \).

(ii) If \( m \geq 2 \), then \( \Phi \) is generated by \( e_{m-1} \). Explicitly, for \( 1 \leq i \leq m - 1 \),

\[e_i = (m - i)e_{m-1},\]
\[g_i = (i(nm + 1) - (2i - 1)m)e_{m-1},\]
\[z = me_{m-1}.\]

(iii) \( \Phi/\langle z \rangle \cong \mathbb{Z}/m\mathbb{Z} \).

**Proof.** When \( m = 1 \) the claim is obvious, so assume \( m \geq 2 \). By [2, Prop. 9.6/10], \( \Phi \) has order

\[\sum_{i=1}^{n} \prod_{j \neq i} m_j = m^2(n - 2) + 2m.\]

From the relations

\[e_{m-2} - 2e_{m-1} = 0,\]
\[e_i - 2e_{i+1} + e_{i+2} = 0, \quad 1 \leq i \leq m - 3,\]
\[z - 2e_1 + e_2 = 0\]

it follows inductively that \( e_i = (m - i)e_{m-1} \) for \( 1 \leq i \leq m - 1 \), and \( z = me_{m-1} \). Next, from the relations

\[-nz + e_1 + g_1 = 0 \quad \text{and} \quad z - 2g_1 + g_2 = 0\]

we get \( g_1 = (nm - m + 1)e_{m-1} \) and \( g_2 = (2nm - 3m + 2)e_{m-1} \). Finally, if \( m \geq 4 \), the relations \( g_i - 2g_{i+1} + g_{i+2} = 0, 1 \leq i \leq m - 3 \), show inductively that

\[g_i = (i(nm + 1) - (2i - 1)m)e_{m-1}, \quad 1 \leq i \leq m - 1.\]

This proves (i) and (ii), and (iii) is an immediate consequence of (ii). \( \square \)
Remark 4.2. Note that by the formula in Theorem 4.1
\[ g_{m-1} = (m^2(n - 2) + 2m - (m(n - 2) + 1))e_{m-1} = -(m(n - 2) + 1)e_{m-1}. \]

It is easy to see that \( m(n - 2) + 1 \) is coprime to the order \( m(m(n - 2) + 2) \) of \( \Phi \). Hence \( g_{m-1} \) is also a generator. This is of course not surprising since the relations defining \( \Phi \) remain the same if we interchange \( e_i \)'s and \( g_i \)'s.

4.3. Grothendieck’s theorem

Grothendieck gave another description of \( \Phi \) in [20]. This description will be useful for us when studying maps between the component groups induced by isogenies of abelian varieties.

Let \( A_K \) be an abelian variety over \( K \) with semi-abelian reduction. Denote by \( \hat{A}_K \) the dual abelian variety of \( A_K \). As discussed in [20], there is a non-degenerate pairing \( u_A : M_A \times M_{\hat{A}} \to \mathbb{Z} \) (called monodromy pairing) having nice functorial properties, which induces an exact sequence

\[ 0 \to M_A \xrightarrow{u_A} \text{Hom}(M_A, \mathbb{Z}) \to \Phi_A \to 0. \tag{4.1} \]

Let \( H \subset A_K(K) \) be a finite subgroup of order coprime to the characteristic of \( k \). Since \( A(R) = A_K(K) \), \( H \) extends to a constant étale subgroup-scheme \( H \) of \( A \). The restriction to the special fiber gives a natural injection \( H_k \cong H \to A_k(k) \), cf. [2, Prop. 7.3/3]. Composing this injection with \( A_k \to \Phi_A \), we get the canonical homomorphism \( \phi : H \to \Phi_A \). Denote \( H_0 := \ker(\phi) \) and \( H_1 := \operatorname{im}(\phi) \), so that there is a tautological exact sequence

\[ 0 \to H_0 \to H \xrightarrow{\phi} H_1 \to 0. \]

Let \( B_K \) be the abelian variety obtained as the quotient of \( A_K \) by \( H \). Let \( \varphi_K : A_K \to B_K \) denote the isogeny whose kernel is \( H \). By the Néron mapping property, \( \varphi_K \) extends to a morphism \( \varphi : A \to B \) of the Néron models. On the special fibers we get a homomorphism \( \varphi_k : A_k \to B_k \), which induces an isogeny \( \varphi_k^0 : A_k^0 \to B_k^0 \) and a homomorphism \( \varphi_\Phi : \Phi_A \to \Phi_B \). The isogeny \( \varphi_k^0 \) restricts to an isogeny \( \varphi_1 : T_A \to T_B \), which corresponds to an injective homomorphisms of character groups \( \varphi^* : M_B \to M_A \) with finite cokernel.

**Theorem 4.3.** Assume \( A_K \) has toric reduction. There is an exact sequence

\[ 0 \to H_1 \to \Phi_A \xrightarrow{\varphi_\Phi} \Phi_B \to H_0 \to 0. \]

**Proof.** The kernel of \( \varphi_k \) is \( H_k \equiv H \). It is clear that \( \ker(\varphi_\Phi) = H_1 \). Let \( \hat{\varphi}_K : \hat{B}_K \to \hat{A}_K \) be the isogeny dual to \( \varphi_K \). Using (4.1), one obtains a commutative diagram with exact rows (cf. [34, p. 8]):

\[
\begin{array}{cccccc}
0 & \longrightarrow & M_A & \longrightarrow & \text{Hom}(M_A, \mathbb{Z}) & \longrightarrow & \Phi_A & \longrightarrow & 0 \\
& & \downarrow{\hat{\varphi}}^* & & \downarrow{\text{Hom}(\varphi^*, \mathbb{Z})} & & \downarrow{\varphi_\Phi} & \downarrow & \\
0 & \longrightarrow & M_B & \longrightarrow & \text{Hom}(M_B, \mathbb{Z}) & \longrightarrow & \Phi_B & \longrightarrow & 0.
\end{array}
\]

From this diagram we get the exact sequence

\[ 0 \to \ker(\varphi_\Phi) \to M_B/\hat{\varphi}^*(M_A) \to \text{Ext}^1_\mathbb{G}_m(M_A/\varphi^*(M_B), \mathbb{Z}) \to \text{coker}(\varphi_\Phi) \to 0. \]
Using the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, it is easy to show that
\[ \operatorname{Ext}^1_{\mathbb{Z}}(M_A/\varphi^*(M_B), \mathbb{Z}) \cong \operatorname{Hom}(M_A/\varphi^*(M_B), \mathbb{Q}/\mathbb{Z}) =: (M_A/\varphi^*(M_B))^\vee, \]
so there is an exact sequence of abelian groups
\[ 0 \to \ker(\varphi_\phi) \to M_B/\hat{\varphi}^*(M_A) \to (M_A/\varphi^*(M_B))^\vee \to \operatorname{coker}(\varphi_\phi) \to 0. \quad (4.2) \]

So far we have not used the assumption that $A_K$ has toric reduction. Under this assumption, $B_K$ also has toric reduction, and $H_0$ is the kernel of $\varphi_t : T_A \to T_B$. Hence $(M_A/\varphi^*(M_B))^\vee \cong H_0$. Next, [5, Thm. 8.6] implies that $M_B/\hat{\varphi}^*(M_A) \cong H_1$. Thus, we can rewrite (4.2) as
\[ 0 \to \ker(\varphi_\phi) \to H_1 \to H_0 \to \operatorname{coker}(\varphi_\phi) \to 0. \]

Since $\ker(\varphi_\phi) = H_1$, this implies that $\operatorname{coker}(\varphi_\phi) \cong H_0$. \qed

5. Component groups of $J_0(x,y)$

5.1. Component groups at $x$ and $y$

We return to the notation in Section 3. As we mentioned in Section 2.1, $X_0(x,y)$ is smooth over $A[1/xy]$.

**Proposition 5.1.**

(i) $X_0(x,y)_{\mathbb{F}_x}$ has a semi-stable model over $\mathcal{O}_x$ such that $X_0(x,y)_{\mathbb{F}_x}$ consists of two irreducible components both isomorphic to $X_0(x)_{\mathbb{F}_x} \cong \mathbb{P}^1_{\mathbb{F}_q}$ intersecting transversally in $q + 1$ points. Two of these singular points have thickness $q + 1$, and the other $q - 1$ points have thickness 1.

(ii) $X_0(x,y)_{\mathbb{F}_y}$ has a semi-stable model over $\mathcal{O}_y$ such that $X_0(x,y)_{\mathbb{F}_y}$ consists of two irreducible components both isomorphic to $X_0(x)_{\mathbb{F}_y} \cong \mathbb{P}^1_{\mathbb{F}_q^2}$ intersecting transversally in $q + 1$ points. All these singular points have thickness 1.

**Proof.** The fact that $X_0(x,y)_F$ has a model over $\mathcal{O}_x$ and $\mathcal{O}_y$ with special fibers of the stated form follows from the same argument as in the case of $X_0(v)_F$ over $\mathcal{O}_v$ ($v \in [F] - \infty$) discussed in [11, §5]. We only clarify why the number of singular points and their thickness are as stated.

