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On the degree of modular parametrizations over function fields

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Abstract

Let E be an elliptic curve over $\mathbf{F}_q(T)$ with conductor $N \cdot \infty$. Let $\wp : X_0(N) \to E$ be the modular parametrization by the Drinfeld modular curve of level N. Assuming that E is a strong Weil curve we prove upper and lower bounds on $\deg \wp$. These bounds are the analogs of well-known (partially conjectural) bounds in the case of rational numbers.

Keywords: Drinfeld modular curves; ABC conjecture; Elliptic curves over function fields

1. Introduction

Let E be an optimal semi-stable elliptic curve over $\mathbf{Q}$ with conductor N_E, and let $X_0(N_E)$ be the modular curve parametrizing E

$$\wp : X_0(N_E) \to E,$$

where $\wp$ is non-trivial and of minimal possible degree. The *degree conjecture* claims that

$$\deg \wp \ll_\varepsilon N_E^{2+\varepsilon}.$$

It is well known that the degree conjecture is equivalent to the *ABC-conjecture*; see [2,13,14,19].

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One can prove a lower bound

$$\deg \wp \gg_{\varepsilon} N_E^{7/6-\varepsilon},$$

using a result of Hoffstein and Lockhart [10] (cf. [14,19]).

Now let $\mathbf{F}_q$ be the finite field of q elements, $A = \mathbf{F}_q[T]$ the polynomial ring, and $K = \mathbf{F}_q(T)$ the rational function field. Choose the place at ∞ to be $\frac{1}{T}$.

Let E be a non-isotrivial (i.e., $j_E \notin \mathbf{F}_q$) semi-stable elliptic curve over K. If E has a split-multiplicative reduction at ∞ (conductor $N_E = N \cdot \infty$) then, as a consequence of deep results of Deligne, Drinfeld, and Zarhin, one has a non-trivial morphism

$$\wp : X_0(N) \to E,$$

where $X_0(N)$ is the Drinfeld modular curve of level N (this is a moduli space of rank-2 Drinfeld modules with a certain level structure), for details see for example [8].

In this paper we will be interested in finding bounds on $\deg \wp$ analogous to the case of rational numbers. We again try to find a connection between such bounds and ABC. We show that such a connection indeed exists, and since some version of the ABC-conjecture is a theorem for $A = \mathbf{F}_q[T]$, these bounds are not conjectural. Throughout the paper we assume that E is an optimal curve (or a strong Weil curve), i.e., it has minimal modular degree in its isogeny class. With this assumption we prove (see Theorem 6.1):

$$\frac{1}{\deg_{\mathrm{ns}}(j_E)} |N|_{\infty}^{1-\varepsilon} \ll_{\varepsilon} \deg \wp \ll_{\varepsilon} |N|_{\infty}^{1+\varepsilon}, \tag{1}$$

where $|N|_\infty = q^{\deg N}$, j_E is the j-invariant of E, and $\deg_{\mathrm{ns}}(j_E)$ is the non-separable degree of the finite morphism induced by $j_E : \mathbf{P}^1 \to \mathbf{P}^1$, or which is the same, the non-separable degree of the finite extension $\mathbf{F}_q(T)/\mathbf{F}_q(j_E)$.

The non-separable degree $\deg_{\mathrm{ns}}(j_E)$ shows up because we use *Szpiro's conjecture* for function fields of positive characteristic [17] for the lower bound. It is not hard to show that $\deg_{\mathrm{ns}}(j_E)$, in general, cannot be removed from Szpiro's bound. On the other hand, the question whether it can be removed from (1) is very closely related to how large the Parshin–Faltings height of strong Weil curves can be (see Section 6 for more details; I would like to think that it indeed can be removed from (1)).

The bounds on $\deg \wp$ are obtained using the same strategy as over $\mathbf{Q}$. As a consequence of the first four sections we prove that

$$\deg \wp = \frac{q^{\deg N - 1}}{-val_{\infty}(j_E)} L(Sym^2\, T_\ell E, 2), \tag{2}$$

where $L(Sym^2\, T_\ell E, s)$ is the L-function attached to the symmetric square of the ℓ-adic Tate module of E, and $(-val_\infty\, j_E)$ is the number of geometrically irreducible components of the Néron model of E at ∞. Then in the rest of the paper we obtain bounds on the entries of the above expression for $\deg \wp$. Grothendieck's cohomological interpretation of L-functions over function fields, the Ramanujan conjecture, and the knowledge of the Riemann hypothesis for $L(Sym^2\, T_\ell E, s)$, coupled with some analytic techniques [10,13] (for the lower bound), give

$$|N|_\infty^{-\varepsilon} \ll_\varepsilon |L(Sym^2\, T_\ell E, 2)| \ll_\varepsilon |N|_\infty^{\varepsilon}.$$

In fact we prove a stronger result: If f is a normalized automorphic cusp form of level N which is an eigenform for all the Hecke operators (or in other words is a newform) then we show

$$|N|_\infty^{-\varepsilon} \ll_\varepsilon |L(Sym^2 f, 2)| \ll_\varepsilon |N|_\infty^{\varepsilon}.$$

When f corresponds to our elliptic curve, in particular it has rational eigenvalues and $L(Sym^2 f, s) = L(Sym^2\, T_\ell E, s)$, then the upper bound can be proved without appealing to the convexity estimates (see Section 5.2).

For the lower bound on $(-val_\infty\, j_E)$ we take the trivial $1 \leqslant (-val_\infty\, j_E)$, and for the upper bound we have $(-val_\infty\, j_E) \leqslant 6\deg_{\mathrm{ns}}(j_E)\deg N$, using the Pesenti–Szpiro theorem [17].

We also would like to remark that (2) can be used to compute very efficiently $\deg \wp$, as $L(Sym^2\, T_\ell E, 2)$ is the value at 2 of a certain polynomial in q^{-s} which is easily computable, see Section 6 for some examples.

The organization of the paper is as follows: In Section 2, we recall some standard facts about Bruhat–Tits tree $\mathscr{T}$ of $PGL_2(K_\infty)$, and the Fourier expansion of $\mathbf{C}$-valued functions on $\mathscr{T}$. We use ∞-local formulae from Fourier analysis as in [5] instead of adelic ones as in Weil [24]; the former allows explicit computations (e.g. of the residues of functions on $\mathscr{T}$). We also carry out some basic computations which are used in the next two sections. In Section 3, we define *Eisenstein series* $E(e,s)$ for the full modular group $\Gamma = GL_2(A)$. This is a modified version of the definition given by Gekeler [5]. Gekeler's definition is not suitable for our purposes. Similar Eisenstein series for $SL_2(K_\infty)$ have been used in [11]. We then compute the Fourier coefficients of $E(e,s)$ and prove that $E(e,s)$ has properties very similar to the classical situation: in particular, it is absolutely convergent for $\mathrm{Re}(s) > 1$, has an analytic continuation to the whole complex plane with a simple pole at $s = 1$, and the residue is a constant function. Moreover we prove its functional equation. We also define the Eisenstein series $E_N(e,s)$ for the Hecke congruence subgroup $\Gamma_0(N)$ and relate it to $E(e,s)$. In Section 4, by computing a Rankin–Selberg integral of two automorphic forms convolved with E_N and then taking the residues we arrive at an expression relating the Petersson inner product (f,f) of a newform f with the special value of $L(Sym^2 f, s)$ at $s = 2$; this is again very similar to the classical case as given, for example, in [21, 2.5]. Combined with a result of Gekeler [6] relating $\deg \wp$ with (f,f) gives (2) when f corresponds to our elliptic curve.

We also derive a functional equation for $L(Sym^2 f, s)$ which is used in Section 5. This is essentially done by computing the local constants. We have restricted ourselves to square-free N only to avoid technical difficulties in this step. To prove the functional equation in general one has to verify certain properties of the twists of newforms by characters in the case of function fields, cf. [12]. As far as I know this has not been done yet, and proving such results was not in the scope of this paper. In Section 5, we derive upper and lower bounds on $L(Sym^2 f, 2)$ using Phragmén–Lindelöf and Siegel-type theorems and the Riemann hypothesis for function fields. When f has rational eigenvalues we compute the degree of $L(Sym^2 f, s)$ as a polynomial in q^{-s} in terms of $\deg N$. This also implies the upper bound but without using analytic methods. In Section 6, we combine the results of the previous sections to prove the theorem claimed at the beginning of the Introduction. We use to our great advantage the fact that some version of the famous Szpiro conjecture over the rational numbers is a theorem in the function field setting.

2. Preliminary computations

Let $A = \mathbf{F}_q[T]$, $K = \mathbf{F}_q(T)$. Also take $\pi = T^{-1}$ to be the uniformizer at infinity, and $K_\infty = \mathbf{F}_q((\pi))$, $\mathcal{O}_\infty = \mathbf{F}_q[[\pi]]$, the ∞-adic integers. For $n \in \mathbf{F}_q[T]$, $|n| := |n|_\infty = q^{\deg n}$ $(\deg 0 = -\infty)$. For a divisor $m = m_f \cdot \infty^k$ of K, where $supp(m_f) \subset Spec\, A$, write $|m|$ for $q^{\deg m}$.

Put $\mathcal{K} = GL_2(\mathcal{O}_\infty)$, and $\mathcal{I} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{K} | c \equiv 0 \,(\mathrm{mod}\, \pi) \right\}$, the Iwahori subgroup. If $\mathcal{T}$ is the Bruhat–Tits tree of $PGL_2(K_\infty)$ then the sets of vertices $X(\mathcal{T})$ and of oriented edges $Y(\mathcal{T})$ are isomorphic to

$$X(\mathcal{T}) \cong GL_2(K_\infty)/\mathcal{K} \cdot K_\infty^*,$$
$$Y(\mathcal{T}) \cong GL_2(K_\infty)/\mathcal{I} \cdot K_\infty^*.$$

We denote by $o(e)$, $t(e)$, $\bar{e}$ the origin, terminus and the inverse of an edge e. We have a canonical map from $Y(\mathcal{T})$ to $X(\mathcal{T})$ which associates to each edge its origin. $\mathcal{T}$ is a $(q+1)$-regular tree. Multiplication from the right by $\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$ corresponds to the map $e \mapsto \bar{e}$ on $Y(\mathcal{T})$. Each non-oriented edge e of $Y(\mathcal{T})$ can be represented by a matrix $\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix}$, where $u \in K_\infty \bmod \pi^k \mathcal{O}_\infty$, and $k \in \mathbf{Z}$. For any edge $e \in Y(\mathcal{T})$ represented in this form define

$$k(e) = k.$$

This function is obviously invariant under the change of orientation of e. It measures the distance of e to ∞, where the shift toward ∞ is the map $\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} \pi^{k-1} & u \\ 0 & 1 \end{pmatrix}$. Moreover, if we denote by $\Gamma = GL_2(A)$ and $\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma \right\}$ (the stabilizer of the end ∞) then k is invariant under Γ_∞. Put

$$\varphi(e, s) = q^{-k(e) \cdot s},$$

where $s \in \mathbf{C}$. (This is the analog of $\varphi(z, s) = \mathrm{Im}(z)^s$, $\mathrm{Im}(z) > 0$, over the complex numbers.)

