

Genus formula for modular curves of \mathcal{D} -elliptic sheaves

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Abstract. We prove a genus formula for modular curves of \mathcal{D} -elliptic sheaves. We use this formula to show that the reductions of modular curves of \mathcal{D} -elliptic sheaves attain the Drinfeld-Vladut bound as the degree of the discriminant of \mathcal{D} tends to infinity.

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1. Introduction. The genus formulas for classical modular curves and Shimura curves are well-known and quite useful in many arithmetic problems.

Drinfeld modular curves [4] play a role over function fields of positive characteristic similar to classical modular curves over \mathbb{Q} . Many invariants of Drinfeld modular curves, including their genera, were calculated by Gekeler in the 80's, cf. [5], [8]. The function field analogue of Shimura curves was introduced by Laumon, Rapoport and Stuhler in [10]. These curves are moduli spaces of certain objects, called \mathcal{D} -elliptic sheaves, which generalize the notion of Drinfeld modules. The primary purpose of the present paper is to produce a genus formula for modular curves of \mathcal{D} -elliptic sheaves (Theorem 5.4). We also compute some other invariants of these curves, such as the number of supersingular and elliptic points. It turns out, maybe not surprisingly, that the genus formula for modular curves of \mathcal{D} -elliptic sheaves is strikingly similar to the genus formula for Shimura curves. In the final section of the paper we discuss an application of main results: we construct a new sequence of curves over finite fields which is asymptotically optimal.

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In view of the fact that only a few general families of such sequences are known, this result is of independent interest.

The paper is organized as follows: In Section 2, we fix the notation and conventions which are used throughout the article. In Section 3, we recall the definition of \mathcal{D} -elliptic sheaves and the main geometric properties of their moduli schemes. In Section 4, we study the points on modular curves of \mathcal{D} -elliptic sheaves over finite fields. This relies on an analogue of the Honda-Tate theory developed in [10]. Here we compute the number of \mathcal{D} -elliptic sheaves having extra automorphisms or large endomorphism algebras. These calculations are used in the proof of the genus formula, and also in the construction of asymptotically optimal sequences of curves. In Section 5, we compute the genus of modular curves of \mathcal{D} -elliptic sheaves. Finally, in Section 6, we recall the definition of asymptotically optimal sequences of curves over finite fields and discuss how to construct such sequences using the modular curves of \mathcal{D} -elliptic sheaves.

2. Notation. Let \mathbb{F}_q be the finite field with q elements. Let $C := \mathbb{P}_{\mathbb{F}_q}^1$ be the projective line over \mathbb{F}_q . Denote by $F = \mathbb{F}_q(T)$ the field of rational functions on C . The set of closed points on C (equivalently, places of F) is denoted by $|C|$. For each $x \in |C|$, we denote by \mathcal{O}_x and F_x the completions of $\mathcal{O}_{C,x}$ and F at x , respectively. The residue field of \mathcal{O}_x is denoted by \mathbb{F}_x , the cardinality of \mathbb{F}_x is denoted by q_x , the degree m extension of \mathbb{F}_x is denoted by $\mathbb{F}_x^{(m)}$, and $\deg(x) := \dim_{\mathbb{F}_q}(\mathbb{F}_x)$. We assume that the valuation $\text{ord}_x : F_x \rightarrow \mathbb{Z}$ is normalized by $\text{ord}_x(\varpi_x) = 1$, where ϖ_x is a fixed uniformizer of \mathcal{O}_x ; the norm $|\cdot|_x$ on F_x is $q_x^{-\text{ord}_x(\cdot)}$. Denote the adèle ring of F by $\mathbb{A} := \prod'_{x \in |C|} F_x$.

Let $\infty := 1/T$ and $A := \mathbb{F}_q[T]$ be the subring of F consisting of functions which are regular away from ∞ . The completion of an algebraic closure of F_∞ is denoted \mathbb{C}_∞ . For each $x \in |C| - \infty$, we denote by $\mathfrak{p}_x \triangleleft A$ the corresponding prime ideal of A . For an ideal $\mathfrak{n} \triangleleft A$, we define its degree as $\deg(\mathfrak{n}) := \dim_{\mathbb{F}_q}(A/\mathfrak{n})$. Note that $\deg(\mathfrak{p}_x) = \deg(x)$. The adèle ring outside of ∞ is $\mathbb{A}^\infty := \prod'_{x \in |C| - \infty} F_x$. In particular, $\mathbb{A} = \mathbb{A}^\infty \times F_\infty$.

Given a ring H , we denote by H^\times the group of its units.

Let D be a quaternion algebra over F , i.e., a 4-dimensional central simple algebra over F . For $x \in |C|$, we let $D_x := D \otimes_F F_x$. We assume throughout the paper that D is split at ∞ , i.e., $D_\infty \cong \mathbb{M}_2(F_\infty)$. (Here \mathbb{M}_2 is the ring of 2×2 matrices.) Let R be the set of places where D is ramified. It is a well-known fact that the cardinality of R is even, and conversely, for any choice of a finite set $R \subset |C|$ of even cardinality there is a unique quaternion algebra ramified exactly at the places in R ; see [17, p. 74]. If $R \neq \emptyset$, then D is a division algebra; if $R = \emptyset$, then $D \cong \mathbb{M}_2(F)$. For $R \neq \emptyset$, define the ideal $\mathfrak{r} := \prod_{x \in R} \mathfrak{p}_x$. Let D^\times be the algebraic group over F defined by $D^\times(B) = (D \otimes_F B)^\times$ for any F -algebra B ; this is the multiplicative group of D .