(i) The special fiber $X_0(x,y)_{\mathbb{F}_x}$ consists of two copies of $X_0(x)_{\mathbb{F}_x}$. The set of points $Y_0(x)_{\mathbb{F}_x}$ is in bijection with the isomorphism classes of pairs $(\phi, C_y)$, where $\phi$ is a rank-2 Drinfeld $A$-module over $\overline{\mathbb{F}}_x$ and $C_y \cong A/y$ is a cyclic subgroup of $\phi$. The two copies of $X_0(x)_{\mathbb{F}_x}$ intersect exactly at the points corresponding to $(\phi, C_y)$ with $\phi$ supersingular; more precisely, $(\phi, C_y)$ on the first copy is identified with $(\chi^{(x)}, C_y^{(x)})$ on the second copy where $\phi^{(x)}$ is the image of $\phi$ under the Frobenius isogeny and $C_y^{(x)}$ is subgroup of $\phi^{(x)}$ which is the image of $C_y$, cf. [11].

Now, by Lemma 2.1, up to an isomorphism over $\overline{\mathbb{F}}_x$, there is a unique supersingular Drinfeld module $\phi$ in characteristic $x$ and $j(\phi) = 0$. It is easy to see that $\phi$ has $q_y + 1 = q^2 + 1$ cyclic subgroups isomorphic to $A/y$, so the set $S = \{ (\phi, C_y) \mid C_y \subset \phi(y) \}$ has cardinality $q^2 + 1$. By Lemma 2.2, $\operatorname{Aut}(\phi) \cong \mathbb{F}_q^x$. This group naturally acts $S$, and the orbits are in bijection with the singular points of $X_0(x,y)_{\mathbb{F}_x}$. Since the genus of $X_0(x,y)_F$ is $q$, the arithmetic genus of $X_0(x,y)_{\mathbb{F}_x}$ is also $q$ due to the flatness of $X_0(x,y) \to \operatorname{Spec}(A)$; see [21, Cor. III.9.10]. Using the fact that the genus of $X_0(x,y)_F$ is zero, a simple calculation shows that the number of singular points of $X_0(x,y)_{\mathbb{F}_x}$ is $q + 1$, cf. [21, p. 298]. Next, by Lemma 2.3, the stabilizer in $\operatorname{Aut}(\phi)$ of $(\phi, C_y)$ is either $\mathbb{F}_q^x$ or $\mathbb{F}_{q^2}^x$. Let $s$ be the number of
pairs \((\phi, C_y)\) with stabilizer \(\mathbb{P}^\times_q\). Let \(t\) be the number of orbits of pairs with stabilizers \(\mathbb{P}^\times_q\); each such orbit consists of \(#(\mathbb{P}^\times_q/\mathbb{P}^\times_q) = q + 1\) pairs \((\phi, C_y)\). Hence we have
\[(q + 1)t + s = q^2 + 1 \quad \text{and} \quad t + s = q + 1.\]

This implies that \(t = q - 1\) and \(s = 2\). Finally, as is explained in [11], the thickness of the singular point corresponding to an isomorphism class of \((\phi, C_y)\) is equal to \(#(\text{Aut}(\phi, C_y)/\mathbb{P}^\times_q)\).

(ii) Similar to the previous case, \(X_0(xy)_{\overline{\mathbb{F}}_q}\) consists of two copies of \(X_0(x)_{\overline{\mathbb{F}}_q} \cong \mathbb{P}^1_{\overline{\mathbb{F}}_q}\). The two copies of \(X_0(x)_{\overline{\mathbb{F}}_q}\) intersect exactly at the points corresponding to the isomorphism classes of pairs \((\phi, C_x)\) with \(\phi\) supersingular. Again by Lemma 2.1, up to an isomorphism over \(\overline{\mathbb{F}}_q\), there is a unique supersingular \(\phi\) and \(j(\phi) \neq 0\). Hence, by Lemma 2.3, \(\text{Aut}(\phi, C_x) \cong \mathbb{F}^\times_q\) for any \(C_x\). There are \(q_x + 1 = q + 1\) cyclic subgroups in \(\phi\) isomorphic to \(A/x\). The rest of the argument is the same as in the previous case. \(\Box\)

**Theorem 5.2.** Let \(\Phi_v\) denote the group of connected components of \(J_0(xy)\) at \(v \in |F|\). Let \(Z\) and \(Z'\) be the irreducible components in Proposition 5.1 with the convention that the reduction of \([\infty]\) lies on \(Z'\). Let \(z = Z - Z'\).

(i) \(\Phi_x \cong \mathbb{Z}/(q^2 + 1)(q + 1)\mathbb{Z}\).

(ii) \(\Phi_y \cong \mathbb{Z}/(q + 1)\mathbb{Z}\).

(iii) Under the canonical specialization map \(\phi_x : C \to \Phi_x\) we have
\[\phi_x(c_x) = 0 \quad \text{and} \quad \phi_x(c_y) = z.\]

In particular, \(q^2 + 1\) divides the order of \(c_y\).

(iv) Under the canonical specialization map \(\phi_y : C \to \Phi_y\) we have
\[\phi_y(c_x) = z \quad \text{and} \quad \phi_y(c_y) = 0.\]

In particular, \(q + 1\) divides the order of \(c_x\).

**Proof.** (i) and (ii) follow from Theorem 4.1 and Proposition 5.1.

(iii) The cusps reduce to distinct points in the smooth locus of \(X_0(xy)_{\overline{\mathbb{F}}_q}\), cf. [41]. Since by Theorem 4.1 we know that \(z\) has order \(q^2 + 1\) in the component group \(\Phi_x\), it is enough to show that the reductions of \([y]\) and \([\infty]\) lie on distinct components \(Z\) and \(Z'\) in \(X_0(xy)_{\overline{\mathbb{F}}_q}\), but the reductions of \([x]\) and \([\infty]\) lie on the same component. The involution \(W_x\) interchanges the two components \(X_0(y)_{\overline{\mathbb{F}}_q}\), cf. [11, (5.3)]. Since \(W_x([\infty]) = [y]\), the reductions of \([\infty]\) and \([y]\) lie on distinct components. On the other hand, \(W_y\) acts on \(X_0(xy)_{\overline{\mathbb{F}}_q}\) by acting on each component \(X_0(y)_{\overline{\mathbb{F}}_q}\) separately, without interchanging them. Since \(W_y([\infty]) = [x]\), the reductions of \([\infty]\) and \([x]\) lie on the same component.

(iv) The argument is similar to (iii). Here \(W_y\) interchanges the two components \(X_0(x)_{\overline{\mathbb{F}}_q}\) and \(W_x\) maps the components to themselves. Hence \([\infty]\) and \([y]\) lie on one component and \([0]\) and \([x]\) on the other component. \(\Box\)

**Theorem 5.3.** The cuspidal divisor group

\[\mathcal{C} \cong \mathbb{Z}/(q + 1)\mathbb{Z} \oplus \mathbb{Z}/(q^2 + 1)\mathbb{Z}\]

is the direct sum of the cyclic subgroups generated by \(c_x\) and \(c_y\), which have orders \((q + 1)\) and \((q^2 + 1)\), respectively. (Note that \(\mathcal{C}\) is cyclic if \(q\) is even, but it is not cyclic if \(q\) is odd.)
Proof. By Lemma 3.3 and Theorem 5.2, \( C \) is generated by \( c_x \) and \( c_y \), which have orders \( (q + 1) \) and \( (q^2 + 1) \), respectively. If the subgroup of \( C \) generated by \( c_x \) non-trivially intersects with the subgroup generated by \( c_y \), then, by Lemma 3.4, \( q \) must be odd and \( \frac{q^2 + 1}{2} c_y = \frac{q^2 + 1}{2} c_y \). Applying \( \phi_y \) to both sides of this equality, we get \( \frac{q^2 + 1}{2} z = 0 \), which is a contradiction since \( z \) generates \( \Phi_y \cong \mathbb{Z} / (q + 1) \mathbb{Z} \). \( \square \)

Remark 5.4. The divisor class \( c_0 \) has order \( (q + 1) (q^2 + 1) \) (resp. \( (q + 1) (q^2 + 1) / 2 \)) if \( q \) is even (resp. odd).

5.2. Component group at \( \infty \)

To obtain a model of \( X_0(\text{xy}) \) over \( \mathcal{O}_\infty \), instead of relying on the moduli interpretation of \( X_0(\text{xy}) \), one has to use the existence of analytic uniformization for this curve; see [28, §4.2]. As far as the structure of the special fiber \( X_0(\text{xy})_{\mathcal{F}_\infty} \) is concerned, it is more natural to compute the dual graph of \( X_0(\text{xy})_{\mathcal{F}_\infty} \) directly using the quotient \( \Gamma_0(\text{xy}) / \mathcal{T} \) of the Bruhat–Tits tree \( \mathcal{T} \) of \( \text{PGL}_2(\mathcal{F}_\infty) \). For the definition of \( \mathcal{T} \), and more generally for the basic theory of trees and groups acting on trees, we refer to [40].

The quotient graph \( \Gamma_0(\text{xy}) / \mathcal{T} \) was first computed by Gekeler [10, (5.2)]. For our purposes we will need to know the relative position of the cusps on \( \Gamma_0(\text{xy}) / \mathcal{T} \) and also the stabilizers of the edges. To obtain this more detailed information, and for the general sake of completeness, we recompute \( \Gamma_0(\text{xy}) / \mathcal{T} \) in this subsection using the method in [16].