Then $\varphi(e,s)$ satisfies the following s-harmonicity condition ($Y^+(\mathcal{T})$ are the positively oriented edges, i.e., those pointing to ∞)

$$\sum_{\substack{e' \in Y^+(\mathcal{T}) \\ t(e')=o(e)}} \varphi(e',s) = q^{-(s-1)}\varphi(e,s).$$

We would like to know the behavior of $\varphi(e,s)$ under the action of Γ on $Y(\mathcal{T})$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $\gcd(c,d)=1$. Put $\varphi_{c,d}(e,s) = \varphi(\gamma(e),s)$ when $c \neq 0$, and $\varphi_{c,d}(e,s) = \varphi(e,s)$ otherwise.

Lemma 2.1. *Assume $c \neq 0$. Let e be represented by $e(k,u) := \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix}$. Write $\omega = val_\infty(cu+d)$, and $k_1 = -\deg c - k\left(\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix} e\right)$. Then*

$$\varphi_{c,d}(e,s) = q^{k_1} = \begin{cases} q^{(k-2\deg c-1)\cdot s}, & \omega \geqslant k - \deg c, \\ q^{(2\omega-k)\cdot s}, & \omega < k - \deg c. \end{cases}$$

Proof. See [5, p. 379]. $\square$

We now take the formula in the lemma to *define* $\varphi_{c,d}$ for an arbitrary pair $(c,d) \in A \times A$ with $c \neq 0$. Then we have

$$\varphi_{tc,td}(e,s) = q^{-2\deg(t)s}\varphi_{c,d}(e,s) \quad (0 \neq t \in A),$$

$$\varphi_{c,d}(e,s) = q^{-\deg(c)s}\varphi_{1,d}\left(\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} e, s\right)$$

and the formula in the lemma remains valid for arbitrary (c,d). Note that $\varphi_{c,d}$ is Γ_∞ invariant.

As discussed in Weil [24] (see also [5]), any function on $Y^+(\Gamma_\infty \backslash \mathcal{T})$ (positively oriented edges of the quotient tree) may be written as a Fourier series. Let β be a non-negative divisor on K,

$$\beta = div(\alpha) \cdot \infty^{\deg \beta} = div(\alpha)_f \cdot \infty^{\deg\beta - \deg\alpha},$$

where $div(\alpha)$ is the principal divisor of $\alpha \in A$ with finite part $div(\alpha)_f$.

If F is a function on $Y^+(\Gamma_\infty \backslash \mathcal{T})$ then (see [5, 2.6–2.8])

$$F\left(\begin{pmatrix} \pi^k & y \\ 0 & 1 \end{pmatrix}\right) = c_0(F,\pi^k) + \sum_{\substack{0 \neq \alpha \in A \\ \deg \alpha \leqslant k-2}} c(F, div(\alpha) \cdot \infty^{k-2})\eta(\alpha y),$$

where

$$c_0(F,\pi^k) = \begin{cases} q^{1-k}\sum_{y\in(\pi)/(\pi^k)} F\left(\begin{pmatrix}\pi^k & y\\ 0 & 1\end{pmatrix}\right), & k\geqslant 1,\\ F\left(\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix}\right), & k\leqslant 1,\end{cases}$$

$$c(F,\beta) = q^{-1-\deg\beta}\sum_{y\in(\pi)/(\pi^{2+\deg\beta})} F\left(\begin{pmatrix}\pi^{2+\deg\beta} & y\\ 0 & 1\end{pmatrix}\right)\eta(-\alpha y),$$

$\eta : K_\infty \mapsto \mathbf{C}^*$ is $\sum a_i\pi^i \mapsto \eta_0(tr(a_1))$ with η_0 a non-trivial additive character of $\mathbf{F}_p$, and tr is the trace map $\mathbf{F}_q \mapsto \mathbf{F}_p$.

As a consequence of these definitions, if F and G are functions on $Y^+(\Gamma_\infty\backslash\mathscr{T})$ such that $F(e) = G\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}e\right)$ for $0\neq a\in A$, then $c(F,\beta)=0$ when $a\nmid\beta_f$ ([5, Corollary 2.11]).

Consider for $0\neq c\in A$

$$F_c(e,s) = \sum_{d\in A}\varphi_{c,d}(e,s).$$

From the properties of $\varphi_{c,d}(e,s)$ one has

$$F_c(e,s) = q^{-\deg c\cdot s}F_1\left(\begin{pmatrix}c & 0\\ 0 & 1\end{pmatrix}e,s\right).$$

Lemma 2.2. *$F_c(e,s)$ absolutely converges for* $\mathrm{Re}(s)>\frac{1}{2}$ *and is Γ_∞ invariant.*

Proof. It is enough to prove that $F_1(e,s)$ is absolutely convergent for $\mathrm{Re}(s)>\frac{1}{2}$.

Recall that

$$F_1(e,s) = \sum_{d\in A}\varphi_{1,d}(e,s).$$

Let $e = \begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}$. If $d\neq 0$ then $val_\infty(u+d) = val_\infty(d) = -\deg d$ as $u\in(\pi)/(\pi^k)$. And since for a fixed k, $-\deg d\leqslant k$ for almost all $d\in A$, it is enough, using Lemma 2.1, to prove that

$$\sum_{0\neq d\in A} q^{-2\deg d\cdot s}$$

is absolutely convergent for $\mathrm{Re}(s)>\frac{1}{2}$. But

$$\sum_{0\neq d\in A} q^{-2\deg d\cdot s} = \sum_{n=0}^{\infty}\sum_{\deg d=n} q^{-2ns} = (q-1)\sum_{n=0}^{\infty} q^{(1-2s)n}.$$

The last sum is indeed absolutely convergent for $\mathrm{Re}(s)>\frac{1}{2}$.

Let $\gamma = \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} \in \Gamma_\infty$. Since r, t are units, $cs + dt$ runs through A if d does. Therefore, using the absolute convergence,

$$\begin{aligned} F_c(\gamma e, s) &= \sum_{d \in A} \varphi_{c,d}\left(\begin{pmatrix} r & s \\ 0 & t \end{pmatrix} e, s \right) = \sum_{d \in A} \varphi_{cr, cs+dt}(e, s) \\ &= F_{cr}(e, s) = F_c(e). \qquad \square \end{aligned}$$

We are interested in the Fourier expansion of $F_c(e, s)$. By Fourier transform, for any non-negative divisor β of K, with $\beta = div(\alpha) \cdot \infty^{\deg \beta}$, $\alpha \subset A$,

$$c(F_c(e, s), \beta) = q^{-1-\deg \beta} \sum_{y \in (\pi)/(\pi^{2+\deg \beta})} F_c\left(\begin{pmatrix} \pi^{2+\deg \beta} & y \\ 0 & 1 \end{pmatrix}, s \right) \eta(-\alpha y).$$

Hence

$$c(F_c(e, s), \beta) = q^{-\deg(c) \cdot s} c(F_1(e, s), \beta \cdot (c)_f^{-1})$$

(which vanishes, in particular, if $c \nmid \beta_f$) and

$$c_0(F_c(e, s), \pi^k) = q^{-\deg(c) \cdot s} c_0(F_1(e, s), \pi^{k - \deg c}).$$

So it is enough to compute the Fourier coefficients of F_1.

Proposition 2.3.

$$c_0(F_1, \pi^k) = \frac{(q^{s-1} + 1)(q - q^{1-s})}{q^{2s-1} - 1} q^{k(s-1)}.$$

Proof. First, suppose $k \geqslant 1$. Then

$$\begin{aligned} c_0(F_1, \pi^k) &= q^{1-k} \sum_{u \in (\pi)/(\pi^k)} F_1(e(k, u), s) \\ &= q^{1-k} \sum_{u \in (\pi)/(\pi^k)} \sum_{d \in A} \varphi_{1,d}(e(k, u), s). \end{aligned}$$

Let

$$\omega = val_\infty(u + d) = \begin{cases} -\deg d, & d \neq 0, \\ val_\infty u, & d = 0. \end{cases}$$

Since we assumed $k \geqslant 1$, when $d \neq 0$, $-\deg d$ is *never* $\geqslant k$, hence $\varphi_{1,d}(e(k, u), s) = q^{(-2 \deg d - k)s}$. When $d = 0$, $val_\infty u < k$ except when $u \equiv 0 \bmod \pi^k$, and then $val_\infty u = k$. Thus

$$c_0(F_1, \pi^k) = q^{1-k} \left(\sum_{u \in (\pi)/(\pi^k)} \varphi_{1,0}(e(k, u), s) + \sum_{u \in (\pi)/(\pi^k)} \sum_{\substack{d \in A \\ d \neq 0}} \varphi_{1,d}(e(k, u), s) \right).$$

Computing the partial sums,

$$\sum_{u\in(\pi)/(\pi^k)} \varphi_{1,0}(e(k,u),s) = q^{(k-1)s} + \sum_{n=1}^{k-1} \sum_{\substack{u\in(\pi)/(\pi^k)\\ val_\infty\ u=n}} q^{(2n-k)s}$$
$$= q^{(k-1)s} + (q-1)q^{k-1-ks}q^{2s-1}\frac{q^{(2s-1)(k-1)}-1}{q^{2s-1}-1}$$

and

$$\sum_{u\in(\pi)/(\pi^k)} \sum_{\substack{d\in A\\ d\neq 0}} \varphi_{1,d}(e(k,u),s) = q^{k-1}\sum_{n=0}^{\infty}\sum_{\deg d=n} q^{(-2n-k)s}$$
$$= q^{k-1}\sum_{n=0}^{\infty}(q-1)q^n q^{(-2n-k)s}$$
$$= (q-1)q^{k-1-ks}\frac{1}{1-q^{1-2s}}.$$

Combining both sums and simplifying gives the answer in the case of $k\geqslant 1$.