For a closed subscheme I of C with ideal sheaf \mathcal{I} , let $\mathcal{O}_I := \mathcal{O}_C/\mathcal{I}$. Fix a locally free sheaf \mathcal{D} of \mathcal{O}_C -algebras with stalk at the generic point equal to D and such that $\mathcal{D}_x := \mathcal{D} \otimes_{\mathcal{O}_C} \mathcal{O}_x$ is a maximal order in D_x . Denote $\mathcal{D}^\infty := \prod_{x \in |C| - \infty} \mathcal{D}_x$. For a finite nonempty closed subscheme I of $C - R - \infty$, let

$$\mathcal{D}_I := \mathcal{D} \otimes_{\mathcal{O}_C} \mathcal{O}_I \quad \text{and} \quad \mathcal{K}_I^\infty := \ker((\mathcal{D}^\infty)^\times \rightarrow \mathcal{D}_I^\times).$$

3. \mathcal{D} -elliptic sheaves and their moduli schemes. Let W be an \mathbb{F}_q -scheme. Denote by Frob_W its Frobenius endomorphism, which is the identity on the points and the q -th power map on the functions. Denote by $C \times W$ the fibred product $C \times_{\text{Spec}(\mathbb{F}_q)} W$. For a sheaf \mathcal{F} on C and \mathcal{G} on W , the sheaf $\text{pr}_1^*(\mathcal{F}) \otimes \text{pr}_2^*(\mathcal{G})$ on $C \times W$ is denoted by $\mathcal{F} \boxtimes \mathcal{G}$.

Let $z : W \rightarrow C$ be a morphism of \mathbb{F}_q -schemes such that $z(W) \subset C - R - \infty$. A \mathcal{D} -elliptic sheaf over W , with pole ∞ and zero z , is a sequence $\mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i \in \mathbb{Z}}$, where each \mathcal{E}_i is a locally free sheaf of $\mathcal{O}_{C \times W}$ -modules of rank 4 equipped with a right action of \mathcal{D} compatible with the \mathcal{O}_C -action, and where

$$\begin{aligned} j_i &: \mathcal{E}_i \rightarrow \mathcal{E}_{i+1} \\ t_i &: {}^\tau \mathcal{E}_i := (\text{Id}_C \times \text{Frob}_W)^* \mathcal{E}_i \rightarrow \mathcal{E}_{i+1} \end{aligned}$$

are injective $\mathcal{O}_{C \times W}$ -linear homomorphisms compatible with the \mathcal{D} -action. The maps j_i and t_i are sheaf modifications at ∞ and z , respectively, which satisfy certain conditions. We refer to [10, §2] for the precise definition.

Let I be a closed subscheme of $C - R - \infty$. Let $\mathbb{E} = (\mathcal{E}_i, j_i, t_i)$ be a \mathcal{D} -elliptic sheaf over W . Assume $z(W)$ is disjoint from I . The restriction $\mathcal{E}_I := \mathcal{E}_i|_{I \times W}$ is independent of i , and t induces an isomorphism ${}^\tau \mathcal{E}_I \cong \mathcal{E}_I$. A *level- I structure* on \mathbb{E} is an $\mathcal{O}_{I \times W}$ -linear isomorphism $\iota : \mathcal{D}_I \boxtimes \mathcal{O}_W \cong \mathcal{E}_I$, compatible with the action of \mathcal{D}_I , which makes the following diagram commutative:

$$\begin{array}{ccc} {}^\tau \mathcal{E}_I & \xrightarrow{t} & \mathcal{E}_I \\ & \swarrow \tau_\iota & \searrow \iota \\ & \mathcal{D}_I \boxtimes \mathcal{O}_W & \end{array}$$

Denote by $\mathbf{Ell}_I^{\mathcal{D}}(W)$ the category whose objects are the \mathcal{D} -elliptic sheaves over W with level I -structures and whose morphisms are isomorphisms. (If $I = \emptyset$, then $\mathbf{Ell}_I^{\mathcal{D}}(W)$ is simply the category of \mathcal{D} -elliptic sheaves over W .) There is a natural free action of \mathbb{Z} on $\mathbf{Ell}_I^{\mathcal{D}}(W)$: $n \in \mathbb{Z}$ acts by

$$[n](\mathcal{E}_i, j_i, t_i; \iota) = (\mathcal{E}_{i+n}, j_{i+n}, t_{i+n}; \iota).$$

We denote the quotient category by $\mathbf{Ell}_I^{\mathcal{D}}(W)/\mathbb{Z}$. There is also an action of \mathcal{D}_I^\times on $\mathbf{Ell}_I^{\mathcal{D}}(W)$: $g \in \mathcal{D}_I^\times$ acts by

$$(\mathcal{E}_i, j_i, t_i; \iota)g = (\mathcal{E}_i, j_i, t_i; \iota \circ g),$$

where g acts on $\mathcal{D}_I \boxtimes \mathcal{O}_W$ via right multiplication on \mathcal{D}_I . The actions of \mathbb{Z} and \mathcal{D}_I^\times obviously commute with each other.