Denote

\[
G_0 = \text{GL}_2(\mathbb{F}_q)
\]

and

\[
G_i = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \text{GL}_2(A) \mid \text{deg}(b) \leq i \right\}, \quad i \geq 1.
\]

As is explained in [16], \( \Gamma_0(\text{xy}) / \mathcal{T} \) can be constructed in “layers”, where the vertices of the \( i \)th layer (in [16] called type-\( i \) vertices) are the orbits

\[
X_i := G_i \backslash \mathbb{P}^1(A/\text{xy})
\]

and the edges connecting type-\( i \) vertices to type-(\( i + 1 \)) vertices, called type-\( i \) edges, are the orbits

\[
Y_i := (G_i \cap G_{i+1}) \backslash \mathbb{P}^1(A/\text{xy}).
\]

There are obvious maps \( Y_i \to X_i \), \( Y_i \to X_{i+1} \) and \( X_i \to X_{i+1} \) which are used to define the adjacencies of vertices in \( X_i \) and \( X_{i+1} \); see [16, 17]. The graph \( \Gamma_0(\text{xy}) / \mathcal{T} \) is isomorphic to the graph with set of vertices \( \bigsqcup_{i \geq 0} X_i \) and set of edges \( \bigsqcup_{i \geq 0} Y_i \) with the adjacencies defined by these maps.

Note that \( \mathbb{P}^1(A/\text{xy}) = \mathbb{P}^1(\mathbb{F}_x) \times \mathbb{P}^1(\mathbb{F}_y) \). We will represent the elements of \( \mathbb{P}^1(A/\text{xy}) \) as couples \([P; Q]\) where \( P \in \mathbb{P}^1(\mathbb{F}_x) \) and \( Q \in \mathbb{P}^1(\mathbb{F}_y) \). With this notation, \( G_i \) acts diagonally on \([P; Q]\) via its images in \( \text{GL}_2(\mathbb{F}_x) \) and \( \text{GL}_2(\mathbb{F}_y) \), respectively.

The group \( G_0 \) acting on \( \mathbb{P}^1(A/\text{xy}) \) has 3 orbits, whose representatives are

\[
\begin{bmatrix} (1) : (1) \\ (0) : (0) \end{bmatrix}, \quad \begin{bmatrix} (1) : (0) \\ (0) : (1) \end{bmatrix}, \quad \begin{bmatrix} (1) : (x) \\ (0) : (1) \end{bmatrix},
\]

where in the last element we write \( x \) for the image in \( \mathbb{F}_y \) of the monic generator of \( x \) under the canonical homomorphism \( A \to A/\text{y} \). The orbit of \([1; 0] \) has length \( q + 1 \), the orbit of \([1; 1] \) has length \( q(q + 1) \), and the orbit of \([1; 1] \) has length \( q(q^2 - 1) \), cf. [16, Prop. 2.10]. Next, note
that \( G_0 \cap G_1 \) is the subgroup \( B \) of the upper-triangular matrices in \( \text{GL}_2(\mathbb{F}_q) \). The \( G_0 \)-orbit of \([\begin{pmatrix} 1 \\ 0 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}]\) splits into two \( B \)-orbits with representatives:

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\] (5.1)

The lengths of these \( B \)-orbits are 1 and \( q \), respectively. The \( G_0 \)-orbit of \([\begin{pmatrix} 1 \\ 0 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}]\) splits into three \( B \)-orbits with representatives:

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\] (5.2)

The lengths of these \( B \)-orbits are \( q \), \( q \), \( q(q-1) \), respectively. Finally, the \( G_0 \)-orbit of \([\begin{pmatrix} 1 \\ 0 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}]\) splits into \( (q+1) \) \( B \)-orbits each of length \( q(q-1) \). The previous statements can be deduced from Proposition 2.11 in [16]. It turns out that the elements of \( \mathbb{P}^1(\mathbb{F}_x) \times \mathbb{P}^1(\mathbb{F}_y) \) listed in (5.1) and (5.2) combined form a complete set of \( G_1 \)-orbit representatives. For \( i \geq 1 \), the set of \( G_i \)-orbit representatives obviously contains a complete set of \( G_{i+1} \)-orbit representatives. A small calculation shows that

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\] (5.3)

is a complete set of \( G_i \)-orbit representatives for any \( i \geq 2 \). Moreover, the elements \([\begin{pmatrix} 1 \\ 0 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}]\) and \([\begin{pmatrix} 0 \\ 1 \end{pmatrix} : (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}]\) are in the same \( G_2 \)-orbit. We recognize the elements in (5.3) as the cusps \( \infty \), \([0]\), \([x]\), \([y]\), respectively. Overall, the structure of \( \Gamma_0(xy) \setminus T \) is described by the diagram in Fig. 2. In the diagram
the broken line --- indicates that there are \((q - 1)\) distinct edges joining the corresponding vertices, and an arrow \(\rightarrow\) indicates an infinite half-line.

Now we compute the stabilizers of the edges. Let \(e\) be an edge in \(\Gamma_0(xy) \setminus T\) of type \(i\). Let

\[ \mathcal{O}(e) = (G_i \cap G_{i+1})[P; Q] \]

be its corresponding orbit in \((G_i \cap G_{i+1}) \setminus \mathbb{P}^1(A/xy)\). Then for a preimage \(\tilde{e}\) of \(e\) in \(T\) we have

\[ \#\text{Stab}_{\Gamma_0(xy)}(\tilde{e}) = \#	ext{Stab}_{G_i \cap G_{i+1}}([P; Q]) = \frac{\#(G_i \cap G_{i+1})}{\#\mathcal{O}(e)}. \]

Using this observation, we conclude from our previous discussion that the edges connecting \([1, 0]: \bigl(1', 1\bigr)\) \(\in X_0\) to any vertex in \(X_1\) have preimages whose stabilizers have order \(#B/q(q - 1) = q - 1\).

The preimages of the edges connecting \([1, 0]: \bigl(1', 1\bigr)\) \(\in X_0\) to \([0, 1]: \bigl(0', 0\bigr)\) \(\in X_1\) and \([1, 0]: \bigl(0', 0\bigr)\) \(\in X_1\) have stabilizers of orders \(q - 1\) and \((q - 1)^2\), respectively. (Note that if a stabilizer has order \((q - 1)\) then it is equal to the center \(Z(\Gamma_0(xy)) \cong \mathbb{F}_q^x\) of \(\Gamma_0(xy)\), as the center is a subgroup of any stabilizer.)

The valency of a vertex \(v\) in \(\Gamma_0(xy)\) is \(3\). Consider the vertex \(v = \bigl(1', 1\bigr)\) \(\in X_1\). Its valency is \((q + 1)\). Let \(\tilde{v}\) be a preimage of \(v\) in \(T\). Since the valency of \(\tilde{v}\) is also \(q + 1\), \(\text{Stab}_{\Gamma_0(xy)}(\tilde{v})\) acts trivially on all edges having \(\tilde{v}\) as an endpoint. Hence the stabilizer of any such edge is equal to \(\text{Stab}_{\Gamma_0(xy)}(\tilde{v})\). We already determined that the stabilizer of a preimage of an edge connecting \(v\) to a type-0 vertex is \(\mathbb{F}^x_q\). This implies that the stabilizer in \(\Gamma_0(xy)\) of a preimage of the edge connecting \(v\) to \([1, 0]: \bigl(1', 1\bigr)\) \(\in X_2\) is also \(\mathbb{F}^x_q\). Finally, consider the vertex \(w = \bigl(0', 0\bigr)\) \(\in X_1\). Its valency is \(3\). Let \(S, S_1, S_2, S_3\) be the orders of stabilizers in \(\Gamma_0(xy)\) of a preimage \(\tilde{w}\) of \(w\) in \(T\), and the edges connecting \(w\) to \([1, 0]: \bigl(1', 1\bigr)\) \(\in X_0\), \([1, 0]: \bigl(0', 0\bigr)\) \(\in X_0\), \([1, 0]: \bigl(1', 1\bigr)\) \(\in X_2\), respectively. From our discussion of the lengths of orbits of type-0 edges, we have \(S_1 = (q - 1)^2\) and \(S_2 = (q - 1)\). Obviously, \(S_i’s\) divide \(S\). On the other hand, counting the lengths of orbits of \(\text{Stab}_{\Gamma_0(xy)}(\tilde{w})\) acting on the set of (non-oriented) edges in \(T\) having \(\tilde{w}\) as an endpoint, we get

\[ q + 1 = \frac{S}{S_1} + \frac{S}{S_2} + \frac{S}{S_3} = \frac{S}{(q - 1)^2} + \frac{S}{(q - 1)} + \frac{S}{S_3}. \]

This implies \(S = S_3 = (q - 1)^2\). To summarize, in Fig. 2 a wavy line \(\sim\) indicates that a preimage of the corresponding edge in \(T\) has a stabilizer in \(\Gamma_0(xy)\) of order \((q - 1)^2\). The edges connecting \([1, 0]: \bigl(1', 1\bigr)\) or \([1, 0]: \bigl(0', 0\bigr)\) to any other vertex have preimages in \(T\) whose stabilizers in \(\Gamma_0(xy)\) are isomorphic to \(\mathbb{F}^x_q\).