Now suppose $k<1$. Then

$$c_0(F_1,\pi^k) = \sum_{d\in A}\varphi_{1,d}(e(k,0),s) = \varphi_{1,0}(e(k,0),s) + \sum_{\substack{d\in A\\ d\neq 0}}\varphi_{1,d}(e(k,0),s).$$

Using Lemma 2.1, the above is equal to

$$q^{(k-1)s} + \sum_{\substack{d\in A\\ -\deg d\geqslant k}} q^{(k-1)s} + \sum_{\substack{d\in A\\ -\deg d<k}} q^{(-2\deg d-k)s}.$$

Since

$$\sum_{\substack{d\in A\\ -\deg d\geqslant k}} q^{(k-1)s} = q^{(k-1)s}\sum_{\substack{d\in A\\ -\deg d\geqslant k}} 1 = q^{(k-1)s}(q^{-k+1}-1)$$

and

$$\sum_{\substack{d\in A\\ -\deg d<k}} q^{(-2\deg d-k)s} = q^{-ks}\sum_{n=-k+1}^{\infty}\sum_{\deg d=n} q^{-2ns}$$
$$= (q-1)q^{-ks}\sum_{n=-k+1}^{\infty} q^{n(1-2s)}$$
$$= (q-1)q^{-ks}q^{(1-2s)(-k+1)}\frac{1}{1-q^{1-2s}},$$

combining these sums and simplifying again gives the claimed result. □

Proposition 2.4. *Let* $\beta = div(\alpha)\cdot\infty^k$, $\alpha \in A$, *be a positive divisor of degree* k, *and* $\deg\alpha = n$. *In particular* $n \leqslant k$. *Then*

$$c(F_1,\beta) = q^{(k+1)(s-1)} + q^{-(k+2)s}\left(-q^{(2s-1)(n+1)} + (q-1)q^{(2s-1)(n+2)}\frac{q^{(2s-1)(k-n)}-1}{q^{2s-1}-1}\right).$$

Proof.

$$\begin{aligned} c(F_1,\beta) &= q^{-1-k}\sum_{y\in(\pi)/(\pi^{2+k})} F_1\left(\begin{pmatrix}\pi^{2+k} & y\\ 0 & 1\end{pmatrix}\right)\eta(-\alpha y)\\ &= q^{-1-k}\sum_{y\in(\pi)/(\pi^{2+k})} F_1(e(2+k,y),s)\eta(-\alpha y). \end{aligned}$$

First, let us compute $F_1(e(2+k,y),s)$, $y\in(\pi)/(\pi^{2+k})$. We have

$$F_1(e(2+k,y),s) = \sum_{d\in A}\varphi_{1,d}(e(2+k,y),s).$$

Since $val_\infty(y+d) = -\deg d$ (unless $d=0$), and $-\deg d$ is never $\leqslant k+2$ (since $k \geqslant 0$), by Lemma 2.1 we get

$$\varphi_{1,d}(e(2+k,y),s) = q^{(-2\deg d-(2+k))s}\quad \text{when } d\neq 0.$$

But this expression does not depend on y, which, along with $\sum_{y\in(\pi)/(\pi^{2+k})}\eta(-\alpha y) = 0$ $(\alpha\neq 0)$, implies that it does not contribute to $c(F_1,\beta)$.

Now

$$\varphi_{1,0}(e(2+k,y),s) = \begin{cases} q^{(2val_\infty\, y-(k+2))s}, & val_\infty\, y < k+2,\\ q^{(k+1)s}, & val_\infty\, y = k+2\end{cases}$$

and

$$c(F_1,\beta) = q^{-1-k}\sum_{y\in(\pi)/(\pi^{2+k})}\varphi_{1,0}(e(2+k,y),s)\eta(-\alpha y).$$

When $val_\infty(y) = k+2$ (i.e., $y=0$ in $(\pi)/(\pi^{2+k})$), the value $\eta(-\alpha y)$ equals 1 since $\deg\alpha \leqslant k$. Hence

$$c(F_1,\beta) = q^{-(1+k)+s(k+1)} + q^{-(1+k)-s(k+2)}\sum_{\substack{y\in(\pi)/(\pi^{2+k})\\ val_\infty\, y\leqslant k+1}} q^{2val_\infty\, y\cdot s}\eta(-\alpha y).$$

Now we compute

$$\begin{aligned}
&\sum_{\substack{y\in(\pi)/(\pi^{2+k})\\ val_\infty\ y\leqslant k+1}} q^{2val_\infty\ y\cdot s}\eta(-\alpha y)\\
&= \sum_{r=1}^{k+1} q^{2rs}\sum_{u\in\mathbf{F}_q^\times}\ \sum_{x_0\in(\pi^{r+1})/(\pi^{2+k})} \eta(-\alpha(uT^{-r}+x_0))\\
&= \sum_{r=1}^{k+1} q^{2rs}\sum_{u\in\mathbf{F}_q^\times}\eta(-\alpha uT^{-r}) \sum_{x_0\in(\pi^{r+1})/(\pi^{2+k})} \eta(-\alpha x_0)\\
&= \sum_{r=n+1}^{k+1} q^{2rs}q^{k+1-r}\sum_{u\in\mathbf{F}_q^\times}\eta(-\alpha uT^{-r})\\
&= q^{k+1}\left(-q^{(2s-1)(n+1)}+(q-1)\sum_{r=n+2}^{k+1} q^{(2s-1)r}\right)\\
&= q^{k+1}\left(-q^{(2s-1)(n+1)}+(q-1)q^{(2s-1)(n+2)}\frac{q^{(2s-1)(k-n)}-1}{q^{(2s-1)}-1}\right).
\end{aligned}$$

If we substitute this into the expression for $c(F_1,\beta)$ we get the result. □

3. Eisenstein series

3.1. The Eisenstein series for the full modular group $GL_2(A)$

Define the *Eisenstein series* as

$$E(e,s)=\sum_{\substack{c\in A\\ \text{monic}}}\ \sum_{\substack{d\in A\\ \gcd(c,d)=1}} \varphi_{c,d}(e,s)+\varphi(e,s). \tag{3}$$

Note that since we have a bijection

$$\begin{aligned}
\Gamma_\infty\backslash\Gamma &\simeq \{(c,d)\in A\times A\mid \gcd(c,d)=1\}/\mathbf{F}_q^\times\\
&\simeq \{(c,d)\in A\times A\mid \gcd(c,d)=1,\ c\ \text{monic}\}\cup\{(0,1)\}
\end{aligned}$$

induced by $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto(c,d)$ we can rewrite

$$E(e,s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\varphi(\gamma(e),s). \tag{4}$$

The possibility of rewriting $E(e,s)$ in this form is justified by the next proposition.

Proposition 3.1. *$E(e,s)$ converges absolutely for* $\mathrm{Re}(s)>1$, *and is Γ invariant.*

Proof. Γ-invariance follows from (4) once we prove the absolute convergence.

It is well known that the graph $\Gamma\backslash\mathcal{T}$ is a half-line, represented by the matrices $\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix}$, with $k\leqslant 0$. So each edge e is in a Γ-orbit of some $\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix}$, with $k\leqslant 0$, and we can assume that e is of that form. For such e, by Lemma 2.1

$$\varphi_{c,d}(e,s)=\begin{cases} q^{(k-2\deg c-1)s}, & \deg d\leqslant \deg c-k,\\ q^{(-2\deg d-k)s}, & \deg d>\deg c-k.\end{cases}$$

Hence

$$\begin{aligned}E(e,s)&=\sum_{c\text{ monic}}\ \sum_{\substack{\gcd(c,d)=1\\ \deg d>\deg c-k}} q^{(-2\deg d-k)s}\\ &\quad+\sum_{c\text{ monic}}\ \sum_{\substack{\gcd(c,d)=1\\ \deg d\leqslant\deg c-k}} q^{(k-2\deg c-1)s}\\ &=q^{-ks}\sum_{c\text{ monic}}\ \sum_{\substack{\gcd(c,d)=1\\ \deg d>\deg c-k}} q^{(-2\deg d)s}+q^{(k-1)s}\\ &\quad\times\sum_{c\text{ monic}}\ \sum_{\substack{\gcd(c,d)=1\\ \deg d\leqslant\deg c-k}} q^{(-2\deg c)s}.\end{aligned}$$

It is enough to prove the absolute convergence of both summands, assuming s is real and $s>1$. For example,

$$\begin{aligned}\sum_{c\text{ monic}}\ \sum_{\substack{\gcd(c,d)=1\\ \deg d>\deg c-k}} q^{-2\deg d\cdot s}&\leqslant\sum_{c\text{ monic}}\ \sum_{\deg d>\deg c-k} q^{-2\deg d\cdot s}\\ &=(q-1)\sum_{n=0}^{\infty} q^n\sum_{m=n-k+1}^{\infty} q^{m(1-2s)}.\end{aligned}$$

The inner sum converges absolutely for $s>\frac{1}{2}$, and the whole expression equals

$$\frac{q-1}{1-q^{1-2s}}\,q^{(1-k)(1-2s)}\sum_{n=0}^{\infty} q^{2n(1-s)}.$$

The last expression is absolutely convergent for $s>1$. Similarly, for the second sum. $\square$

Now we turn to the Fourier expansion of $E(e,s)$. Consider

$$\begin{aligned}\sum_{c\text{ monic}} F_c(e,s)&=\sum_{c\text{ monic}}\ \sum_{d\in A}\varphi_{c,d}(e,s)=\sum_{\substack{0\neq t\in A\\ \text{monic}}}\ \sum_{c\text{ monic}}\ \sum_{\substack{d\in A\\ \gcd(c,d)=t}}\varphi_{c,d}(e,s)\\ &=\sum_{\substack{0\neq t\in A\\ \text{monic}}}\ \sum_{c\text{ monic}}\ \sum_{\substack{d\in A\\ \gcd(c,d)=1}}\varphi_{tc,td}(e,s)\end{aligned}$$

$$= \sum_{\substack{0 \neq t \in A \\ \text{monic}}} \sum_{c \text{ monic}} \sum_{\substack{d \in A \\ \gcd(c,d)=1}} q^{-2 \deg t \cdot s} \varphi_{c,d}(e,s)$$
$$= \sum_{\substack{0 \neq t \in A \\ \text{monic}}} |t|^{-2s}(E(e,s) - \varphi(e,s)) = \zeta(2s)(E(e,s) - \varphi(e,s)),$$

where $\zeta(s) = \frac{1}{1-q^{1-s}}$ is the zeta function of A. Hence

$$E(e,s) = \zeta^{-1}(2s) \sum_{c \text{ monic}} F_c(e,s) + \varphi(e,s), \tag{5}$$

and since $F_c(e,s) = q^{-\deg c \cdot s} F_1\left(\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} e, s\right)$, the Fourier expansion of $E(e,s)$ can be derived from that of $F_1(e,s)$.

Proposition 3.2.

$$c_0(E, \pi^k) = q^{-sk} + \frac{q^{1-s}(1-q^{-s})}{1-q^{1-s}} q^{-k(1-s)}.$$

Proof. From the definition $c_0(\varphi, \pi^k) = q^{-ks}$. Also, as we found,

$$c_0(F_1, \pi^k) = \frac{(q^{s-1}+1)(q-q^{1-s})}{q^{2s-1}-1} q^{-k(1-s)}.$$

On the other hand,

$$k\left(\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} e\right) = k(e) - \deg c.$$

Hence

$$\begin{aligned}
c_0(E, \pi^k) &= q^{-ks} + \zeta^{-1}(2s) \sum_{c \text{ monic}} q^{-\deg c \cdot s} c_0(F_1, \pi^{k - \deg c}) \\
&= q^{-ks} + \zeta^{-1}(2s) \sum_{c \text{ monic}} q^{-\deg c \cdot s} q^{-(1-s)(k - \deg c)} \\
&\quad \times \left(\frac{(q^{s-1}+1)(q-q^{1-s})}{q^{2s-1}-1}\right) \\
&= q^{-ks} + \zeta^{-1}(2s) \left(\frac{(q^{s-1}+1)(q-q^{1-s})}{q^{2s-1}-1}\right) q^{-(1-s)k} \\
&\quad \times \sum_{c \text{ monic}} q^{\deg c(1-2s)} \\
&= q^{-ks} + \zeta^{-1}(2s) \left(\frac{(q^{s-1}+1)(q-q^{1-s})}{q^{2s-1}-1}\right) q^{-(1-s)k} \frac{1}{1-q^{2-2s}}.
\end{aligned}$$

Simplifying gives the answer. □

Now we compute the non-constant Fourier coefficients. Let $\beta = div(\alpha) \cdot \infty^k$ be a positive divisor with $\alpha \in A$, and $\deg(\alpha) \leqslant k$. Then since $c(\varphi, \beta) = 0$ we have

$$c(E,\beta)=\zeta^{-1}(2s)\sum_{c\ \text{monic}}c(F_c,\beta)=\zeta^{-1}(2s)\sum_{c\ \text{monic}}q^{-\deg c\cdot s}c(F_1,\beta\cdot c^{-1}),$$

which is zero unless $c\,|\,\alpha$. Hence

$$c(E,\beta)=\zeta^{-1}(2s)\sum_{\substack{c\ \text{monic}\\ c\,|\,\alpha}}q^{-\deg c\cdot s}c(F_1,\beta\cdot c^{-1}).$$

One can combine this with our previous computation of $c(F_1,\beta)$ to write down an explicit (and messy) expression for $c(E,\beta)$. Since it is not essential for our purposes we do not do that. What is important though is that now it is clear that each $c(E,\beta)$ can be meromorphically continued to the whole plane with a possible simple pole at $s=\frac{1}{2}$. This implies that $E(e,s)$ itself can be meromorphically continued to the whole complex plane with possible simple poles at $s=\frac{1}{2}$ and $s=1$ (the latter coming from $c_0(E)$).