We have the following fundamental theorem:

Theorem 3.1. *Fix some $I \neq \emptyset$. The functor $W \mapsto \mathbf{Ell}_I^{\mathcal{D}}(W)/\mathbb{Z}$ is representable by a smooth quasi-projective scheme $X_I^{\mathcal{D}}$ over $C' := C - I - R - \infty$ of pure relative dimension 1. If D is a division algebra, then $X_I^{\mathcal{D}}$ is proper over C' .*

Proof. This is a consequence of (4.1), (5.1), and (6.2) in [10]. □

Corollary 3.2. *The functor $W \mapsto \mathbf{Ell}_\emptyset^{\mathcal{D}}(W)/\mathbb{Z}$ has a coarse moduli scheme $X_\emptyset^{\mathcal{D}}$, which is smooth of pure relative dimension 1 over $C - R - \infty$ with geometrically irreducible fibres. If D is a division algebra, then $X_\emptyset^{\mathcal{D}}$ is proper. For $I \neq \emptyset$, there is a functorial morphism $X_I^{\mathcal{D}} \rightarrow X_\emptyset^{\mathcal{D}}$ over C' which is finite flat of degree $\#(\mathcal{D}_I^\times)/(q - 1)$.*

Proof. The action of \mathcal{D}_I^\times on the stack $\mathbf{Ell}_I^{\mathcal{D}}$ induces an action of \mathcal{D}_I^\times on the scheme $X_I^{\mathcal{D}}$. The quotient scheme $X_I^{\mathcal{D}}/\mathcal{D}_I^\times$ is the coarse moduli scheme for $W \mapsto \mathbf{Ell}_\emptyset^{\mathcal{D}}(W)/\mathbb{Z}$ over $C - I - R - \infty$. The subgroup of $\mathcal{D}_I^\times \cong \mathrm{GL}_2(\mathcal{O}_I)$ which fixes all isomorphism classes of objects in $\mathbf{Ell}_I^{\mathcal{D}}(W)$ is \mathbb{F}_q^\times embedded diagonally into $\mathrm{GL}_2(\mathcal{O}_I)$. Hence the morphism $X_I^{\mathcal{D}} \rightarrow X_I^{\mathcal{D}}/\mathcal{D}_I^\times$ is finite flat of degree $\#(\mathcal{D}_I^\times)/(q - 1)$. To extend $X_\emptyset^{\mathcal{D}}$ over I , take J disjoint from I . The quotient $X_J^{\mathcal{D}}/\mathcal{D}_J^\times$ is defined over $C - J - R - \infty$ and is isomorphic to $X_I^{\mathcal{D}}/\mathcal{D}_I^\times$ over $C - J - I - R - \infty$. One obtains $X_\emptyset^{\mathcal{D}}$ by gluing these two schemes over $C - J - I - R - \infty$. Finally, the geometric fibres of

$$X_\emptyset^{\mathcal{D}} \rightarrow C - R - \infty$$

are irreducible by [14, Proposition 3.2]. □

Remark 3.3. Using the Morita equivalence and a result of Drinfeld, one can show that when $D = \mathbb{M}_2$ the category $\mathbf{Ell}_I^{\mathcal{D}}(W)/\mathbb{Z}$ is equivalent to the category of rank-2 Drinfeld A -modules over W equipped with level I -structures, cf. [2]. Hence in that case $X_I^{\mathcal{D}}$ are the modular curves constructed in [4].

Assume D is a division algebra. Denote by $X_{I,F}^{\mathcal{D}} := X_I^{\mathcal{D}} \times_{C'} \mathrm{Spec}(F)$ the generic fibre of $X_I^{\mathcal{D}}$. For a closed point o of $C - I - R - \infty$ denote $X_{I,o}^{\mathcal{D}} := X_I^{\mathcal{D}} \times_{C'} \mathrm{Spec}(\mathbb{F}_o)$. Let \bar{F} (resp. $\bar{\mathbb{F}}_o$) be a fixed algebraic closure of F (resp. \mathbb{F}_o). Fix a prime number ℓ not equal to the characteristic of F . Consider the ℓ -adic cohomology groups

$$H_{I,F}^i := H^i(X_{I,F}^{\mathcal{D}} \otimes_F \bar{F}, \bar{\mathbb{Q}}_\ell)$$

and

$$H_{I,o}^i := H^i(X_{I,o}^{\mathcal{D}} \otimes_{\mathbb{F}_o} \bar{\mathbb{F}}_o, \bar{\mathbb{Q}}_\ell).$$

These are finite dimensional $\bar{\mathbb{Q}}_\ell$ -vector spaces which are non-zero only if $0 \leq i \leq 2$. The dimensions of these spaces do not depend on the choice of ℓ . As a consequence of the proper base change theorem, there are canonical isomorphisms of $\bar{\mathbb{Q}}_\ell$ -vector

spaces $H_{I,F}^i \cong H_{I,o}^i$, $0 \leq i \leq 2$, cf. [10, p. 287]. The *Euler-Poincaré characteristic* of $X_{I,F}^{\mathcal{D}}$ is

$$\chi(X_{I,F}^{\mathcal{D}}) = \sum_{i=0}^2 (-1)^i \dim_{\overline{\mathbb{Q}_\ell}}(H_{I,F}^i).$$

Similarly, define

$$\chi(X_{I,o}^{\mathcal{D}}) = \sum_{i=0}^2 (-1)^i \dim_{\overline{\mathbb{Q}_\ell}}(H_{I,o}^i).$$

$\chi(X_{I,o}^{\mathcal{D}})$ is independent of the choice of o , since it is equal to $\chi(X_{I,F}^{\mathcal{D}})$.