Now from [28, §4.2] one deduces the following. The quotient graph \(\Gamma_0(xy) \setminus T\), without the infinite half-lines, is the dual graph of the special fiber of a semi-stable model of \(X_0(xy)_{\mathbb{F}_q}\) over \(\text{Spec}(\mathbb{Q}_\infty)\). The special fiber \(X_0(xy)_{\mathbb{F}_q}\) has 6 irreducible components \(Z, Z', E, E', G, G'\), all isomorphic to \(\mathbb{P}^1_{\mathbb{F}_q}\), such that \(Z\) and \(Z'\) intersect in \(q - 1\) points, \(E\) intersects \(Z\) and \(E'\) intersects \(Z'\) and \(E\), \(G\) intersects \(Z\) and \(G'\) intersects \(Z'\) and \(G\). Moreover, all intersection points are ordinary double singularities. By [28, Prop. 4.3], the thickness of the singular point corresponding to an edge \(e \in \Gamma_0(xy) \setminus T\) is

\[ \#\left(\text{Stab}_{\Gamma_0(xy)}(\tilde{e})/\mathbb{F}^x_q\right), \]

hence all intersection points on \(Z\) or \(Z'\) have thickness 1, but the intersection points of \(E\) and \(E'\), and of \(G\) and \(G'\) have thickness \((q - 1)\), cf. Fig. 3. From the structure of \(\Gamma_0(xy) \setminus T\), one also concludes that the reductions of the cusps are smooth points in \(X_0(xy)_{\mathbb{F}_q}\). Moreover, \([\infty], [0], [x], [y]\) reduce to points on \(E, E', G, G'\) respectively.
Blowing up $X_0(xy)_{C_\infty}$ at the intersection points of $E$, $E'$, and $G$, $G'$, $(q-2)$-times each, we obtain the minimal regular model of $X_0(xy)_{\overline{F}}$ over $\text{Spec}(C_{\infty})$. This is a curve of the type discussed in Section 4.2 with $m = n = (q + 1)$, and we enumerate its irreducible components so that $E_1 = E$, $E_q = E'$, $G_1 = G$, $G_q = G'$.

**Theorem 5.5.** Let $\phi_\infty : C \rightarrow \Phi_\infty$ denote the canonical specialization map.

(i) $\Phi_\infty \cong \mathbb{Z}/(q^2 + 1)(q + 1)\mathbb{Z}$

(ii) $\phi_\infty(c_x) = (q^2 + 1)e_q$ and $\phi_\infty(c_y) = -q(q + 1)e_q = (q^3 + 1)e_q$.

(iii) If $q$ is even, then $\phi_\infty : C \xrightarrow{\sim} \Phi_\infty$ is an isomorphism.

(iv) If $q$ is odd, then there is an exact sequence

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow C \overset{\phi_\infty}{\rightarrow} \Phi_\infty \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$ 

**Proof.** Part (i) is an immediate consequence of the preceding discussion and Theorem 4.1. We have determined the reductions of the cusps at $\infty$, so using Theorem 4.1, we get

$$\phi_\infty(c_x) = g_1 - e_1 = (q^2 + q + 1)e_q - qe_q = (q^2 + 1)e_q$$

and

$$\phi_\infty(c_y) = g_q - e_1 = -q^2e_q - qe_q = -q(q + 1)e_q,$$

which proves (ii). Since $\gcd(q^2 + 1, q(q + 1)) = 1$ (resp. 2) if $q$ is even (resp. odd), cf. Lemma 3.4, the subgroup of $\Phi_\infty$ generated by $\phi_\infty(c_x)$ and $\phi_\infty(c_y)$ is $(e_q)$ (resp. $2e_q$) if $q$ is even (resp. odd). On the other hand, we know that $e_q$ generates $\Phi_\infty$. Therefore, if $q$ is even, then $\phi_\infty$ is surjective, and if $q$ is odd, then the cokernel of $\phi_\infty$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The claims (iii) and (iv) now follow from Theorem 5.3. □

**Remark 5.6.** We note that (iii) and a slightly weaker version of (iv) in Theorem 5.5 can be deduced from Theorem 5.3 and a result of Gekeler [14]. In fact, in [14, p. 366] it is proven that for an arbitrary $n$ the kernel of the canonical homomorphism from the cuspidal divisor group of $X_0(n)_{\overline{F}}$ to $\Phi_\infty$ is a quotient of $(\mathbb{Z}/(q-1)\mathbb{Z})^{c-1}$, where $c$ is the number of cusps of $X_0(n)_{\overline{F}}$. In our case, this result says that $\ker(\phi_\infty)$ is a quotient of $(\mathbb{Z}/(q-1)\mathbb{Z})^2$. Now suppose $q$ is even. Then $C \cong \mathbb{Z}/(q^2 + 1)(q + 1)\mathbb{Z}$. Since for even $q$, $\gcd(q - 1, (q^2 + 1)(q + 1)) = 1$, $\phi_\infty$ must be injective. But by (i), $\#\Phi_\infty = (q^2 + 1)(q + 1) = 2C$, so $\phi_\infty$ is also surjective. When $q = 2$, the fact that $\#\Phi_\infty = 15$ and $\phi_\infty$ is an isomorphism is already contained in [14, (5.3.1)].

Now suppose $q$ is odd. Then $C \cong \mathbb{Z}/(q^2 + 1)\mathbb{Z} \oplus \mathbb{Z}/(q + 1)\mathbb{Z}$. Since

$$\gcd(q - 1, q + 1) = \gcd(q - 1, q^2 + 1) = 2,$$

$\ker(\phi_\infty) \subset (\mathbb{Z}/2\mathbb{Z})^2$. Since $\Phi_\infty$ is cyclic but $C$ is not, $\ker(\phi_\infty)$ is not trivial, hence it is either $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$. (Theorem 5.5 implies that the second possibility does not occur.)

**Notation 5.7.** Let $C_0$ be the subgroup of $C$ generated by $c_y$. 

> Fig. 3. $X_0(xy)_{C_{\infty}}$ for $q = 3$. 

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Corollary 5.8. The cyclic group $C_0$ has order $q^2 + 1$. Under the canonical specializations $C_0$ maps injectively into $\Phi_x$ and $\Phi_\infty$, and $C_0$ is the kernel of $\phi_y$.

Proof. The claims easily follow from Theorems 5.2, 5.3 and 5.5. \qed

6. Component groups of $J^x y$

6.1. A class number formula

Let $H$ be a quaternion algebra over $F$. Let $\text{Ram} \subset |F|$ be the set of places where $H$ ramifies. Assume $\infty \in \text{Ram}$. Denote $\mathcal{R} = \text{Ram} - \infty$. Note that $\mathcal{R} \neq \emptyset$ since $\#\text{Ram}$ is even.

Let $\mathcal{O}$ be a hereditary $A$-order in $H$. Let $I_1, \ldots, I_h$ be the isomorphism classes of left $\mathcal{O}$-ideals. It is known that $\#(\mathcal{O}) := h$, called the class number of $\mathcal{O}$, is finite. For $i = 1, \ldots, h$ we denote by $\mathcal{O}_i$ the right order of the respective $I_i$. (For the definitions see [42].) Denote

$$M(\mathcal{O}) = \sum_{i=1}^{h} (\mathcal{O}_i^{\times} : \mathbb{F}_q^{\times})^{-1}.$$ 

It is not hard to show that each $\mathcal{O}_i^{\times}$ is isomorphic to either $\mathbb{F}_q^{\times}$ or $\mathbb{F}_q^{2\times}$; see [7, p. 383]. Let $U(\mathcal{O})$ be the number of right orders $\Theta_i$ such that $\Theta_i^{\times} \cong \mathbb{F}_q^{2\times}$. In particular,

$$h(\mathcal{O}) = M(\mathcal{O}) + U(\mathcal{O}) \left( 1 - \frac{1}{q+1} \right).$$

Definition 6.1. For a subset $S$ of $|F|$, let

$$\text{Odd}(S) = \begin{cases} 1, & \text{if all places in } S \text{ have odd degrees;} \\ 0, & \text{otherwise.} \end{cases}$$

Let $S \subset |F| - \infty$ be a finite (possibly empty) set of places such that $\mathcal{R} \cap S = \emptyset$. Let $\mathfrak{n} \triangleleft A$ be the square-free ideal whose support is $S$. Let $\mathcal{O}$ be an Eichler $A$-order of level $\mathfrak{n}$. (When $S = \emptyset$, $\mathcal{O}$ is a maximal $A$-order in $H$.) The formulae that follow are special cases of (1), (4) and (6) in [7]:

$$M^S(H) := M(\mathcal{O}) = \frac{1}{q^2 - 1} \prod_{v \in \mathcal{R}} (q_v - 1) \prod_{w \in S} (q_w + 1).$$

$$U^S(H) := U(\mathcal{O}) = 2^{\#R + \#S - 1} \text{Odd}(\mathcal{R}) \prod_{w \in S} (1 - \text{Odd}(w)).$$

Denote

$$h^S(H) = M^S(H) + U^S(H) \cdot \frac{q}{q+1}.$$ 

6.2. Component groups at $x$ and $y$

Let $D$ and $R$ be as in Section 2.2. Recall that we assume $\infty \notin R$. Fix a place $w \in R$. Let $D^w$ be the quaternion algebra over $F$ which is ramified at $(R - w) \cup \infty$. Fix a maximal $A$-order $\mathcal{O}$ in $D^w$, and denote
\[ A^w = A[w^{-1}]; \]
\[ \mathcal{D}^w = \mathcal{D} \otimes_A A^w; \]
\[ \Gamma^w = \{ \gamma \in (\mathcal{D}^w)^\times \mid \text{ord}_w(\text{Nr}(\gamma)) \in 2\mathbb{Z} \}; \]

Here \( w^{-1} \) denotes the inverse of a generator of the ideal in \( A \) corresponding to \( w \), and \( \text{Nr} \) denotes the reduced norm on \( D^w \).