To find the functional equation for $E(e,s)$ we employ a clever trick used in [11]. First, recall that $\Gamma\backslash\mathcal{T}$ is a half-line represented by the matrices $\begin{pmatrix}T^k & 0\\ 0 & 1\end{pmatrix}$, $k\geqslant 0$ (i.e., this is the *fundamental domain*). Next, any function on $\Gamma\backslash\mathcal{T}$ is supported only on its zeroth Fourier coefficient (this is from the definition of Fourier expansion), in particular

$$E(e,s)=c_0(E,\pi^{-k})\quad\text{for } e=\begin{pmatrix}T^k & 0\\ 0 & 1\end{pmatrix},\ k\geqslant 0.$$

Hence the functional equation satisfied by $c_0(E,\pi^{-k})$ will be the functional equation of $E(e,s)$ itself. Substituting $s\mapsto 1-s$ in Proposition 3.2 one easily derives that the functional equation is

$$\Lambda_E(e,1-s)=-\Lambda_E(e,s),\tag{6}$$

where $\Lambda(e,s)=-\zeta(s+1)\cdot E(e,s)$ and $\zeta(s)=\frac{1}{1-q^{1-s}}$ as before.

We summarize the results of this section in the following:

Theorem 3.3. *The Eisenstein series defined as*

$$E(e,s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\varphi(\gamma(e),s)$$

converges absolutely for $\mathrm{Re}(s)>1$, *has an analytic continuation to* $\mathbf{C}$ *with a simple pole at* $s=1$ *and residue*

$$\underset{s=1}{Res}\ E(e,s)=\frac{1}{\zeta(2)\log q},\tag{7}$$

and satisfies a functional equation as in (6).

3.2. The Eisenstein series for the Hecke congruence subgroups

Let $\Gamma_0(N)$ be the Hecke congruence subgroup of Γ of level N, i.e.,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \bmod N \right\},$$

where N is some *monic* polynomial of A.

We have a bijection

$$\begin{aligned} \Gamma_\infty \backslash \Gamma_0(N) &\simeq \{(c,d) \in A \times A \mid \gcd(c,d) = 1,\ c \equiv 0 \bmod N\}/\mathbf{F}_q^\times \\ &\simeq \{(c,d) \in A \times A \mid \gcd(c,d) = 1,\ c \text{ monic}, \\ &\quad c \equiv 0 \bmod N\} \cup \{(0,1)\} \end{aligned}$$

induced by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c,d)$.

Define the *Eisenstein series of level N as*

$$E_N(e,s) = \sum_{\substack{c \in A \\ c \text{ monic} \\ c \equiv 0 \bmod N}} \sum_{\substack{d \in A \\ \gcd(c,d)=1}} \varphi_{c,d}(e,s) + \varphi(e,s), \tag{8}$$

which by the above bijection equals

$$E_N(e,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \varphi(\gamma(e), s). \tag{9}$$

We are interested in expressing $E_N(e,s)$ in terms of $E(e,s)$. Let

$$D_N(s) := \frac{\zeta(s)}{\zeta_N(s)} = \prod_{\substack{\mathfrak{p} \mid N \\ \mathfrak{p} \text{ monic} \\ \text{prime}}} (1 - |\mathfrak{p}|^{-s})^{-1},$$

where $\zeta_N(s)$ is the zeta-function of the affine line without the Euler factors corresponding to the primes dividing N. Then we have

Lemma 3.4.

$$E_N(e,s) = \frac{D_N(2s)}{|N|^s} \sum_{\substack{r \mid N \\ r \text{ monic}}} \frac{\mu(r)}{|r|^s} E\left(\begin{pmatrix} N/r & 0 \\ 0 & 1 \end{pmatrix} e, s \right),$$

where $\mu(r)$, $r \in A$, is the Möbius function.

Proof. Write

$$E_N(e,s) = \frac{1}{q-1} \sum_{\substack{c,d \in A \\ \gcd(cN,d)=1}} \varphi_{cN,d}(e,s).$$

Since $\varphi_{tc,td}(e,s) = q^{-2\deg(t)s}\varphi_{c,d}(e,s)$ we can rewrite

$$\begin{aligned}
(q-1)\zeta_N(2s)E_N(e,s) &= \sum_{\substack{c,d\in A\\ \gcd(N,d)=1}} \varphi_{cN,d}(e,s)\\
&= \sum_{c,d\in A}\varphi_{cN,d}(e,s)\sum_{\substack{r\mid d,N\\ r\ \text{monic}}}\mu(r)\\
&= \sum_{\substack{r\mid N\\ r\ \text{monic}}}\mu(r)\sum_{c,d\in A}\varphi_{cN,rd}(e,s)\\
&= \frac{1}{|N|^s}\sum_{\substack{r\mid N\\ r\ \text{monic}}}\frac{\mu(r)}{|r|^s}\sum_{c,d\in A}\varphi_{c,d}\left(\begin{pmatrix} N/r & 0\\ 0 & 1\end{pmatrix}e,s\right)\\
&= \frac{\zeta(2s)(q-1)}{|N|^s}\sum_{\substack{r\mid N\\ r\ \text{monic}}}\frac{\mu(r)}{|r|^s}E\left(\begin{pmatrix} N/r & 0\\ 0 & 1\end{pmatrix}e,s\right),
\end{aligned}$$

where in the end we again used the transformation rule for $\varphi_{tc,td}(e,s)$. $\square$

From the proved properties of $E(e,s)$ it is clear that $E_N(e,s)$ converges absolutely for $\mathrm{Re}(s)>1$, and can be meromorphically continued to $\mathbf{C}$ with a simple pole at $s=1$.

4. The Rankin–Selberg integral

Consider the following conditions on $\mathbf{C}$-valued functions F on $Y(\mathscr{T})$:

(i) $F(e)+F(\bar{e})=0 \quad \forall e\in Y(\mathscr{T})$,
(ii) $\sum_{\substack{e\in Y(\mathscr{T})\\ t(e)=v}} F(e)=0 \quad \forall v\in X(\mathscr{T})$,
(iii) $F(\gamma e)=F(e) \quad \forall e\in Y(\mathscr{T}), \quad \forall \gamma\in\Gamma_0(N)$,
(iv) F has compact (=finite) support modulo $\Gamma_0(N)$ (this means that F vanishes eventually on each of the half-lines (=cusps) of $\Gamma_0(N)\backslash\mathscr{T}$).

Functions satisfying (i)–(iv) are called *automorphic cusp forms of level N* (of Jacquet–Langlands–Drinfeld type) [8]. Denote them by $H_!(\mathscr{T},\mathbf{C})^{\Gamma_0(N)}$. Their arithmetic importance will be explained in the next section. As for now observe that condition (iv) forces the constant Fourier coefficient c_0 of F to vanish.

The space $H_!(\mathscr{T},\mathbf{C})^{\Gamma_0(N)}$ is equipped with a Petersson scalar product defined by

$$(f,g) = \int_{Y(\Gamma_0(N)\backslash\mathscr{T})} f(e)\cdot\bar{g}(e)\,d\mu e,$$

where $f, g \in H_!(\mathcal{T}, \mathbf{C})^{\Gamma_0(N)}$ and μe is the Haar measure on the discrete set $Y(\Gamma_0(N)\backslash\mathcal{T})$ given by $\frac{q-1}{2}\#(Stab_{\Gamma_0(N)}\, e)^{-1}$ (here $Stab_{\Gamma_0(N)}\, e$ is the stabilizer of $e \in Y(\mathcal{T})$).

One can also define Hecke operators T_m for each divisor m of A, and Atkin–Lehner involutions acting on the space of automorphic forms $H_!(\mathcal{T}, \mathbf{C})^{\Gamma_0(N)}$ (the T_m are derived from correspondences on the double coset space $\Gamma_0(N)\backslash GL_2(K_\infty)/\mathcal{I}\cdot K_\infty^* \cong Y(\Gamma_0(N)\backslash\mathcal{T})$ in a standard way).

Functions F in $H_!(\mathcal{T}, \mathbf{C})^{\Gamma_0(N)}$ have Fourier expansions and one can associate an L-function to F as

$$L(F, s+1) = \sum_{n \text{ pos. div.}} c(F, n)|n|^{-s},$$

where the sum is over all non-negative divisors, including those with an ∞-component. The purpose of this section is to find a relation between $(\cdot,\cdot)$ and a special value of the L-function of a certain convolution of two automorphic cusp forms. This is done by computing the following Rankin–Selberg integral:

$$\begin{aligned} R &= \int_{Y(\Gamma_0(N)\backslash\mathcal{T})} E_N(e,s)\cdot f(e)\cdot \bar{g}(e)\, d\mu e \\ &= \int_{Y(\Gamma_0(N)\backslash\mathcal{T})} \left(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \varphi(\gamma(e), s) \right)\cdot f(e)\cdot \bar{g}(e)\, d\mu e \\ &= \int_{Y(\Gamma_0(N)\backslash\mathcal{T})} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \varphi(\gamma(e), s)\cdot f(\gamma e)\cdot \bar{g}(\gamma e)\, d\mu e. \end{aligned}$$

The last equality follows from f and g being $\Gamma_0(N)$ invariant.

Since $\#(Stab_{\Gamma_0(N)}\, e) < \infty$ for each $e \in Y(\mathcal{T})$ and since $f(\gamma e)$, $g(\gamma e)$ have compact support mod $\Gamma_0(N)$, only for finitely many $\gamma \in \Gamma_\infty\backslash\Gamma_0(N)$ the inequality $f(\gamma e)\cdot\bar{g}(\gamma e) \neq 0$ holds; thus

$$R = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \int_{\gamma\mathcal{D}_0} \varphi(e,s)\cdot f(e)\cdot \bar{g}(e)\, d\mu e,$$

where $\mathcal{D}_0$ is a fundamental domain for $Y(\Gamma_0(N)\backslash\mathcal{T})$.