4. \mathcal{D} -elliptic sheaves of finite characteristic. Let $o \in |C| - R - \infty$ be fixed, and let k be a fixed algebraic closure of \mathbb{F}_o . A \mathcal{D} -elliptic sheaf of characteristic o over k is a \mathcal{D} -elliptic sheaf \mathbb{E} over $\text{Spec}(k)$ such that the zero is the canonical morphism

$$z : \text{Spec}(k) \rightarrow \text{Spec}(\mathbb{F}_o) \hookrightarrow C.$$

In [10, Chapter 9], the authors, following Drinfeld, develop a Honda-Tate type theory for \mathcal{D} -elliptic sheaves. The basis of this theory is a construction which attached to each \mathcal{D} -elliptic sheaf of characteristic o over k a (D, ∞, o) -type. A (D, ∞, o) -type is a pair $(\tilde{F}, \tilde{\Pi})$ having the following properties, cf. [10, (9.11)]:

1. \tilde{F} is a separable field extension of F of degree dividing 2;
2. $\tilde{\Pi} \in \tilde{F}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$, but $\tilde{\Pi} \notin F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ unless $\tilde{F} = F$;
3. ∞ does not split in \tilde{F} ;
4. The valuations of F naturally extend to the group $\tilde{F}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. There are exactly two places $\tilde{x} = \tilde{\infty}$ and $\tilde{x} = \tilde{o}$ of \tilde{F} such that $\text{ord}_{\tilde{x}}(\tilde{\Pi}) \neq 0$. The place $\tilde{\infty}$ is the unique place of \tilde{F} over ∞ and \tilde{o} divides o . Moreover,

$$\text{ord}_{\tilde{\infty}}(\tilde{\Pi}) \cdot \deg(\tilde{\infty}) = -[\tilde{F} : F]/2.$$

5. For each place x of F and each place \tilde{x} of \tilde{F} dividing x , we have

$$(2[\tilde{F}_{\tilde{x}} : F_x]/[\tilde{F} : F]) \cdot \text{inv}_x(D) \in \mathbb{Z}.$$

6. $h := 2[\tilde{F}_{\tilde{o}} : F_o]/[\tilde{F} : F]$ is an integer satisfying $1 \leq h \leq 2$.

Next, using (1)–(6), we deduce some additional properties of \tilde{F} depending on whether $h = 1$ or 2.

Lemma 4.1. *If $h = 1$, then \tilde{F} is a separable quadratic extension of F in which o splits and each $x \in R$ does not split. If $h = 2$, then $\tilde{F} = F$*

Proof. Let $h = 1$. Since $1 = 2[\tilde{F}_{\tilde{o}} : F_o]/[\tilde{F} : F]$ and $[\tilde{F} : F] \leq 2$, we must have $[\tilde{F} : F] = 2$ and $[\tilde{F}_{\tilde{o}} : F_o] = 1$. From $[\tilde{F}_{\tilde{x}} : F_x] \cdot \text{inv}_x(D) \in \mathbb{Z}$, we conclude that $[\tilde{F}_{\tilde{x}} : F_x] = 2$ for every $x \in R$.

Now let $h = 2$. Suppose $\tilde{F} \neq F$. Since $2 = 2[\tilde{F}_{\tilde{o}} : F_o]/[\tilde{F} : F]$, we must have $[\tilde{F}_{\tilde{o}} : F_o] = 2$. Hence \tilde{o} is the unique place of \tilde{F} over o . Let g be the generator of $\text{Gal}(\tilde{F}/F)$. Let N be a non-zero integer such that $\tilde{\Pi}^N \in \tilde{F}$. Consider $F' := F[\tilde{\Pi}^N]$. Since $\tilde{\Pi} \in F'^{\times} \otimes \mathbb{Q}$, we must have $F' = \tilde{F}$. Since $\tilde{\Pi}^N$ has non-zero valuations only at \tilde{o} and $\tilde{\infty}$, and g fixes both places as neither of them splits, $\tilde{\Pi}^N/g(\tilde{\Pi}^N)$ has zero valuation at all places of \tilde{F} . This implies that $\tilde{\Pi}^N/g(\tilde{\Pi}^N)$ is a root of unity, so there is an integer $M \geq N$ such that $\tilde{\Pi}^M/g(\tilde{\Pi}^M) = 1$. By the same argument as above, $\tilde{F} = F[\tilde{\Pi}^M]$. But now g fixes $\tilde{\Pi}^M$, so $\tilde{F} = F$, which is a contradiction. \square

Definition 4.2. Following the terminology for elliptic curves, we call a \mathcal{D} -elliptic sheaf of characteristic o over k *ordinary* or *supersingular* depending on whether its (D, ∞, o) -type has $h = 1$ or $h = 2$, respectively. Denote by $\mathfrak{S}_o^{\mathcal{D}} \subset X_{\tilde{o}, o}^{\mathcal{D}}(k)$ the subset of closed points corresponding to supersingular \mathcal{D} -elliptic sheaves.

To an \mathbb{E} with (D, ∞, o) -type $(\tilde{F}, \tilde{\Pi})$ there is a canonically associated central division algebra $\text{End}^0(\mathbb{E})$ over \tilde{F} , called the *endomorphism algebra of \mathbb{E}* ; see [10, (9.10)]. There is also a natural notion of *endomorphism ring* $\text{End}(\mathbb{E})$ of \mathbb{E} [12], which is an A -order in $\text{End}^0(\mathbb{E})$.