By fixing an isomorphism \( D^w \otimes_F F_w \cong M_2(F_w) \), one can consider \( \Gamma^w \) as a subgroup of \( \text{GL}_2(F_w) \) whose image in \( \text{PGL}_2(F_w) \) is discrete and cocompact. Hence \( \Gamma^w \) acts on the Bruhat–Tits tree \( T^w \) of \( \text{PGL}_2(F_w) \). It is not hard to show that \( \Gamma^w \) acts without inversions, so the quotient graph \( \Gamma^w \backslash T^w \) is a finite graph without loops. We make \( \Gamma^w \backslash T^w \) into a graph with lengths by assigning to each edge \( e \) of \( \Gamma^w \backslash T^w \) the length \( \#(\text{Stab}_e) / \mathbb{F}_q \), where \( \hat{e} \) is a preimage of \( e \) in \( T^w \). The graph with lengths \( \Gamma^w \backslash T^w \) does not depend on the choice of isomorphism \( D^w \otimes_F F_w \cong M_2(F_w) \), since such isomorphisms differ by conjugation.

As follows from the analogue of Cherednik–Drinfeld uniformization for \( X^R_{F_w} \), proven in this context by Hausberger [22], \( X^R_{F_w} \) is a twisted Mumford curve: Denote by \( \mathcal{O}^{(2)}_w \) the quadratic unramified extension of \( \mathcal{O}_w \) and denote by \( \mathbb{F}^{(2)}_w \) the residue field of \( \mathcal{O}^{(2)}_w \). Then \( X^R_{F_w} \) has a semi-stable model \( X^R_{\mathcal{O}^{(2)}_w} \) over \( \mathcal{O}^{(2)}_w \) such that the irreducible components of \( X^R_{\mathcal{O}^{(2)}_w} \) are projective lines without self-intersections, and the dual graph \( G(X^R_{\mathcal{O}^{(2)}_w}) \), as a graph with lengths, is isomorphic to \( \Gamma^w \backslash T^w \).

On the other hand, as is done in [25] for the quaternion algebras over \( \mathbb{Q} \), the structure of \( \Gamma^w \backslash T^w \) can be related to the arithmetic to \( D^w \): The number of vertices of \( \Gamma^w \backslash T^w \) is \( 2h^0(D^w) \), the number of edges is \( h^1(D^w) \), each edge has length 1 or \( q + 1 \), and the number of edges of length \( q + 1 \) is \( U^w(D^w) \) (the notation here is as in Section 6.1). Hence, using the formulae in Section 6.1, we get the following:

**Proposition 6.2.** \( X^R_{F_w} \) has a semi-stable model \( X^R_{\mathcal{O}^{(2)}_w} \) over \( \mathcal{O}^{(2)}_w \) such that \( X^R_{\mathcal{O}^{(2)}_w} \) is a union of projective lines without self-intersections. The number of vertices of the dual graph \( G(X^R_{\mathcal{O}^{(2)}_w}) \) is

\[ \frac{2}{q^2 - 1} \prod_{v \in R - w} (q_v - 1) + 2\#R - 1 \text{Odd}(R - w) \frac{q}{q + 1}; \]

the number of edges is

\[ \frac{(q_w + 1)}{q^2 - 1} \prod_{v \in R - w} (q_v - 1) + 2\#R - 1 \text{Odd}(R - w)(1 - \text{Odd}(w)) \frac{q}{q + 1}. \]

The edges of \( G(X^R_{\mathcal{O}^{(2)}_w}) \) have length 1 or \( q + 1 \). The number of edges of length \( q + 1 \) is

\[ 2\#R - 1 \text{Odd}(R - w)(1 - \text{Odd}(w)). \]

This proposition has an interesting corollary:

**Corollary 6.3.** Let \( g(R) \) be the genus of \( X^R_{F_w} \). Then

\[ g(R) = 1 + \frac{1}{q^2 - 1} \prod_{v \in R} (q_v - 1) - \frac{q}{q + 1} 2\#R - 1 \text{Odd}(R). \]
Proof. Let $h_1$ be the dimension of the first simplicial homology group of $G(X_{\mathcal{O}_w}^{R(2)})$ with $\mathbb{Q}$-coefficients. Let $V$, $E$ be the number of vertices and edges of this graph, respectively. By Euler’s formula, $h_1 = E - V + 1$. Proposition 6.2 gives formulae for $V$ and $E$ from which it is easy to see that $h_1$ is given by the above expression. Since the irreducible components of $X_{\mathcal{O}_w}^{R(2)}$ are projective lines, it is not hard to show that $h_1$ is the arithmetic genus of $X_{\mathcal{O}_w}^{R(2)}$; cf. [21, p. 298]. On the other hand, $X_{\mathcal{O}_w}^{R(2)}$ is flat over $\mathcal{O}_w$, so the genus $g(R)$ of its generic fiber is equal to the arithmetic genus of the special fiber; see [21, p. 263]. (Note that the special role of $w$ in the formula for $V$ and $E$ disappears in $g(R)$, as expected. This formula for $g(R)$ was obtained in [30] by a different argument.)

Theorem 6.4. Let $\Phi_\nu'$ denote the group of connected components of $J_{XY}$ at $\nu \in |F|$.

(i) $\Phi_\nu' \cong \mathbb{Z}/(q + 1)\mathbb{Z}$;
(ii) $\Phi_\nu' \cong \mathbb{Z}/(q^2 + 1)(q + 1)\mathbb{Z}$.

Proof. In general, the information supplied by Proposition 6.2 is not sufficient for determining the graph $G(X_{\mathcal{O}_w}^{R(2)})$ uniquely. Nevertheless, in the case when $R = \{x, y\}$ Proposition 6.2 does uniquely determine $G(X_{\mathcal{O}_w}^{R(2)}); G(X_{\mathcal{O}_w}^{xy})$ is a graph without loops, which has 2 vertices, $q + 1$ edges, and all edges have length 1. Similarly, $G(X_{\mathcal{O}_w}^{xy})$ is a graph without loops, which has 2 vertices, $q + 1$ edges, two of the edges have length $q + 1$ and all others have length 1. Hence, in both cases, the dual graph is the graph with two vertices and $q + 1$ edges connecting them, cf. Fig. 4.

Now Theorem 4.1 can be used to conclude that the component groups are as stated. □

6.3. Component group at $\infty$

Here we again rely on the existence of analytic uniformization. Let $\Lambda$ be a maximal $A$-order in $D$. Let

$$\Gamma^\infty := \Lambda^\times.$$ 

Since $D$ splits at $\infty$, by fixing an isomorphism $D \otimes F_\infty \cong \mathbb{M}_2(F_\infty)$, we get an embedding $\Gamma^\infty \hookrightarrow \text{GL}_2(F_\infty)$. The group $\Gamma^\infty$ is a discrete, cocompact subgroup of $\text{GL}_2(F_\infty)$, well defined up to conjugation. Let $T^\infty$ be the Bruhat–Tits tree of $\text{PGL}_2(F_\infty)$. The group $\Gamma^\infty$ acts on $T^\infty$ without inversions, so the quotient $\Gamma^\infty \backslash T^\infty$ is a finite graph without loops which we make into a graph with lengths by assigning to an edge $e$ of $\Gamma^\infty \backslash T^\infty$ the length $\#(\text{Stab}_{\Gamma^\infty}(\tilde{e})/\mathbb{P}_q^\times)$, where $\tilde{e}$ is a preimage of $e$ in $T^\infty$. By a theorem of Blum and Stuhler [1, Thm. 4.4.11],

$$(X_{F_\infty}^R)^{an} \cong \Gamma^\infty \backslash \Omega.$$ 

From this one deduces that $X_{F_\infty}^R$ has a semi-stable model $X_{\mathcal{O}_\infty}^R$ over $\mathcal{O}_\infty$ such that the dual graph of $X_{\mathcal{O}_\infty}^R$, as a graph with lengths, is isomorphic to $\Gamma^\infty \backslash T^\infty$, cf. [25]. The structure of $\Gamma^\infty \backslash T^\infty$ can be related to the arithmetic of $D$; see [32].
Proposition 6.5. \( X^R_f \) has a semi-stable model \( X^R_{O_\infty} \) over \( O_\infty \) such that the special fiber \( X^R_{O_\infty} \) is a union of projective lines without self-intersections. The number of vertices of the dual graph \( G(X^R_{O_\infty}) \) is

\[
\frac{2}{q-1}(g(R) - 1) + \frac{q}{q-1}2^{#R-1} \text{Odd}(R);
\]

the number of edges is

\[
\frac{q + 1}{q-1}(g(R) - 1) + \frac{q}{q-1}2^{#R-1} \text{Odd}(R).
\]

All edges have length 1.

**Proof.** See Proposition 5.2 and Theorem 5.5 in [32]. \( \square \)

Theorem 6.6. \( \Phi_\infty^* \cong \mathbb{Z}/(q+1)\mathbb{Z} \).