Let $\nu e = \frac{q-1}{2}\#(Stab_e\, \Gamma_\infty)^{-1}$ be the measure on $Y(\Gamma_\infty\backslash\mathcal{T})$. Then

$$R = \int_{Y(\Gamma_\infty\backslash\mathcal{T}} \varphi(e,s)\cdot f(e)\cdot \bar{g}(e)\, d\nu e.$$

Write $\nu_+(e) = \frac{q-1}{\# Stab_e\, \Gamma_\infty}$, then (since $f(e)\cdot\bar{g}(e)$ is orientation invariant)

$$R = \int_{Y^+(\Gamma_\infty\backslash\mathcal{T})} \varphi(e,s)\cdot f(e)\cdot \bar{g}(e)\, d\nu_+ e.$$

Γ_∞ has a nice fundamental domain on $Y^+(\mathcal{T})$ given by the set of matrices $\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix}$, $k \in \mathbf{Z}$ and $u \in ((\pi)/(\pi^k))/\mathbf{F}_q^\times$ (i.e., well defined up to the action of $\mathbf{F}_q^{\times}$), cf. [5, p. 375]. One easily computes that

$$v_+(e) = \frac{q-1}{\#Stab_e\,\Gamma_\infty} = \begin{cases} 1, & k>1,\ u\neq 0,\\ \dfrac{1}{q-1}, & k>1,\ u=0,\\ \dfrac{q^{k-1}}{q-1}, & k\leqslant 1. \end{cases}$$

Hence we can rewrite the last integral as a sum

$$\begin{aligned} R = &\frac{1}{q-1}\sum_{k\leqslant 1} q^{k(1-s)-1} f\left(\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix}\right)\cdot \bar{g}\left(\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix}\right)\\ &+\frac{1}{q-1}\sum_{k\geqslant 2}\sum_{u\in(\pi)/(\pi^k)} q^{-ks} f\left(\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\right)\cdot \bar{g}\left(\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\right)\\ = &\frac{1}{q-1}\sum_{k\geqslant 2} q^{-ks}\sum_{u\in(\pi)/(\pi^k)} f\left(\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\right)\cdot \bar{g}\left(\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\right), \end{aligned}$$

where the last equality follows from the fact that, by the cusp form assumption, $f\left(\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix}\right) = c_0(f,\pi^k) = 0$ when $k\leqslant 1$.

Write

$$f\left(\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\right) = c_0(f,\pi^k) + \sum_{\substack{0\neq\alpha\in A\\ \deg\alpha\leqslant k-2}} c(f, div(\alpha)\infty^{k-2})\eta(\alpha u)$$

$$\bar{g}\left(\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\right) = \overline{c_0}(g,\pi^k) + \sum_{\substack{0\neq\beta\in A\\ \deg\beta\leqslant k-2}} \bar{c}(g, div(\beta)\infty^{k-2})\eta(-\beta u).$$

Since $c_0(f,\pi^k) = \overline{c_0}(g,\pi^k) = 0$, we get

$$\begin{aligned} &\sum_{u\in(\pi)/(\pi^k)} f\left(\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\right)\cdot \bar{g}\left(\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\right)\\ &= \sum_{u\in(\pi)/(\pi^k)}\sum_{\substack{0\neq\alpha,\beta\in A\\ \deg\alpha\leqslant k-2\\ \deg\beta\leqslant k-2}} c(f, div(\alpha)\infty^{k-2})\cdot\bar{c}(g, div(\beta)\infty^{k-2})\eta((\alpha-\beta)u)\\ &= q^{k-1}\sum_{\substack{0\neq\alpha\in A\\ \deg\alpha\leqslant k-2}} c(f, div(\alpha)\infty^{k-2})\cdot\bar{c}(g, div(\alpha)\infty^{k-2}), \end{aligned}$$

where the last equality follows from the fact that $\sum_{u\in(\pi)/(\pi^k)}\eta((\alpha-\beta)u) = 0$ unless $\alpha=\beta$, in which case it is equal to q^{k-1}.

Substituting this into the expression for R we get

$$R = \frac{1}{q-1} \sum_{k \geqslant 2} q^{-ks} q^{k-1} \sum_{\substack{0 \neq \alpha \in A \\ \deg \alpha \leqslant k-2}} c(f, div(\alpha) \infty^{k-2}) \cdot \bar{c}(g, div(\alpha) \infty^{k-2})$$

$$= \frac{1}{q-1} \sum_{k \geqslant 0} q^{-(k+2)s} q^{k+1} \sum_{\substack{0 \neq \alpha \in A \\ \deg \alpha \leqslant k}} c(f, div(\alpha) \infty^{k}) \cdot \bar{c}(g, div(\alpha) \infty^{k})$$

$$= q^{1-2s} \sum_{n \text{ pos. div.}} c(f, n) \cdot \bar{c}(g, n) |n|^{-(s-1)} = q^{1-2s} L(f \otimes \bar{g}, s+1), \tag{10}$$

since every effective divisor can be written in the form $div(\alpha)\infty^k$ with $\deg \alpha \leqslant k$, well defined up to a non-zero scalar. Here we use the last equation as a definition for $L(f \otimes \bar{g}, s)$.

Now assume f and g are newforms (i.e., normalized eigenforms for the Hecke algebra which do not arise from cusp forms of lower level). We would like to derive a functional equation for $L(f \otimes \bar{g}, s)$ and relate its special value to the Petersson inner product (f, g). We will assume that the level N is square-free for technical reasons, as was explained in the Introduction.

For any $m \mid N$ (which satisfies $(m, N/m) = 1$ in view of our assumption) there exists the (partial) Atkin–Lehner involution W_m on $X_0(N)$. It is given on $\Gamma_0(N) \backslash \mathscr{T}$ by multiplication from the left with any matrix $\left(\begin{smallmatrix} ma & b \\ Nc & md \end{smallmatrix}\right)$ with $a, b, c, d \in A$ and determinant γm for some $\gamma \in \mathbf{F}_q^{\times}$.

We will especially be interested in W_m when $m = \mathfrak{p}$ is prime. For simplicity represent $W_\mathfrak{p}$ by

$$\beta = \begin{pmatrix} a\mathfrak{p} & -b \\ N & \mathfrak{p} \end{pmatrix}, \quad det\, \beta = \mathfrak{p}.$$

Let $d \mid \frac{N}{\mathfrak{p}}$, then

$$\begin{pmatrix} \frac{N}{\mathfrak{p}d} & 0 \\ 0 & 1 \end{pmatrix} \beta \begin{pmatrix} \frac{N}{d} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} s & \frac{-rN}{d\mathfrak{p}} \\ d & \mathfrak{p} \end{pmatrix} \in GL_2(A). \tag{11}$$

Also one easily verifies that β normalizes $\Gamma_0(N)$. The usefulness of β (among other things) is that newforms of level N are stable under the action of β with eigenvalues ± 1. Write

$$R = \int_{\mathscr{D}_0(N)} E_N(e, s) \cdot f(e) \cdot \bar{g}(e)\, d\mu e$$

$$= \frac{D_N(2s)}{|N|^s} \sum_{\substack{r \mid N \\ r \text{ monic}}} \frac{\mu(r)}{|r|^s} \int_{\mathscr{D}_0(N)} E\left(\begin{pmatrix} N/r & 0 \\ 0 & 1 \end{pmatrix} e, s \right) f(e) \cdot \bar{g}(e)\, d\mu e.$$

Let $d \mid \frac{N}{\mathfrak{p}}$ and consider

$$R_{\mathfrak{p}} = \int_{\mathcal{D}_0(N)} E\left(\begin{pmatrix} N/d\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} e, s\right) f(e) \cdot \bar{g}(e)\, d\mu e$$
$$= \int_{\beta^{-1}\mathcal{D}_0(N)} E\left(\begin{pmatrix} N/d\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} \beta e, s\right) f(\beta e) \cdot \bar{g}(\beta e)\, d\mu \beta e.$$

Since β normalizes $\Gamma_0(N)$, $\beta^{-1}\mathcal{D}_0(N)$ is again a fundamental domain for $\Gamma_0(N)$ and $\#Stab_{\Gamma_0(N)}\, e = \#Stab_{\Gamma_0(N)}\, \beta e$, so the Haar measure does not change. Moreover, as $f(e)$, $g(e)$ are newforms,

$$f(\beta e) \cdot \bar{g}(\beta e) = c_{\mathfrak{p}} f(e) \cdot \bar{g}(e),$$

where $c_{\mathfrak{p}} = \pm 1$.

Using (11), and the fact that $E(e, s)$ is invariant under the action of Γ, we also get

$$E\left(\begin{pmatrix} N/d\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} \beta e, s\right) = E\left(\begin{pmatrix} N/d & 0 \\ 0 & 1 \end{pmatrix} e, s\right).$$

Putting all together,

$$R_{\mathfrak{p}} = c_{\mathfrak{p}} \int_{\mathcal{D}_0(N)} E\left(\begin{pmatrix} N/d & 0 \\ 0 & 1 \end{pmatrix} e, s\right) f(e) \cdot \bar{g}(e)\, d\mu e,$$

and combining with (10)

$$q^{1-2s} L(f \otimes \bar{g}, s+1)$$
$$= D_N(2s) \prod_{\mathfrak{p} \mid N} (1 - c_{\mathfrak{p}} |\mathfrak{p}|^{-s}) \frac{1}{|N|^s}$$
$$\times \int_{\mathcal{D}_0(N)} E\left(\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} e, s\right) f(e) \cdot \bar{g}(e)\, d\mu e. \tag{12}$$

Now assume that $f = g$, and put $a_n := c(f, n)|n|$. In this situation all the $c_{\mathfrak{p}}$ equal 1, in particular $\prod_{\mathfrak{p} \mid N}(1 - c_{\mathfrak{p}}|\mathfrak{p}|^{-s}) = D_N(s)^{-1}$. The assumption that f is a newform implies that $a_1 = 1$, and because the $c(f, n)$ are eigenvalues of the Hecke operators, which are self-adjoint with respect to the Petersson inner product, they are real, so $a_n = \overline{a_n}$. Moreover $L(f, s)$ has Euler product expansion

$$L(f, s) = \sum_{n \text{ pos. div.}} \frac{a_n}{|n|^s} = \prod_{\mathfrak{p}} \left(1 - \frac{\alpha_{\mathfrak{p}}}{|\mathfrak{p}|^s}\right)^{-1} \left(1 - \frac{\beta_{\mathfrak{p}}}{|\mathfrak{p}|^s}\right)^{-1}, \tag{13}$$

where $\alpha_{\mathfrak{p}} = \beta_{\mathfrak{p}} = 0$ if $\mathfrak{p}^2 \mid N$, $\alpha_{\mathfrak{p}} = 0$, $\beta_{\mathfrak{p}} = \pm 1$ if $\mathfrak{p} \parallel N$, and $\alpha_\infty = 0$, $\beta_\infty = 1$. (These formulas hold, and are given for the reader's convenience, in full generality, although the case $\mathfrak{p}^2 \mid N$ is excluded.)