As is proven in [10], if \mathbb{E} is ordinary, then $\text{End}^0(\mathbb{E}) = \tilde{F}$. If \mathbb{E} is supersingular, then $\text{End}^0(\mathbb{E})$ is the quaternion algebra \bar{D} over F with invariants

$$\text{inv}_x \bar{D} = \begin{cases} 1/2, & \text{if } x = \infty; \\ 1/2, & \text{if } x = o; \\ \text{inv}_x(D), & \text{otherwise.} \end{cases}$$

Definition 4.3. Let $\text{Aut}(\mathbb{E})$ be the set of all non-zero elements of $\text{End}(\mathbb{E})$ which are algebraic over \mathbb{F}_q .

Lemma 4.4. *The elements of $\text{Aut}(\mathbb{E})$ form a group isomorphic to either \mathbb{F}_q^{\times} or $\mathbb{F}_{q^2}^{\times}$.*

Proof. This is obvious if \mathbb{E} is ordinary. If \mathbb{E} is supersingular then this follows from [12, §§ 6–7]. \square

Definition 4.5. We say that \mathbb{E} has *extra automorphisms* if $\text{Aut}(\mathbb{E}) \cong \mathbb{F}_{q^2}^{\times}$.

To proceed further, we need to recall some standard facts about maximal orders in quaternion algebras. The definitions and the proofs can be found in [17]. Fix some maximal A -order \mathcal{O} in \bar{D} . Let $\mathcal{S} := \{I_1, \dots, I_m\}$ be the right ideal classes of \mathcal{O} , and let $\mathcal{O}_i := \mathcal{O}_{\ell}(I_i)$ be the left order of I_i , $1 \leq i \leq m$. The number $\mathfrak{c}(\bar{D}, \infty) := m$, called the *class number* of \bar{D} with respect to ∞ , does not depend on the choice of \mathcal{O} . Moreover, the orders \mathcal{O}_i , $1 \leq i \leq m$, are maximal, and all conjugacy classes of maximal A -orders in \bar{D} appear in $\{\mathcal{O}_1, \dots, \mathcal{O}_m\}$. The *mass* of \bar{D} with respect to ∞ is

$$\mathfrak{m}(\bar{D}, \infty) := (q - 1) \sum_{i=1}^m (\#\mathcal{O}_i^{\times})^{-1}.$$

Finally, the two-sided ideals of \mathcal{O} form a commutative group generated by the ideals of A and ideals Π_x , $x \in R \cup o$, such that $\Pi_x^2 = \mathfrak{p}_x$ (see [17, pp. 86-87]).

For a finite set $S \subset |C|$, let

$$\wp(S) = \begin{cases} 0, & \text{if some place in } S \text{ has even degree;} \\ 1, & \text{otherwise.} \end{cases}$$

In particular, $\wp(\emptyset) = 1$.

Theorem 4.6. *With above notation,*

$$\mathfrak{m}(\bar{D}, \infty) = \frac{1}{q^2 - 1} \prod_{x \in R \cup o} (q_x - 1)$$

and

$$\mathfrak{c}(\bar{D}, \infty) = \frac{1}{q^2 - 1} \prod_{x \in R \cup o} (q_x - 1) + \frac{q}{q + 1} \cdot 2^{\#R} \cdot \wp(R \cup o).$$

Proof. See (1), (4), and (6) in [3]. □

Theorem 4.7. *With above notation, there exists a bijection*

$$\mathfrak{S}_o^{\mathcal{D}} \xrightarrow{\sim} \mathcal{S}$$

such that if $\mathbb{E}_j \in \mathfrak{S}_o^{\mathcal{D}}$ corresponds to $I_j \in \mathcal{S}$ then $\text{End}(\mathbb{E}_j) \cong O_\ell(I_j)$. The Galois group $\text{Gal}(k/\mathbb{F}_o)$ preserves the set $\mathfrak{S}_o^{\mathcal{D}}$. Moreover, the action of the geometric Frobenius Frob_o on $\mathfrak{S}_o^{\mathcal{D}}$ corresponds to the action of Π_o on \mathcal{S} given by right multiplication $I_j \mapsto I_j \Pi_o$.

Proof. First note that since $O_r(I_j) = O_\ell(\Pi_o)$ the product $I_j \Pi_o$ makes sense, and it is a right ideal of \mathcal{O} , cf. [17, p. 22]. For the proof of the first statement of the theorem see [12, §7]. (When D is the matrix algebra, i.e., in the case of Drinfeld modules, this is due to Gekeler [7, Theorem 4.3].) The second statement can be deduced from the discussion in [10, §10] (see p. 274, in particular). □

Corollary 4.8. *All points in $\mathfrak{S}_o^{\mathcal{D}}$ are rational over $\mathbb{F}_o^{(2)}$, and*

$$\#\mathfrak{S}_o^{\mathcal{D}} = \frac{1}{q^2 - 1} \prod_{x \in R \cup o} (q_x - 1) + \frac{q}{q + 1} \cdot 2^{\#R} \cdot \wp(R \cup o).$$

Proof. The formula for $\#\mathfrak{S}_o^{\mathcal{D}}$ is an immediate consequence of Theorems 4.6 and 4.7. For the first statement, note that the action of Frob_o^2 on $\mathfrak{S}_o^{\mathcal{D}}$ corresponds to the action of \mathfrak{p}_o on the ideal classes \mathcal{S} . But \mathfrak{p}_o is a principal ideal of A since $\text{Pic}(A) = 1$, so it acts trivially on the ideal classes. □