**Proof.** Applying Proposition 6.5 in the case \( R = \{x, y\} \), one easily concludes that \( X^\nu_f \) has a semi-stable model over \( O_\infty \) whose dual graph looks like Fig. 4: it has 2 vertices, \( q + 1 \) edges, and all edges have length 1. The structure of \( \Phi_\infty^* \) now follows from Theorem 4.1. \( \square \)

7. Jacquet–Langlands isogeny

Let \( D \) and \( R \) be as in Section 2.2. Let \( X := X^R_f \), \( X' := X_0(R)_F \), \( J := F^R \), \( J' := J_0(R) \). Fix a separable closure \( F^\text{sep} \) of \( F \) and let \( G_F := \text{Gal}(F^\text{sep}/F) \). Let \( p \) be the characteristic of \( F \) and fix a prime \( \ell \neq p \). Denote by \( V_\ell(J) \) the Tate vector space of \( J \); this is a \( \mathbb{Q}_\ell \)-vector space of dimension \( 2g(R) \) naturally equipped with a continuous action of \( G_F \). Let \( V_\ell(J)^* \) be the linear dual of \( V_\ell(J) \).

Theorem 7.1. There is a surjective homomorphism \( J' \to J \) defined over \( F \).

**Proof.** Let \( \mathbb{A} = \prod_{v \in |F|} F_v \) denote the Adele ring of \( F \) and let \( \mathbb{A}^\infty = \prod_{v \in |F| - \infty} F_v \), so \( \mathbb{A} = \mathbb{A}^\infty \times F_\infty \).

Fix a uniformizer \( \pi_\infty \) at \( \infty \). Let \( A(D^\times(F) \setminus D^\times(\mathbb{A})/\pi_\infty^\infty) \) be the space of \( \mathbb{Q}_\ell \)-valued locally constant functions on \( D^\times(\mathbb{A})/\pi_\infty^\infty \) which are invariant under the action of \( D^\times(F) \) on the left. This space is equipped with the right regular representation of \( D^\times(\mathbb{A})/\pi_\infty^\infty \). Since \( D \) is a division algebra, the coset space \( D^\times(F) \setminus D^\times(\mathbb{A})/\pi_\infty^\infty \) is compact and decomposes as a sum of irreducible admissible representations \( \Pi \) with finite multiplicities \( m(\Pi) > 0 \), cf. [26, §13]:

\[
\mathcal{A}_D := A(D^\times(F) \setminus D^\times(\mathbb{A})/\pi_\infty^\infty) = \bigoplus_{\Pi} m(\Pi) \cdot \Pi.
\]

Moreover, as follows from the Jacquet–Langlands correspondence and the multiplicity-one theorem for automorphic cuspidal representations of \( \text{GL}_2(\mathbb{A}) \), the multiplicities \( m(\Pi) \) are all equal to 1; see [18, Thm. 10.10]. The representations appearing in the sum (7.1) are called automorphic. Each automorphic representation \( \Pi \) decomposes as a restricted tensor product \( \Pi = \bigotimes_{v \in |F|} \Pi_v \) of admissible irreducible representations of \( D^\times(F_v) \). We denote \( \Pi^\infty = \bigotimes_{v \neq \infty} \Pi_v \), so \( \Pi = \Pi^\infty \times \Pi_\infty \). If \( \Pi \) is finite dimensional, then it is of the form \( \Pi = \chi \circ \text{Nr} \), where \( \chi \) is a Hecke character of \( \mathbb{A}^\times \) and \( \text{Nr} \) is the reduced norm on \( D^\times \), cf. [26, Lem. 14.8]. If \( \Pi \) is infinite dimensional, then \( \Pi_v \) is infinite dimensional for every \( v \notin R \).

Let \( \psi_v \) be a character of \( F^\nu_v \). Denote by \( \text{Sp}_v \otimes \psi_v \) the unique irreducible quotient of the induced representation

\[
\text{Ind}^{\text{GL}_2}_B \left( \chi \cdot \frac{1}{|v|} \psi_v \oplus \frac{1}{|v|} \psi_v \right).
\]
where $B$ is the subgroup of upper-triangular matrices in $GL_2$. The representation $Sp_v \otimes \psi_v$ is called the special representation of $GL_2(F_v)$ twisted by $\psi_v$. If $\psi_v = 1$, then we simply write $Sp_v$.

For $v \in R$, let $D_v$ be the maximal order in $D(F_v)$. Let

$$K := \prod_{v \in R} D_v^\times \times \prod_{v | F | - R - \infty} GL_2(O_v) \subset D^\times(A^\infty).$$

Taking the $K$-invariants in Theorems 14.9 and 14.12 in [26], we get an isomorphism of $G_F$-modules

$$V_f(f)^* \otimes_{Q,\ell} \overline{Q}_\ell = H^1_{\text{ét}}(X \otimes F^{\text{sep}}, \overline{Q}_\ell) = \bigoplus_{\Pi \in A_0 \atop \Pi_\infty \cong \text{Sp}_\infty} (\Pi_\infty)^K \otimes \sigma(\Pi),$$

(7.2)

where $\sigma(\Pi)$ is a 2-dimensional irreducible representation of $G_F$ over $\overline{Q}_\ell$ with the following property: If $(\Pi_\infty)^K \neq 0$, then for all $v \in |F| - R - \infty$, $\sigma(\Pi)$ is unramified at $v$ and there is an equality of $L$-functions

$$L \left( s - \frac{1}{2}, \Pi_v \right) = L(s, \sigma(\Pi)_v);$$

here $\sigma(\Pi)_v$ denotes the restriction of $\sigma(\Pi)$ to a decomposition group at $v$. This uniquely determines $\sigma(\Pi)$ by the Chebotarev density theorem [39, Ch. I, pp. 8–11]. Next, we claim that the dimension of $(\Pi_\infty)^K$ is at most one. Indeed, if $v \in |F| - R - \infty$, then $\Pi_\infty^D(\psi_v)$ is at most one-dimensional by [3, Thm. 4.6.2]. On the other hand, note that $D_v^\times$ is normal in $D^\times(F_v)$ and $D^\times(F_v)/D_v^\times \cong \mathbb{Z}$ for $v \in R$. Hence $\Pi_v^D \neq 0$ implies $\Pi_v = \psi_v \circ \text{Nr}$ for some unramified character of $F_v^\times$ ($\psi_v$ is unramified because the reduced norm maps $D_v^\times$ surjectively onto $O_v^\times$).

Let $I_v$ be the Iwahori subgroup of $GL_2(O_v)$, i.e., the subgroup of matrices which maps to $B(F_v)$ under the reduction map $GL_2(O_v) \to GL_2(F_v)$. Let

$$I = \prod_{v \in R} I_v \times \prod_{v | F | - R - \infty} GL_2(O_v) \subset GL_2(A^\infty).$$

Let $A_0 := A_0(GL_2(F) \setminus GL_2(A))$ be the space of $\overline{Q}_\ell$-valued cusp forms on $GL_2(A)$; see [17, §4] or [3, §3.3] for the definition. Taking the $I$-invariants in Theorem 2 of [8], we get an isomorphism of $G_F$-modules

$$V_f(f)^* \otimes_{Q,\ell} \overline{Q}_\ell = H^1_{\text{ét}}(X' \otimes F^{\text{sep}}, \overline{Q}_\ell) = \bigoplus_{\Pi \in A_0 \atop \Pi_\infty \cong \text{Sp}_\infty} (\Pi_\infty)^I \otimes \rho(\Pi),$$

(7.3)

where $\rho(\Pi)$ is 2-dimensional irreducible representation of $G_F$ over $\overline{Q}_\ell$ with the following property: If $(\Pi_\infty)^I \neq 0$, then for all $v \in |F| - R - \infty$, $\rho(\Pi)$ is unramified at $v$ and

$$L \left( s - \frac{1}{2}, \Pi_v \right) = L(s, \rho(\Pi)_v).$$

In this case, $(\Pi_\infty)^I$ is finite dimensional, but its dimension might be larger than one (due to the existence of old forms).

The global Jacquet–Langlands correspondence [24, Ch. III] associates to each infinite dimensional automorphic representation $\Pi$ of $D^\times(A)$ a cuspidal representation $\Pi' = JL(\Pi)$ of $GL_2(A)$ with the following properties:
(1) if \( v \not\in R \) then \( \Pi_v \cong \Pi'_v \);  
(2) if \( v \in R \) and \( \Pi_v \cong \psi_v \circ \text{Nr} \) for a character \( \psi \) of \( F_v^\times \), then

\[
\Pi'_v \cong \text{Sp}_v \otimes \psi_v.
\]

As we observed above, for \( \Pi \in \mathcal{A}_D \) such that \( (\Pi^\infty)^K \neq 0 \), the characters \( \psi_v \) at the places in \( R \) are unramified. Thus, for \( v \in R \), \( \Pi'_v \) is a twist of \( \text{Sp}_v \) by an unramified character. On the other hand, the representations of the form \( \text{Sp}_v \otimes \psi_v \), with \( \psi_v \) unramified, can be characterized by the property that they have a unique 1-dimensional \( \mathcal{I}_v \)-fixed subspace; see [4]. Hence if \( (\Pi^\infty)^K \neq 0 \), then \((\Pi')^\infty)^2 \neq 0\).