$L(f \otimes f, s)$ also has an Euler product expansion of the following form:

$$\begin{aligned} L(f \otimes f, s) &= \sum_{n \text{ pos. div.}} \frac{a_n^2}{|n|^s} \\ &= \prod_{\mathfrak{p}} \left(1 - \frac{\alpha_{\mathfrak{p}}^2 \beta_{\mathfrak{p}}^2}{|\mathfrak{p}|^{2s}}\right)\left(1 - \frac{\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}}}{|\mathfrak{p}|^s}\right)^{-1}\left(1 - \frac{\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}}}{|\mathfrak{p}|^s}\right)^{-1} \\ &\quad \times \left(1 - \frac{\alpha \mathfrak{p}^2}{|\mathfrak{p}|^s}\right)^{-1}\left(1 - \frac{\beta_{\mathfrak{p}}^2}{|\mathfrak{p}|^s}\right)^{-1}. \end{aligned} \tag{14}$$

Now from Drinfeld's work one knows the "Ramanujan conjecture" for $L(f, s)$, i.e., if $\mathfrak{p} \nmid N \cdot \infty$ then $\alpha_{\mathfrak{p}} = \overline{\beta_{\mathfrak{p}}}$, and $|\alpha_{\mathfrak{p}}| = |\beta_{\mathfrak{p}}| = |\mathfrak{p}|^{1/2}$. Thus

$$\zeta_N(2s-2) \cdot L(f \otimes f, s) = \zeta_N(s-1) \cdot L(Sym^2 f, s), \tag{15}$$

where the local factors $L_{\mathfrak{p}}(Sym^2 f, s)$, following Shimura [20], are defined as

$$L_{\mathfrak{p}}(Sym^2 f, s) = \begin{cases} 1, & \mathfrak{p}^2 \mid N \cdot \infty, \\ \left(1 - \frac{1}{|\mathfrak{p}^s|}\right)^{-1}, & \mathfrak{p} \| N \cdot \infty, \\ \left(1 - \frac{\alpha_{\mathfrak{p}}^2}{|\mathfrak{p}|^s}\right)^{-1}\left(1 - \frac{\alpha_{\mathfrak{p}}\overline{\alpha_{\mathfrak{p}}}}{|\mathfrak{p}|^s}\right)^{-1} & \\ \quad \times \left(1 - \frac{\overline{\alpha_{\mathfrak{p}}}^2}{|\mathfrak{p}|^s}\right)^{-1}, & \mathfrak{p} \nmid N \cdot \infty. \end{cases} \tag{16}$$

From (12) we have

$$\begin{aligned} &\zeta(s) \cdot L(Sym^2 f, s+1) \\ &\quad = \zeta(2s) \frac{q^{2s-1}}{|N|^s} \int_{\mathscr{D}_0(N)} E\left(\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} e, s\right) f(e) \cdot \bar{f}(e) \, d\mu e. \end{aligned} \tag{17}$$

By taking residues on both sides at $s = 1$, and using (7), we arrive at

$$L(Sym^2 f, 2) = q \cdot \frac{\|f\|^2}{|N|}, \tag{18}$$

which is the connection between the Petersson inner product $\|f\|^2$ and the special value of $L(Sym^2 f, s)$ we were looking for. (This is where it was important that the residue of $E(e, s)$ at $s = 1$ did not depend on e.) See Section 6 for examples where $L(Sym^2 f, s)$ is explicitly calculated.

Remark. Compare (18) with the formula over $\mathbf{Q}$ (see [13]):

$$L(Sym^2 f, 2) = 288\pi^3 \frac{\|f\|^2}{N}.$$

Here f is a newform of weight 2, and N is square-free.

Finally, using (6) and (17), we deduce a functional equation for $L(Sym^2 f, s)$, which we will need in the next section, viz., putting

$$\Lambda(s) := q^{1-2s}|N|^s \frac{-\zeta(s+1)\zeta(s)}{\zeta(2s)} L(Sym^2 f, s+1),$$

we have

$$\Lambda(1-s) = -\Lambda(s). \tag{19}$$

Remark. When f has rational eigenvalues this functional equation also follows from the Poincaré duality (see Section 5.2).

5. Bounds on $||f||^2$

In this section we still assume that f is a newform.

5.1. Upper bound

From Drinfeld's results it follows that $L(Sym^2 f, s)$ is a quotient of an L-function arising from a certain non-isotrivial abelian variety (this is essentially the "Shimura construction"). On the other hand, from Grothendieck's cohomological interpretation of L-functions, the latter is known to be a polynomial in q^{-s}. Using this one can show that $L(Sym^2 f, s)$ is holomorphic on the whole complex plane $\mathbf{C}$. (Alternatively, one can use the properties of the Eisenstein series to prove the holomorphicity.)

As was mentioned earlier, one knows the Ramanujan conjecture for $L(f, s)$, hence, if we write

$$L(Sym^2 f, s) = \sum_{n \text{ pos. div.}} \frac{b_n}{|n|^s},$$

then from the Euler product expansion for $L(Sym^2 f, s)$ it is easy to see that for any $\varepsilon > 0$ there exists a constant C_ε depending only on ε (and q), but not on f, such that

$$|b_n| \leqslant C_\varepsilon \cdot |n|^{1+\varepsilon}. \tag{20}$$

We would like to estimate $|L(Sym^2 f, 2)|$ from above. The conventions we use in the following proposition and for the rest of the paper are as follows.

If G and H are any functions depending on f and a complex parameter z, we write $|G(f,z)| \ll |H(f,z)|$ if there exists a constant C such that $|G(f,z)| \leqslant C\,|H(f,z)|$ for all z involved and independently of f. If C depends on the choice of some ε, we write $|G(f,z)| \ll_\varepsilon |H(f,z)|$.

Proposition 5.1. *Fix an arbitrary* $\varepsilon > 0$. *Then for any* $1 - 2\varepsilon < \sigma < 2 + 2\varepsilon$ *and arbitrary real* γ *we have*

$$|L(Sym^2 f, \sigma + i\gamma)| \ll_\varepsilon |N|^{(2+2\varepsilon-\sigma)}.$$

In particular (*after replacing* ε *by* $\frac{1}{2}\varepsilon$),

$$|L(Sym^2 f, 2) \ll_\varepsilon |N|^{\varepsilon}.$$

Proof. The estimate will follow from Rademacher's version of the Phragmén–Lindelöf theorem. First, using (20)

$$\begin{aligned} |L(Sym^2 f, 2 + 2\varepsilon + i\gamma)| &\leqslant \sum_{m=0}^{\infty} q^{-(2+2\varepsilon)m} \left(\sum_{\deg n = m} |b_n| \right) \\ &\leqslant C_\varepsilon \sum_{m=0}^{\infty} q^{-(2+2\varepsilon)m} q^{m+1} q^{m(1+\varepsilon)} \leqslant D_\varepsilon, \end{aligned}$$

with some constant D_ε independent of f. That is,

$$|L(Sym^2 f, 2 + 2\varepsilon + i\gamma)| \ll_\varepsilon 1.$$

Next, using (19) we get

$$|L(Sym^2 f, 1 - 2\varepsilon + i\gamma)| \ll_\varepsilon |N|^{1+4\varepsilon}.$$

Applying Theorem 2 in [18],

$$|L(Sym^2 f, \sigma + i\gamma)| \ll_\varepsilon (|N|^{1+4\varepsilon})^{\frac{2+2\varepsilon-\sigma}{1+4\varepsilon}} = |N|^{(2+2\varepsilon-\sigma)}. \qquad \square$$

The above proposition and (18) imply

Corollary 5.2.

$$\|f\|^2 \ll_\varepsilon |N|^{1+\varepsilon}.$$

5.2. *Upper bound on* $\|f\|^2$ *when* f *has rational eigenvalues*

When the eigenvalues of f are rational, $L(Sym^2 f, s)$ is a polynomial in q^{-s} whose degree is not hard to compute. We carry out this calculation in the present section. The knowledge of the degree is useful in practice when one actually tries to compute $L(Sym^2 f, s)$ (see Section 6), and also allows to get Corollary 5.2 avoiding analytic methods.

It is known (see [6] or [8]) that in the case when the eigenvalues of f are rational there is a non-isotrivial elliptic curve E defined over $K = \mathbf{F}_q(T)$, with split multiplicative reduction at ∞, and conductor $N_E = N \cdot \infty$, such that

$$L_\mathfrak{p}(f, s) = L_\mathfrak{p}(E, s). \tag{21}$$

Here $L_{\mathfrak{p}}(f,s)$ are the local Euler factors in (13), and $L_{\mathfrak{p}}(E,s)$ are defined as follows:

Let $\ell \neq p = char(\mathbf{F}_q)$ be a prime number, and let $T_\ell(E) = \varprojlim E_{\ell^n}$ be the ℓ-adic Tate module of E, $V_\ell(E) = T_\ell(E) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$, and $V_\ell(E)^\vee = Hom_{\mathbf{Q}_\ell}(V_\ell(E), \mathbf{Q}_\ell)$. For a place $\mathfrak{p}$ of K one defines

$$L_{\mathfrak{p}}(E,s) = det(1 - q_{\mathfrak{p}}^{-s} Frob_{\mathfrak{p}} \mid (V_\ell(E)^\vee)^{I_{\mathfrak{p}}})^{-1},$$

where $I_{\mathfrak{p}}$ is the inertia subgroup at $\mathfrak{p}$, and $q_{\mathfrak{p}}$ is the order of the residue field at $\mathfrak{p}$, i.e., $q_{\mathfrak{p}} = q^{\deg \mathfrak{p}}$.

Now define

$$\begin{aligned} L(Sym^2\, E, s) &= \prod_{\mathfrak{p}} L_{\mathfrak{p}}(Sym^2\, E, s) \\ &= \prod_{\mathfrak{p}} det(1 - q_{\mathfrak{p}}^{-s} Frob_{\mathfrak{p}} \mid (Sym^2\, V_\ell(E)^\vee)^{I_{\mathfrak{p}}})^{-1}. \end{aligned}$$

Lemma 5.3. *If N is square-free then for all places $\mathfrak{p}$ of K there is an equality*

$$L_{\mathfrak{p}}(Sym^2 f, s) = L_{\mathfrak{p}}(Sym^2\, E, s),$$

where $L_{\mathfrak{p}}(Sym^2 f, s)$ is defined in (16).

Proof. If $\mathfrak{p}$ is a place where E has good reduction, then by (the easy half of) the Néron–Ogg–Shafarevich criterion, $(V_\ell(E)^\vee)^{I_{\mathfrak{p}}} = V_\ell(E)^\vee$, and the statement easily follows from (21).

Now let $\mathfrak{p}$ be a place where E has bad (multiplicative by assumption) reduction. It is easy to see that the ℓ-adic representation $Sym^2\, V_\ell(E)^\vee$ does not change when we replace E by a quadratic twist. So we can assume that the reduction is split multiplicative.

It is well known from the theory of Tate curves that we have a non-split exact sequence of $I_{\mathfrak{p}}$-modules

$$0 \to V_\ell(\mu) \to V_\ell(E) \to \mathbf{Q}_\ell \to 0. \tag{22}$$

Now, for any elliptic curve E, the Weil pairing shows that

$$Sym^2(V_\ell(E)^\vee) = Sym^2(V_\ell(E)) \otimes V_\ell(\mu)^{\otimes(-2)}.$$

This observation and the fact that (22) does not split as a sequence of $I_{\mathfrak{p}}$-modules easily imply that $Sym^2(V_\ell(E)^\vee)^{I_{\mathfrak{p}}} \cong \mathbf{Q}_\ell$. Hence

$$L_{\mathfrak{p}}(Sym^2\, E, s) = (1 - q_{\mathfrak{p}}^{-s})^{-1} = L_{\mathfrak{p}}(Sym^2 f, s). \qquad \square$$

Remark. When N is not square-free then $L(Sym^2 f, s)$ and $L(Sym^2\, E, s)$, as defined above, in general agree only up to the local factors of places of additive reduction.