Corollary 4.9. *The number of points on $X_{\emptyset, o}^{\mathcal{D}}(k)$ corresponding to \mathcal{D} -elliptic sheaves with extra automorphisms is $2^{\#R} \cdot \wp(R)$. Moreover, when $\wp(R) \neq 0$, all \mathcal{D} -elliptic sheaves with extra automorphisms are ordinary (resp. supersingular) if $\text{deg}(o)$ is even (resp. odd).*

Proof. Let w be the number of points on $X_{\emptyset,o}^{\mathcal{D}}(k)$ corresponding to \mathcal{D} -elliptic sheaves with extra automorphisms. Let $\mathbb{E} \in \mathfrak{S}_o^{\mathcal{D}}$, and let $E \in \mathcal{S}$ be the ideal corresponding to \mathbb{E} under the bijection of Theorem 4.7. Since $\text{Aut}(\mathbb{E}) = O_\ell(E)^\times$, the number of points in $\mathfrak{S}_o^{\mathcal{D}}$ with extra automorphisms is equal to

$$(\mathbf{c}(\bar{D}, \infty) - \mathbf{m}(\bar{D}, \infty)) / (1 - (q + 1)^{-1}),$$

which is $2^{\#R} \cdot \wp(R \cup o)$, according to Theorem 4.6. Now suppose \mathbb{E} is ordinary. If $\text{Aut}(\mathbb{E}) \cong \mathbb{F}_{q^2}^\times$, then \mathbb{F}_{q^2} embeds into \tilde{F} . Since \tilde{F} is quadratic, we must have $\tilde{F} = \mathbb{F}_{q^2}F$. On the other hand, by Lemma 4.1, the places in R do not split in \tilde{F} , but o splits. Hence the degree of each place in R must be odd and $\deg(o)$ must be even. Overall, so far, we can make the following conclusion. If some place in R has even degree then $w = 0$. If all places in R have odd degrees and $\deg(o)$ is also odd, then $w = 2^{\#R}$, since in this case the \mathcal{D} -elliptic sheaves with extra automorphisms are necessarily supersingular.

Now suppose D is a division algebra. Fix some non-empty $I \subset C - R - o - \infty$ and consider the covering $\pi : X_{I,o}^{\mathcal{D}} \rightarrow X_{\emptyset,o}^{\mathcal{D}}$. The points which ramify in this covering are exactly the points corresponding to \mathcal{D} -elliptic sheaves with extra automorphisms. Moreover, the ramification index at such a point is equal to $\#(\mathbb{F}_{q^2}^\times / \mathbb{F}_q^\times) = q + 1$. Hence the ramifications are tame, and the Riemann-Hurwitz formula implies

$$(4.1) \quad \chi(X_{I,o}^{\mathcal{D}}) = \deg(\pi) \cdot \chi(X_{\emptyset,o}^{\mathcal{D}}) - \deg(\pi) \frac{q}{q + 1} w.$$

As we mentioned in §2, the Euler-Poincaré characteristic and $\deg(\pi)$ do not depend on the choice of o , so w also does not depend on o . Therefore, if $\deg(o)$ is even, then $w = 2^{\#R} \cdot \wp(R)$ and all \mathcal{D} -elliptic sheaves with extra automorphisms are ordinary.

Finally, suppose $D \cong \mathbb{M}_2(F)$ (equiv. $R = \emptyset$). Then the problem can be reformulated as a problem about rank-2 Drinfeld A -modules over k , cf. Remark 3.3. In those terms, the problem was solved by Gekeler using different techniques; cf. [6, Proposition 7.1]: There is a unique k -isomorphism class of rank-2 Drinfeld A -modules with automorphism group $\mathbb{F}_{q^2}^\times$. The j -invariant of this class is 0, and a Drinfeld module with $j = 0$ is supersingular or ordinary, depending on whether $\deg(o)$ is odd or even; see [8], [6]. □

5. Genus formula. In this section we compute $\chi(X_{I,F}^{\mathcal{D}})$ and $\chi(X_{\emptyset,F}^{\mathcal{D}})$. We will compute the first number using analytic methods, and then deduce the second number from the Riemann-Hurwitz formula. Throughout the section we assume that D is a division algebra.

Let I be a fixed closed non-empty subscheme of $C - R - \infty$. The double coset space $D^\times(F) \backslash D^\times(\mathbb{A}^\infty) / \mathcal{K}_I^\infty$ has finite cardinality. In fact, since D is split at ∞ , the reduced norm induces a bijection

$$D^\times(F) \backslash D^\times(\mathbb{A}^\infty) / \mathcal{K}_I^\infty \xrightarrow{\sim} F^\times \backslash (\mathbb{A}^\infty)^\times / \text{Nr}(\mathcal{K}_I^\infty).$$

(This is a consequence of the Strong Approximation Theorem for D^\times , cf. [17, p. 89].) Choose a system S of representatives for this double coset space. For each $s \in S$, let

$$\Gamma_{I,s} := D^\times(F) \cap s\mathcal{K}_I^\infty s^{-1}.$$

Lemma 5.1. *Under the natural embedding*

$$\Gamma_{I,s} \hookrightarrow D^\times(F)/F^\times \hookrightarrow D^\times(F_\infty)/F_\infty^\times \cong \mathrm{PGL}_2(F_\infty),$$

$\Gamma_{I,s}$ is a discrete, cocompact, torsion-free subgroup of $\mathrm{PGL}_2(F_\infty)$.