Now using (7.2) and (7.3), one concludes that \( V_\ell(J) \) is isomorphic with a quotient of \( V_\ell(J') \) as a \( G_F \)-module. On the other hand, by a theorem of Zarhin (for \( p > 2 \)) and Mori (for \( p = 2 \))

\[
\text{Hom}_F(J', J) \otimes \mathbb{Q}_\ell \cong \text{Hom}_{G_F}(V_\ell(J'), V_\ell(J)). \quad (7.4)
\]

Thus, there is a surjective homomorphism \( J' \to J \) defined over \( F \). \( \square \)

**Corollary 7.2.** \( J_0(xy) \) and \( J^{xy} \) are isogenous over \( F \).

**Proof.** Since \( \dim(J^{xy}) = q = \dim(J_0(xy)) \), the claim follows from Theorem 7.1. \( \square \)

**Conjecture 7.3.** There exists an isogeny \( J_0(xy) \to J^{xy} \) whose kernel is \( C_0 \).

As an initial evidence for the conjecture, note that \( J_0(xy)/C_0 \) has component groups at \( x, y, \infty \) of the same order as those of \( J^{xy} \). This follows from Theorem 4.3, Corollary 5.8, and Table 1 in the introduction. We will show below that Conjecture 7.3 is true for \( q = 2 \).

**Remark 7.4.** The statement of Theorem 7.1 can be refined. The abelian variety \( J \) has toric reduction at every \( v \in R \), so it is isogenous to an abelian subvariety of \( J' \) having the same reduction property. The new subvariety of \( J' \), \( J'^{\text{new}} \), defined as in the case of classical modular Jacobians (cf. [35], [13, p. 248]), is the abelian subvariety of \( J' \) of maximal dimension having toric reduction at every \( v \in R \). Hence \( J \) is isogenous to a subvariety of \( J'^{\text{new}} \). By computing the dimension of \( J'^{\text{new}} \), one concludes that \( J \) and \( J'^{\text{new}} \) are isogenous over \( F \).

**Remark 7.5.** There is just one other case, beside the case which is the focus of this paper, when \( J \) and \( J' \) are actually isogenous. As one easily shows by comparing the genera of modular curves \( X_K \) and \( X_0(R) \), the genera of these curves are equal if and only if \( R = \{ x, y \} \) and \( \{ \deg(x), \deg(y) \} = \{ 1, 1 \}, \{ 1, 2 \}, \{ 2, 2 \} \). Assume \( \deg(x) = \deg(y) = 2 \). Then the genus of both \( X^{xy} \) and \( X_0(xy) \) is \( q^2 \), but neither of these curves is hyperelliptic. The curve \( X_0(xy) \) again has 4 cusps which can be represented as in Section 3. Calculations similar to those we have carried out in earlier sections lead to the following result:

(1) The cuspidal divisor group \( C \) is generated by \( c_0 \) and \( c_x \). The order of \( c_0 \) is \( q^2 + 1 \). The order of \( c_x \) is divisible by \( q^2 + 1 \) and divides \( q^4 - 1 \). The order of \( c_y \) is divisible by \( q^2 + 1 \) and divides \( q^4 - 1 \).

(2) \( \Phi_x \cong \Phi'_x \cong \mathbb{Z}/(q^2 + 1)\mathbb{Z} \).

(3) \( \Phi_y \cong \Phi'_y \cong \mathbb{Z}/(q^2 + 1)\mathbb{Z} \).

(4) The canonical map \( \phi_x : C \to \Phi_x \) is surjective, and

\[
\phi_x(c_0) = z, \quad \phi_x(c_x) = 0, \quad \phi_x(c_y) = z.
\]

(5) The canonical map \( \phi_y : C \to \Phi_y \) is surjective, and

\[
\phi_y(c_0) = z, \quad \phi_y(c_x) = z, \quad \phi_y(c_y) = 0.
\]
The fact that $X_0(xy)$ is not hyperelliptic complicates the calculation of $C$: just the relations between the cuspidal divisors arising from the Drinfeld discriminant function are not sufficient for pinning down the orders of $c_x$ and $c_y$, cf. (3.3). Next, the calculations required for determining $\Phi_\infty$, $\Phi'_\infty$, and $\phi_\infty$ appear to be much more complicated than those in Sections 5.2 and 6.3. Nevertheless, based on the facts that we are able to prove, and in analogy with the case $\deg(x) = 1$, $\deg(y) = 2$, we make the following prediction: The cuspidal divisor group $C \cong (\mathbb{Z}/(q^2 + 1)\mathbb{Z})^2$ is the direct sum of the cyclic subgroups generated by $c_x$ and $c_y$ both of which have order $q^2 + 1$, and there is an isogeny $J_0(xy) \to J^R$ whose kernel is $C$.

**Definition 7.6.** It is known that every elliptic curve $E$ over $F$ with conductor $n_E = n \cdot \infty$, $n \ll A$, and split multiplicative reduction at $\infty$ is isogenous to a subvariety of $J_0(n)$; see [17]. This follows from (7.3), (7.4), and the fact [6, p. 577] that the representation $\rho_E : G_F \to \text{Aut}(V_{\ell}(E)^\vee)$ is automorphic (i.e., $\rho_E = \rho(\Pi)$ for some $\Pi \in \mathcal{A}_0$). The multiplicity-one theorem can be used to show that in the $F$-isogeny class of $E$ there exists a unique curve $E'$ which is isomorphic to a one-dimensional abelian subvariety of $J_0(n)$, thus maps “optimally” into $J_0(n)$. We call $E'$ the $J_0(n)$-optimal curve. Theorem 7.1 and Remark 7.4 imply that $E$ with square-free conductor $R \cdot \infty$ and split multiplicative reduction at $\infty$ is also isogenous to a subvariety of $J^R$. Moreover, in the $F$-isogeny class of $E$ there is a unique elliptic curve $E''$ which is isomorphic to a one-dimensional abelian subvariety of $J^R$. We call $E''$ the $J^R$-optimal curve.

**Notation 7.7.** Let $E$ be an elliptic curve over $F$ given by a Weierstrass equation

$$E : Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6.$$  

Let $E^{(p)}$ be the elliptic curve given by the equation

$$E^{(p)} : Y^2 + a_1^pXY + a_3^pY = X^3 + a_2^pX^2 + a_4^pX + a_6^p.$$  

There is a Frobenius morphism $\text{Frob}_p : E \to E^{(p)}$ which maps a point $(x_0, y_0)$ on $E$ to the point $(x_0^p, y_0^p)$ on $E^{(p)}$. It is clear that the $j$-invariants of these elliptic curves are related by the equation $j(E^{(p)}) = j(E)^p$. If $E$ has semi-stable reduction at $v \in |F|$, then $\Phi_{E,v} \cong \mathbb{Z}/n\mathbb{Z}$, where $\Phi_{E,v}$ denotes the component group of $E$ at $v$ and $n = -\text{ord}_v(j(E)) > 1$. In this case, $\Phi_{E^{(p)},v} \cong \mathbb{Z}/pn\mathbb{Z}$.

**Definition 7.8.** An elliptic curve $E$ over $F$ with $j$-invariant $j(E) \notin \mathbb{F}_q$ is said to be Frobenius minimal if it is not isomorphic to $E^{(p)}$ for some other elliptic curve $\bar{E}$ over $F$. It is easy to check that this is equivalent to $j(E) \notin F^p$, cf. [38].

For $q$ even, Schweizer has completely classified the elliptic curves over $F$ having conductor of degree 4 in terms of explicit Weierstrass equations; see [37]. We are particularly interested in those curves which have conductor $xy\infty$ and split multiplicative reduction at $\infty$.

**Theorem 7.9.** Assume $q = 2^s$. Elliptic curves over $F$ with conductor $xy\infty$ exist only if there exists an $\mathbb{F}_q$-automorphism of $F$ that transforms the conductor into $(T + 1)(T^2 + T + 1)\infty$. In particular, $s$ must be odd.

If $s$ is odd, then there exists two isogeny classes of elliptic curves over $F$ with conductor $(T + 1)(T^2 + T + 1)\infty$ and split multiplicative reduction at $\infty$. The Frobenius minimal curves in each isogeny class are listed in Tables 2 and 3; the last three columns in the tables give the orders of the component groups $\Phi_{E,v}$ of the corresponding curve $E$ at $v = x$, $y$, $\infty$.

**Proof.** Theorem 4.1 in [37].
Next, [37, Prop. 3.5] describes explicitly the isogenies between the curves in classes I and II: There is an isomorphism of étale group-schemes over $F$

$$E_1[3] \cong H_1 \oplus H_2,$$

where $H_1 \cong \mathbb{Z}/3\mathbb{Z}$ and $H_2 \cong \mu_3$. The subgroup-scheme $H_1$ is generated by $(T + 1, 1)$ and $H_2$ is generated by $(T^2, sT^3 + s^2)$, where $s$ is a third root of unity. Then $E_1/H_1 \cong E_1'$ and $E_1/H_2 \cong E_1''$. (It is well known that an elliptic curve over $F$ with conductor of degree 4 has rank 0, so in fact $E_1(F) = H_1 \cong \mathbb{Z}/3\mathbb{Z}$.) Similarly, the subgroup-scheme $H_3$ of $E_2$ generated by $(1, 1)$ is isomorphic to $\mathbb{Z}/5\mathbb{Z}$, $E_2/H_3 \cong E_2'$, and $E_2(F) = H_3 \cong \mathbb{Z}/5\mathbb{Z}$.