$V_\ell(E)^\vee$, and thus its Sym^2, defines a constructible ℓ-adic sheaf over $\mathbf{P}^1_{\mathbf{F}_q}$ which is twisted-constant at the places of $\mathbf{P}^1_{\mathbf{F}_q}$ where E has a good reduction.

Grothendieck's theory of L-functions implies that

$$L(Sym^2\, E, s) = \frac{P_1(q^{-s})}{P_0(q^{-s})P_2(q^{-s})},$$

where

$$P_j(X) = det(1 - X\, Frob_q \mid H^j_{et}(\mathbf{P}^1_{\bar{\mathbf{F}}_q}, Sym^2(V_\ell(E)^\vee))).$$

It is known (thanks to Deligne's results in "Weil II") that for non-isotrivial curves

$$P_j(X) = 1 \quad \text{for } j = 0, 2.$$

We would like to compute the degree of $L(Sym^2\, E, s)$ as a polynomial in q^{-s}, which is equal to

$$d = \deg P_1(X) = dim_{\mathbf{Q}_\ell}(H^1_{et}(\mathbf{P}^1_{\bar{\mathbf{F}}_q}, Sym^2(V_\ell(E)^\vee))).$$

Proposition 5.4.

$$d = 2\deg N_E - 6 = 2\deg N - 4.$$

Proof. To compute d one has to compute the conductor of $Sym^2(V_\ell(E)^\vee)$. To do so we use the Grothendieck–Ogg–Shafarevich formula ([16, Theorem V.2.12]). It gives

$$d = \sum_{\mathfrak{p} \in \mathbf{P}^1_{\mathbf{F}_q}} (\varepsilon_\mathfrak{p} + \delta_\mathfrak{p}) - 6,$$

where $\varepsilon_\mathfrak{p} = \deg(\mathfrak{p})(3 - dim_{\mathbf{Q}_\ell}(Sym^2(V_\ell(E)^\vee)^{I_\mathfrak{p}}))$ is the tame part of the conductor, and the wild part $\delta_\mathfrak{p}$ is 0 since we have assumed E is semi-stable.

Now the claim follows from the proof of the previous lemma. □

This proposition essentially computes the Euler–Poincaré characteristic of $Sym^2\, V_\ell(E)^\vee$. Now the Poincaré duality (cf. [16, Chapter VI]) implies the functional equation of $L(Sym^2\, E, s)$:

$$q^{-\chi(Sym^2\, V_\ell(E)^\vee)\cdot(2s-3/2)} L(Sym^2\, E, s) = L(Sym^2\, E, 3 - s),$$

which explicitly is

$$q^{(\deg N-2)\cdot(2s-3)} L(Sym^2\, E, s) = L(Sym^2\, E, 3 - s).$$

One easily checks that this coincides with (19). (Actually the Poincaré duality does not imply that the sign of the functional equation is always $+1$, but this can be deduced from the theory of local ε-factors.)

Using this proposition we also can deduce Corollary 5.2. Indeed,

$$|L(Sym^2 f,2)| = \left|\sum_{n=0}^{d} q^{-2n}\left(\sum_{\substack{\deg m=n \\ m \text{ pos. div.}}} b_m\right)\right|$$

$$\leqslant C_\varepsilon \sum_{n=0}^{d} q^{-2n} q^{n+1} q^{n(1+\varepsilon)} \leqslant C_\varepsilon q^{(d+1)\varepsilon} = C_\varepsilon q^{(2\deg N-3)\varepsilon},$$

i.e.,

$$|L(Sym^2 f,2)| \ll_\varepsilon |N|^\varepsilon.$$

5.3. *Lower bound*

Using (12) and (7) we have

$$\|f\|^2 \gg |N| \operatorname*{Res}_{s=2} L(f\otimes f,s).$$

This subsection consists of bounding $R := \operatorname{Res}_{s=2} L(f\otimes f,s)$ from below. The idea is the same as in the proof of Siegel's Theorem by Goldfeld [9], see also [10].

Proposition 5.5.

$$R \gg_\varepsilon \frac{1}{\deg N}.$$

Proof. First, we seek a bound on $L(f\otimes f,s)$ on the line $\frac{3}{2}+i\gamma$, $\gamma\in\mathbf{R}$. From Proposition 5.1,

$$|L(Sym^2 f,3/2+i\gamma)| \ll_\varepsilon |N|^{1/2+\varepsilon}.$$

Also

$$\left|1+\frac{1}{|\mathfrak{p}|^{1/2}|\mathfrak{p}|^{i\gamma}}\right| \geqslant 1-\frac{1}{|\mathfrak{p}|^{1/2}} > \frac{1}{5}.$$

Hence by (15),

$$|L(f\otimes f,3/2+i\gamma)| \ll_\varepsilon |N|^{1/2+\varepsilon}.5^{\#\{\mathfrak{p}\,|\,\mathfrak{p}|N\}}$$
$$\ll_\varepsilon |N|^{1/2+\varepsilon}.5^{\deg N/\log \deg N} \ll_\varepsilon |N|^{7/2+\varepsilon}. \qquad (23)$$

Let $\frac{3}{2}<\beta<2$ and $x>0$. Then

$$\frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} \frac{L(f\otimes f,s+\beta)x^s}{s(s+1)}\,ds = \sum_{\substack{\deg n<\log x \\ n \text{ pos. div.}}} \frac{a_n^2}{|n|^\beta}\left(1-\frac{|n|}{x}\right).$$

Since $a_1 = 1$, we have for all $x \geqslant 2$,

$$1 \ll \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} \frac{L(f\otimes f, s+\beta)x^s}{s(s+1)}\,ds. \tag{24}$$

Shifting the line of integration to $\mathrm{Re}(s) = \frac{3}{2} - \beta < 0$, we pick up the residues at $s = 0,\ 2-\beta$.

Using bound (23), we see that the right-hand side of (24) becomes

$$\frac{Rx^{2-\beta}}{(2-\beta)(3-\beta)} + L(f\otimes f, \beta) + O_\varepsilon(|N|^4 x^{3/2-\beta}).$$

Taking $x = |N|^{4\cdot c}$, for a sufficiently large constant c, we get

$$1 \ll_\varepsilon \frac{R|N|^{4\cdot c(2-\beta)}}{2-\beta} + L(f\otimes f, \beta).$$

Take

$$2 - \beta = \frac{1}{4\deg N}.$$

From Deligne's fundamental results in "Weil II" one knows the Riemann hypothesis for $L(f\otimes f, s)$. In particular it has no real zeros in $(\frac{3}{2}, 2)$. And since $L(f\otimes f, s)$ is positive for $\mathrm{Re}\, s > 2$ and has a simple pole at $s = 2$, we must have $L(f\otimes f, 2 - \frac{1}{4\deg N}) < 0$. This yields $1 \ll_\varepsilon R\deg N$, and finally

$$\frac{1}{\deg N} \ll_\varepsilon R. \qquad \square$$

Corollary 5.6. *For any* $\varepsilon > 0$

$$|N|^{1-\varepsilon} \ll_\varepsilon \|f\|^2.$$

6. Main theorem

Let E be an optimal elliptic curve over $\mathbf{Q}$ with conductor N, and let $X_0(N)$ be the modular curve such that

$$\wp : X_0(N) \to E$$

is non-trivial and of minimal possible degree. Optimal means that $\deg \wp$ is minimal for E in the isogeny class of E (such a curve always exists), or alternatively, that any $\wp' : X_0(N) \to E'$, with E' isogenous to E, factors through $\wp$. The *degree conjecture* claims that

$$\frac{\deg \wp}{c_E^2} \ll_\varepsilon N^{2+\varepsilon},$$

(i.e., there exists C_ε, independent of the curve E, such that $\frac{\deg \wp}{c_E^2} \leqslant C_\varepsilon N^{2+\varepsilon}$), where c_E is the Manin constant (conjecturally $= \pm 1$). It is well known that the degree conjecture is equivalent to the ABC-conjecture; see [2,13,14,19].

Combining the results of Frey, Silverman and others one can prove a lower bound (cf. [14,19]:

$$\frac{\deg \wp}{c_E^2} \gg_\varepsilon N^{7/6-\varepsilon}.$$

One can also rewrite these bounds in a more geometric form

$$\log g(X_0(N)) \ll_\varepsilon \log(\deg \wp) \overset{?}{\ll}_\varepsilon \log g(X_0(N)). \tag{25}$$

In this section we will prove the analogs of these bounds for $\mathbf{F}_q(T)$. Let us first recall the analog of modular parametrization.

Let C be the completed algebraic closure of K_∞. The function field analog of the upper-half plane is the *Drinfeld upper-half plane*

$$\Omega = \mathbf{P}^1(C) - \mathbf{P}^1(K_\infty) = C - K_\infty.$$

It has a natural structure of a rigid analytic space over K_∞. Drinfeld proved that there exists a smooth affine algebraic curve $Y_0(N)$ defined over K such that $\Gamma_0(N)\backslash\Omega$ is isomorphic, as an analytic space, to the analytification $Y_0(N)^{\mathrm{an}}$ of $Y_0(N)$. Let $X_0(N)$ be the smooth projective model of $Y_0(N)$; it is called a *Drinfeld modular curve* (of level N):

$$X_0(N) = \Gamma_0(N)\backslash\Omega \cup \{\text{cusps}\}.$$

Let $J_0(N)$ denote the Jacobian variety of $X_0(N)$. As a result of several deep theories there are canonical bijections between the sets of

(a) normalised Hecke eigenforms f in $H_!^{\mathrm{new}}(\mathscr{T}, \mathbf{Q})^{\Gamma_0(N)}$ with rational eigenvalues,
(b) one-dimensional isogeny factors of $J_0^{\mathrm{new}}(N)$,
(c) isogeny classes of elliptic curves E/K with conductor $N_E = N \cdot \infty$, and split multiplicative reduction at ∞.

"New" here has the same meaning as over $\mathbf{Q}$, with $H_!$ being replaced by the space of cusp forms of weight 2, for details see [8]. Moreover, the relation between L-functions of corresponding f and E is

$$L(E, s) = L(f, s).$$

Hence for any elliptic curve E over K with split multiplicative reduction at ∞ and conductor $N \cdot \infty$ there is a non-trivial morphism

$$\wp : X_0(N) \to E.$$

Each isogeny class contains a unique curve for which $\deg \wp$ is minimal. Such a curve is again called *optimal* (or strong Weil curve).

Gekeler and Reversat came up with an explicit description of $\wp : X_0(N) \to E$ (see [6] or [8]).