Proof. See [13, Lemma 6.4]. □

Let Ω denote the Drinfeld upper half-plane over F_∞ ; see [4]. As a set, $\Omega = \mathbb{C}_\infty - F_\infty$. By the previous lemma, each group $\Gamma_{I,s} \subset \mathrm{PGL}_2(F_\infty)$ is a Schottky subgroup which acts on Ω via linear fractional transformations. As follows from the theory of Mumford curves, the quotient $\Gamma_{I,s} \backslash \Omega$ is the analytification of a smooth projective curve $X_{\Gamma_{I,s}}$ over \mathbb{C}_∞ . Denote by $X_I^{\mathcal{D}}(\mathbb{C}_\infty)$ the underlying rigid-analytic variety of $X_{I,F}^{\mathcal{D}} \otimes_F F_\infty$. We have the following fundamental fact:

Theorem 5.2. *There is a canonical isomorphism of analytic spaces over \mathbb{C}_∞*

$$X_I^{\mathcal{D}}(\mathbb{C}_\infty) \cong \bigsqcup_{s \in S} X_{\Gamma_{I,s}}.$$

Proof. See [1, Theorem 4.4.11]. □

Now using the theory of rigid-analytic uniformizations of Jacobians of Mumford curves [11], one concludes that

$$\chi(X_{I,F}^{\mathcal{D}}) = 2 \cdot \sum_{s \in S} \chi(\Gamma_{I,s}),$$

where $\chi(\Gamma_{I,s}) = 1 - \dim_{\mathbb{Q}} H^1(\Gamma_{I,s}, \mathbb{Q})$.

In [15], Serre developed a theory which allows to compute $\chi(\Gamma_{I,s})$ as a volume. Serre's result is reproduced in a convenient form in Proposition 5.3.6 of [9]. Combining this with Proposition 5.3.9 in [9], in our situation we get the following statement:

Let dg be the Haar measure on $\mathrm{GL}_2(F_\infty)$ normalized by $\mathrm{Vol}(\mathrm{GL}_2(\mathcal{O}_\infty), dg) = 1$. Let dz be the Haar measure on F_∞^\times normalized by $\mathrm{Vol}(\mathcal{O}_\infty^\times, dz) = 1$. Fix the Haar measure $dh = dg/dz$ on $\mathrm{PGL}_2(F_\infty)$ and the counting measure $d\delta$ of $\Gamma_{I,s}$. Then

$$(5.1) \quad \chi(\Gamma_{I,s}) = \frac{1}{2}(1 - q) \cdot \mathrm{Vol} \left(\Gamma_{I,s} \backslash \mathrm{PGL}_2(F_\infty), \frac{dh}{d\delta} \right).$$

Now define a Haar measure $d\bar{g}$ on $D^\times(\mathbb{A})$ as follows: For $x \in |C|$, normalize the Haar measure dg_x on $D^\times(F_x)$ by $\text{Vol}(\mathcal{D}_x^\times, dg_x) = 1$, and let $d\bar{g}$ be the restricted product measure. We have

$$D^\times(F) \setminus D^\times(\mathbb{A})/\mathcal{K}_I^\infty F_\infty^\times \cong \bigsqcup_{s \in S} (\Gamma_{I,s} \setminus \text{PGL}_d(F_\infty)),$$

and the push-forward of $d\bar{g}$ on $D^\times(\mathbb{A})$ to the double coset space above induces the measure $dh/d\delta$ on each $\Gamma_{I,s} \setminus \text{PGL}_2(F_\infty)$. Hence

$$\chi(X_{I,F}^{\mathcal{D}}) = (1 - q) \cdot \text{Vol}(D^\times(F) \setminus D^\times(\mathbb{A})/\mathcal{K}_I^\infty F_\infty^\times, d\bar{g}).$$

Consider the homomorphism

$$\|\cdot\| : D^\times(\mathbb{A}) \rightarrow q^{\mathbb{Z}}$$

given by the composition of the reduced norm $\text{Nr} : D^\times(\mathbb{A}) \rightarrow \mathbb{A}^\times$ with the idelic norm $\prod_{x \in |C|} |\cdot|_x : \mathbb{A}^\times \rightarrow q^{\mathbb{Z}}$. Denote the kernel of this homomorphism by $D^1(\mathbb{A})$. The group $D^\times(F)$, under the diagonal embedding into $D^\times(\mathbb{A})$, lies in $D^1(\mathbb{A})$, thanks to the product formula. The quotient $D^\times(F) \setminus D^1(\mathbb{A})$ is compact, hence has finite volume. It is well-known that $\|\cdot\| : D^\times(\mathbb{A}) \rightarrow q^{\mathbb{Z}}$ is surjective. The image of F_∞^\times in $q^{\mathbb{Z}}$ is clearly $q^{2\mathbb{Z}}$. Hence there is an exact sequence

$$0 \rightarrow D^\times(F) \setminus D^1(\mathbb{A})/\mathcal{O}_I^\times \rightarrow D^\times(F) \setminus D^\times(\mathbb{A})/F_\infty^\times \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0,$$

which implies

$$\begin{aligned} & \text{Vol}(D^\times(F) \setminus D^\times(\mathbb{A})/\mathcal{K}_I^\infty F_\infty^\times, d\bar{g}) \\ &= 2 \cdot \#\text{GL}_2(\mathcal{O}_I) \cdot \text{Vol}(D^\times(F) \setminus D^1(\mathbb{A}), d\bar{g}). \end{aligned}$$