**Proposition 7.10.** Assume $q = 2^s$ and $s$ is odd.

(i) $E_1$ and $E_2$ are the $J_{0(xy)}$-optimal curves in the isogeny classes I and II.

(ii) $E_2'$ is the $J^{XY}$-optimal curve in the isogeny class II.

(iii) If Conjecture 7.3 is true, then $E_1$ is the $J^{XY}$-optimal curve in the isogeny class I.

**Proof.** (i) There is a method due to Gekeler and Reversat [12, Cor. 3.19] which can be used to determine $\Phi_{E, \infty}$ of the $J_0(n)$-optimal curve in a given isogeny class. This method is based on the study of the action of Hecke algebra on $H_1(\Gamma_0(n) \setminus T, \mathbb{Z})$. For $\deg(n) = 3$ the Gekeler–Reversat method can be further refined [38, Cor. 1.2]. Applying this method for $n = xy$, one obtains $\Phi_{E, \infty} = 3$ (resp. $\Phi_{E, \infty} = 5$) for the $J_0(xy)$-optimal elliptic curve $E$ in the isogeny class I (resp. II). Since there is a unique curve with this property in each isogeny class, we conclude that $E_1$ and $E_2$ are the $J_0(xy)$-optimal elliptic curves. (For $q = 2$, this is already contained in [12, Ex. 4.4].)

(ii) Assume $q$ is arbitrary. Let $E$ be an elliptic curve over $F$ which embeds into $J^{XY}$. Since $J^{XY}$ has split toric reduction at $\infty$, [29, Cor. 2.4] implies that the kernel of the natural homomorphism

$$\Phi_{E, \infty} \rightarrow \Phi'_{E, \infty} \cong \mathbb{Z}/(q + 1)\mathbb{Z},$$

is a subgroup of $\mathbb{Z}/(q, \infty - 1)\mathbb{Z}$. Hence $\Phi_{E, \infty}$ divides $(q^2 - 1)$. First, this implies that $\Phi_{E, \infty}$ is coprime to $p$, so $E$ must be Frobenius minimal in its isogeny class. Second, if $q = 2^s$ and $s$ is odd, then 5 does not divide $(q^2 - 1)$, so $E_2'$ is not $J^{XY}$-optimal. This leaves $E_2'$ as the only possible $J^{XY}$-optimal curve in the isogeny class II.

(iii) Let $E$ be the $J^{XY}$-optimal curve in the isogeny class I. By the discussion in (ii), this curve is one of the curves in Table 2. Suppose there is an isogeny $\varphi : J_0(xy) \rightarrow J^{XY}$ whose kernel is $C_0$. Restricting $\varphi$ to $E_1 \hookrightarrow J_0(xy)$, we get an isogeny $\varphi' : E_1 \rightarrow E$ defined over $F$ whose kernel is a subgroup of $C_0 \cong \mathbb{Z}/(q^2 + 1)\mathbb{Z}$. Note that 3 does not divide $q^2 + 1$. On the other hand, any isogeny from $E_1$ to $E_1'$ or $E_1''$ must have kernel whose order is divisible by 3. This implies that $\varphi'$ has trivial kernel, so $E = E_1$. □
Remark 7.11. In the notation of the proof of Proposition 7.10, consider the restriction of \( \varphi \) to \( E_2 \leftarrow J_0(xy) \). By part (ii) of the proposition, there results an isogeny \( \varphi'' : E_2 \to E_2' \) whose kernel is a subgroup of \( \mathbb{Z}/(q^2 + 1)\mathbb{Z} \). Since 5 divides \( q^2 + 1 \) when \( s \) is odd, part (ii) of Proposition 7.10 is compatible with Conjecture 7.3.

Theorem 7.12. Conjecture 7.3 is true for \( q = 2 \).

Proof. Assume \( q = 2 \). By Proposition 7.10, \( E_1 \) and \( E_2 \) are the \( J_0(xy) \)-optimal curves. Since the genus of \( X_0(xy) \) is 2, it is hyperelliptic (this is true for general \( q \) by Schweizer’s theorem which we used in Section 3). The genus being 2 also implies that a quotient of \( X_0(xy) \) by an involution has genus 0 or 1. The Atkin–Lehner involutions form a subgroup in \( \text{Aut}(X_0(xy)) \) isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^2 \). Since the hyperelliptic involution is unique, each \( E_1 \) and \( E_2 \) can be obtained as a quotient of \( X_0(xy) \) under the action of an Atkin–Lehner involution. Thus, there are degree-2 morphisms \( \pi_i : X_0(xy) \to E_i, \ i = 1, 2 \). In fact, one obtains the closed immersions \( \pi_i^* : E_i \to J_0(xy) \) from these morphisms by Picard functoriality. Let \( \pi_i^* : J_0(xy) \to E_i \) be the dual morphism. It is easy to show that the composition \( \pi_i^* \circ \pi_i^* : E_i \to E_i \) is the isogeny given by multiplication by \( 2 = \deg(\pi_i) \). This implies that \( E_1 \) and \( E_2 \) intersect in \( J_0(xy) \) in their common subgroup-scheme of 2-division points \( S := \pi_1^* E_1[2] = \pi_2^* E_2[2] \), so

\[
J_0(xy)(F) = H_1 \oplus H_3 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} = C.
\]

Let \( \psi : J_0(xy) \to E_1 \times E_2 \) be the isogeny with kernel \( S \). Note that \( S \) is characterized by the non-split exact sequence of group-schemes over \( F \):

\[
0 \to \mu_2 \to S \to \mathbb{Z}/2\mathbb{Z} \to 0.
\]

By Proposition 7.10, \( E_2' \) is the \( J_{10}^{\psi} \)-optimal elliptic curve in the isogeny class II. Let \( E \) be the \( J_{10}^{\psi} \)-optimal elliptic curves in class I. From the proof of Proposition 7.10, we know that \( E \) is Frobenius minimal, so it is one of the curves listed in Table 2. There are also Atkin–Lehner involutions acting on \( X_{10}^{\psi} \) and they form a subgroup in \( \text{Aut}(X_{10}^{\psi}) \) isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^2 \); see [31]. Now exactly the same argument as above implies that \( E \) and \( E_2' \) intersect in \( J_{10}^{\psi} \) along their common subgroup-scheme of 2-division points \( S' \subseteq S \). Let \( \nu : J_{10}^{\psi} \to E \times E_2' \) be the isogeny with kernel \( S' \). Let \( \hat{\nu} : E \times E_2' \to J_{10}^{\psi} \) be the dual isogeny.

The following argument is motivated by [19]. Consider the composition

\[
\phi : J_0(xy) \xrightarrow{\psi} E_1 \times E_2 \xrightarrow{\phi_1 \times \phi_2} E \times E_2' \xrightarrow{\hat{\nu}} J_{10}^{\psi},
\]

where \( \phi_1 \) is either the identity morphism or has kernel \( H_1, H_2 \), and \( \phi_2 \) has kernel \( H_3 \). Since \( \phi_1 \times \phi_2 \) has odd degree, this morphism maps the kernel of \( \psi \) to the kernel of \( \hat{\nu} \). Indeed, both are the “diagonal” subgroups isomorphic to \( S \) in the corresponding group-schemes \((E_1 \times E_2)[2]\) and \((E \times E_2')[2]\). More precisely, \( \mathcal{H} := \ker(\psi) \) is uniquely characterized as the subgroup-scheme of \( \mathcal{G} := (E_1 \times E_2)[2] \) having the following properties: \( \mathcal{H}^0 \) is the image of the diagonal morphism \( \mu_2 \to \mu_2 \times \mu_2 = \mathcal{G}^0 \) and the image of \( \mathcal{H} \) in \( \mathcal{G}^0 \) under the natural morphism \( \mathcal{G} \to \mathcal{G}^0 \) is the image of the diagonal morphism \( \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). A similar description applies to \( \ker(\hat{\nu}) \subset (E \times E_2')[2] \). Thus, there is an isogeny \( \phi' : J_0(xy) \to J_{10}^{\psi} \) such that \( \phi = \phi'[2] \) and \( \ker(\phi') \cong \ker(\phi_1 \times \phi_2) \). We conclude that \( J_{10}^{\psi} \) is isomorphic to the quotient of \( J_0(xy) \) by one of the following subgroups

\[
H_3, \quad H_1 \oplus H_3, \quad H_2 \oplus H_3.
\]

Now note that \( H_1 \) and \( H_3 \) under the specialization map \( \phi_\infty \) inject into \( \Phi_\infty \), but \( H_2 \) maps to 0 (indeed, \( H_2 \cong \mu_3 \) has non-trivial action by \( \text{Gal}(\overline{\mathbf{F}}/\mathbf{F}) \) whereas \( \Phi_\infty \) is constant). Hence Theorem 4.3 implies that the quotients of \( J_0(xy) \) by the subgroups listed above have component groups at \( \infty \) of orders 3,
1, 9, respectively. Since $\Phi_\infty' \cong \mathbb{Z}/3\mathbb{Z}$, we see that $J^{\infty}$ is the quotient of $J_0(xy)$ by $H_3$ which is $C_0$ in this case. □

References