Let $\widetilde{\Gamma_0(N)} := \Gamma_0(N)/(K_\infty^* \cap \Gamma_0(N))$, and let $\bar{\Gamma} := \Gamma_0(N)^{\mathrm{ab}}/(\Gamma_0(N)^{\mathrm{ab}})_{\mathrm{tors}}$ be the maximal torsion-free abelian quotient of $\Gamma_0(N)$. Let $\omega \in \Omega$ be an arbitrary base point and $\alpha \in \Gamma_0(N)$. Put

$$u_\alpha(z) = \widetilde{\prod_{\gamma \in \Gamma_0(N)}} \left(\frac{z - \gamma\omega}{z - \gamma\alpha\omega} \right).$$

Then one can show that $u_\alpha(z)$ converges locally uniformly to an invertible function u_α on Ω that does not depend on the choice of $\omega \in \Omega$, and depends only on the class of $\bar{\alpha}$ of α in $\bar{\Gamma}$.

Also one can show [7] that there is a canonical isomorphism

$$i : \bar{\Gamma} \cong H_!(\mathcal{T}, \mathbf{Z})^{\Gamma_0(N)}.$$

Let $f \in H_!^{\mathrm{new}}(\mathcal{T}, \mathbf{Z})^{\Gamma_0(N)}$ be primitive (i.e., $f \notin nH_!(\mathcal{T}, \mathbf{Z})^{\Gamma_0(N)}$ for $n > 1$) with rational eigenvalues. Write f also for a representative of its preimage $i^{-1}(f) \in \bar{\Gamma}$ in $\Gamma_0(N)$.

The Gekeler–Reversat theorem is summarized in the following commutative diagram:

$$\begin{array}{ccccc}
\Omega & & \xrightarrow{\quad u_f \quad} & & C^* \\
\downarrow & & & & \downarrow \\
\Gamma_0(N) \backslash \Omega & & & & C^*/q_E^{\mathbf{Z}} \\
\| & & & & \| \\
Y_0(N)(C) & \hookrightarrow & X_0(N)(C) & \xrightarrow{\wp} & E_f(C),
\end{array}$$

where q_E is the Tate period of E at ∞, and E_f is optimal for f. Moreover Gekeler proves in [6, Proposition 3.8] that for such E

$$\deg \wp = \frac{\|f\|^2}{-val_\infty(j_E)}, \tag{26}$$

where j_E is the j-invariant of E.

Remark. If one takes $\int_{i\infty}^{z} f(s)\, ds$, with f the newform for $E/\mathbf{Q}$, to be the analog of $u_f(z)$, then Gekeler's argument has the same flavor as deducing in the classical setting

$$4\pi^2 c_E^2(f,f) = 2\pi^2 i \int_{X_0(N)(\mathbf{C})} f(\tau)\, d\tau \wedge \overline{f(\tau)\, d\tau} = \deg \wp \, \frac{i}{2} \int_{E_f(\mathbf{C})} dz \wedge \overline{dz}$$

$$= \deg \wp \int_{E(\mathbf{C})} \omega_E \wedge \overline{\omega_E} = (\deg \wp) e^{-2h(E/\mathbf{Q})},$$

where $h(E/\mathbf{Q})$ is the Faltings height, and c_E is Manin's constant.

Combining (26) with (18) we already get something interesting, namely

$$\deg \wp = \frac{q^{\deg N - 1}}{-val_\infty(j_E)} L(Sym^2\, E, 2). \tag{27}$$

The usefulness of this formula is that it allows quick computation of $\deg \wp$. Indeed, $L(Sym^2\, E, s)$ is a polynomial in q^{-s} whose degree is $2 \deg N - 4$, and the local Euler factors of $L(Sym^2\, E, s)$ are the Sym^2 of the local Euler factors of E. So to compute $L(Sym^2\, E, 2)$ it is enough to compute the number of points of the reduction of E at the places of degree up to $2 \deg N - 4$ (or degree up to $\deg N - 2$ using the functional equation (19)). We demonstrate this on three concrete examples which can be found at the end of [6]; the L-function calculations below were done using a short program written on Magma.

Example. Fix $q = 2$. Let first $N = T(T^2 + T + 1)$. Then, as shown in [6], $J_0(N)$ is isogeneous to $E_1 \times E_2$ with

$$E_1 : Y^2 + (T+1)XY + Y = X^3 + T(T^2 + T + 1),$$

$$j(E_1) = \frac{(T+1)^{12}}{T^3(T^2 + T + 1)^3},$$

$$E_2 : Y^2 + (T+1)XY + Y = X^3 + X^2 + T + 1,$$

$$j(E_2) = \frac{(T+1)^{12}}{T^5(T^2 + T + 1)}.$$

Now by computer calculations one obtains

$$L(Sym^2\, E_1, s) = 8q^{-2s} + 1, \qquad L(Sym^2\, E_1, 2) = \tfrac{3}{2},$$

$$L(Sym^2\, E_2, s) = 8q^{-2s} + 4q^{-s} + 1, \qquad L(Sym^2\, E_2, 2) = \tfrac{5}{2}.$$

Thus by (27)

$$\deg \wp_1 = \frac{2^2}{3} \cdot \frac{3}{2} = 2,$$

$$\deg \wp_2 = \frac{2^2}{5} \cdot \frac{5}{2} = 2.$$

In fact, Gekeler shows in [6] that E_1 and E_2 are involutory, i.e., quotients of $X_0(N)$ by certain Atkin–Lehner involutions.

Next take $N = T^4 + T^3 + 1$. Then for the optimal curve

$$E : Y^2 + TXY + Y = X^3 + X^2,$$

$$j(E) = \frac{T^{12}}{N},$$

$$L(Sym^2\, E, s) = 64q^{-4s} + 16q^{-3s} + 2q^{-s} + 1, \qquad L(Sym^2\, E, 2) = 2.$$

Thus

$$\deg \wp = \frac{2^3}{8} \cdot 2 = 2.$$

We conclude with an example of an optimal curve in odd characteristic ($q = 7$) (again found by Gekeler):

$$E/\mathbf{F}_7(T) : Y^2 = X^3 - 3T(T^3 + 2)X - 2T^6 + 3T^3 + 1.$$

One computes

$$val_\infty(j_E) = -3, \qquad N_E = (T^3 - 2) \cdot \infty,$$

$$L(Sym^2\, E, s) = 343q^{-2s} - 17q^{-s} + 1, \qquad L(Sym^2\, E, 2) = \tfrac{39}{49}.$$

So

$$\deg \wp = \frac{7^2}{3} \cdot \frac{39}{49} = 13. \qquad \square$$

Now we are ready to prove the main result of this paper. Let Δ be the minimal discriminant of E, and $\deg_{\mathrm{ns}}(j_E)$ be the inseparable degree of $K/\mathbf{F}_q(j_E)$.

Theorem 6.1. *Let E and $\wp$ be as above. Then for any $\varepsilon > 0$*

$$\frac{|N|^{1-\varepsilon}}{\deg_{\mathrm{ns}}(j_E)} \ll_\varepsilon \deg \wp \ll_\varepsilon |N|^{1+\varepsilon}.$$

Proof. By the Tate algorithm, $-val_\infty(j_E)$ is the number of irreducible components of the Néron model of E at ∞. Hence

$$1 \leqslant -val_\infty(j_E) = \deg \Delta \leqslant 6 \deg_{\mathrm{ns}}(j_E) \deg N. \tag{28}$$

The last inequality is the celebrated *Szpiro bound* in the case of rational function fields (see [17] for the proof in the most general situation, without any assumptions on the reduction types of E or the characteristic of K). Now the result follows from (26), Corollaries 5.2, and 5.6. $\square$

Let us rewrite the above theorem in a geometric form, nicely comparable with (25). Again denote by $g(X_0(N))$ the genus of the Drinfeld modular curve.

Corollary 6.2.

$$\log \frac{g(X_0(N))}{\deg_{\mathrm{ns}}(j_E)} \ll_\varepsilon \log(\deg \wp) \ll_\varepsilon \log g(X_0(N))$$

Proof. A formula for the genus $g(X_0(N))$ was derived by Gekeler [3]. From this one easily shows that

$$\deg N \ll \log g(X_0(N)) \ll \deg N,$$

and the result follows. $\square$

It is natural to ask whether the "unpleasant" $\deg_{ns}(j_E)$ can be removed from the lower bound in Theorem 6.1. We conclude with few remarks on this.

First of all, $\deg_{ns}(j_E)$ cannot be removed from Szpiro's bound which is valid for an arbitrary elliptic curve. Indeed, fix an elliptic curve E of conductor N_E and consider the infinitely many elliptic curves E_n isogenous to E by a power of the Frobenius morphism, i.e., $E_n = Frob^n(E)$. Then $\deg \Delta_{E_n} = q^n \deg \Delta_E$, but $N_{E_n} = N_E$.

On the other hand, optimal curves are rather special in their isogeny class. Observe that $\deg \Delta$ is very closely related to the Parshin–Faltings height of E. Over the rational numbers extensive computations show that the optimal curves tend to have Tate–Shafarevich group of minimal size in their isogeny class; see papers by John Cremona. Also Glenn Stevens [22] conjectures that optimal curves for $X_1(N)_{/\mathbf{Q}}$ have minimal Faltings height in their isogeny class (supported by explicit computations for $N \leqslant 200$).

Motivated by this one could expect that for optimal curves $\deg_{ns}(j_E)$ is minimal in its isogeny class, that is, equal to 1 (observe that each isogeny class does contain a curve for which the morphism induced by $j_E : \mathbf{P}^1 \to \mathbf{P}^1$ is separable; this can be seen most easily when the characteristic is larger than 3). Unfortunately, such an optimistic conjecture is false. For example, $X_0(T^3)_{/\mathbf{F}_2(T)}$ is the elliptic curve $y^2 + Txy = x^3 + T^2x$ with $j = T^4$. Nevertheless, the few explicit examples of optimal curves which I know seem to suggest that this is true for semi-stable curves.

Here is a strategy which, if carried out, will prove this for curves with prime conductor, i.e., $N_E = \mathfrak{p} \cdot \infty$ where $\mathfrak{p}$ is prime. First, observe that if j_E has a non-trivial non-separable degree then $j_E = g(T)^p$, where $g(T) \in \mathbf{F}_q(T)$. This implies that $p \mid (-val_{\mathfrak{p}} j_E)$, and the latter quantity is the order of the geometric group of connected components Φ_E of the Néron model of E at $\mathfrak{p}$.

Let $\Phi_{\mathfrak{p}}$ be the group of connected components of the Néron model of $J_0(\mathfrak{p})$ at $\mathfrak{p}$. Let $\Phi_{\mathfrak{p}} \to \Phi_E$ be the map induced from the surjection $J_0(\mathfrak{p}) \to E$. If this map were surjective then we would get a contradiction. Indeed, one can compute the order of $\Phi_{\mathfrak{p}}$ using the results of Raynaud on specialization of Jacobians, see [4]; we have $\#\Phi_{\mathfrak{p}} = g(X_0(\mathfrak{p})) + 1$, and this is always coprime to the characteristic. Hence the surjectivity would imply the same for Φ_E and so p could not divide its order.

The question whether $\Phi_{\mathfrak{p}} \to \Phi_E$ is surjective seems to be rather deep. Over $\mathbf{Q}$ a similar statement is known to be true, cf. [1,15]. The proofs use the full force of Mazur's theory of the Eisenstein ideal. This theory in the function field setting still needs to be developed (Tamagawa [23] has some results in this direction but they do not seem to be enough to approach this question).

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