This last volume can be expressed in terms of the zeta-function of C (see [13, §4]):

$$\text{Vol}(D^\times(F) \setminus D^1(\mathbb{A}), d\bar{g}) = \frac{1}{(q-1)^2(q^2-1)} \prod_{x \in R} (q_x - 1).$$

Combining the previous calculations, one obtains:

Theorem 5.3.

$$\chi(X_{I,F}^{\mathcal{D}}) = -\frac{2 \cdot \#\text{GL}_2(\mathcal{O}_I)}{(q-1)(q^2-1)} \cdot \prod_{x \in R} (q_x - 1).$$

Now consider $\chi(X_{\emptyset,F}^{\mathcal{D}})$. Fix some closed point o on $C - I - R - \infty$. We know that $\chi(X_{\emptyset,F}^{\mathcal{D}}) = \chi(X_{\emptyset,o}^{\mathcal{D}})$, and $\chi(X_{I,F}^{\mathcal{D}}) = \chi(X_{I,o}^{\mathcal{D}})$. Hence, if we combine Corollary 3.2, (4.1), Corollary 4.9 and Theorem 5.3, then we obtain the formula:

$$(5.2) \quad \chi(X_{\emptyset,F}^{\mathcal{D}}) = -\frac{2}{(q^2-1)} \prod_{x \in R} (q_x - 1) + \frac{q}{q+1} \cdot 2^{\#R} \cdot \wp(R).$$

By Corollary 3.2, $X_{\emptyset,F}^{\mathcal{D}}$ is a smooth, projective, geometrically irreducible curve over F . To simplify the notation, denote this curve by X^R and its fibre over o by X_o^R . Let $g(X^R)$ be the genus of X^R . The Euler-Poincaré characteristic $\chi(X^R)$ and $g(X^R)$ are related by the formula $\chi(X^R) = 2 - 2 \cdot g(X^R)$. Hence from (5.2) we get:

Theorem 5.4.

$$g(X^R) = 1 + \frac{1}{q^2 - 1} \left(\prod_{x \in R} (q_x - 1) - q \cdot (q - 1) \cdot 2^{\#R-1} \cdot \wp(R) \right).$$

Corollary 5.5. $g(X^R)$ is always divisible by q . In particular, X^R is never an elliptic curve. $g(X^R) = 0$ if and only if either $\deg(\mathfrak{r}) = 2$, or $q = 4$ and $\mathfrak{r} = (T^4 - T)$.

Remark 5.6. One can also try to classify those X^R which are hyperelliptic. In this direction, we are able to prove that for a fixed q there are only finitely many X^R which are hyperelliptic. If q is odd, then X^R is hyperelliptic if and only if $\deg(\mathfrak{r}) = 3$. The proofs of these results will appear elsewhere.

It is interesting to compare the formula in Theorem 5.4 to the formula for the genus of Shimura curves over \mathbb{Q} . First, we rewrite the formula for $g(X^R)$ in a slightly different form.

Let K be a global field and L be a separable quadratic extension of K . For a place x of K , define the *Artin-Legendre symbol* $\left(\frac{L}{x}\right)$ (cf. [17, p. 94]):

$$\left(\frac{L}{x}\right) = \begin{cases} 1, & \text{if } x \text{ splits in } L; \\ -1, & \text{if } x \text{ is inert in } L; \\ 0, & \text{if } x \text{ ramifies in } L. \end{cases}$$

Using this notation, we have

$$g(X^R) = 1 + \frac{1}{q^2 - 1} \prod_{x \in R} (q_x - 1) - \frac{1}{2} \frac{q}{q + 1} \prod_{x \in R} \left(1 - \left(\frac{\mathbb{F}_{q^2} F}{x}\right) \right).$$

Now let H be the indefinite quaternion division algebra over \mathbb{Q} with reduced discriminant d . This means that $H \otimes \mathbb{R} \cong \mathbb{M}_2(\mathbb{R})$ and H is ramified exactly at the primes dividing d . Let \mathcal{O} be a maximal \mathbb{Z} -order in H . Denote by Γ^d the group of units of positive norm in \mathcal{O} . Under the embedding

$$\Gamma^d \hookrightarrow H \otimes \mathbb{R} \cong \mathbb{M}_2(\mathbb{R})$$

the image of Γ^d lies in $\mathrm{SL}_2(\mathbb{R})$, hence naturally acts on the upper half-plane \mathfrak{h} . The quotient \mathfrak{h}/Γ^d is a smooth, projective, geometrically irreducible curve X^d which has a model over \mathbb{Q} . (In fact, X^d is a moduli space of abelian surfaces equipped with an action of \mathcal{O} .) The genus $g(X^d)$ is given by the formula (see [17, p. 120]):

$$g(X^d) = 1 + \frac{1}{12} \prod_{p|d} (p - 1) - \frac{1}{2} \left(\frac{1}{2} \prod_{p|d} \left(1 - \left(\frac{\mathbb{Q}(\sqrt{-1})}{p}\right) \right) + \frac{2}{3} \prod_{p|d} \left(1 - \left(\frac{\mathbb{Q}(\sqrt{-3})}{p}\right) \right) \right).$$

We see that the formulas for $g(X^d)$ and $g(X^R)$ are strikingly similar. Even the terms $1/12$ and $1/(q^2 - 1)$ are of similar nature: $-\zeta_{\mathbb{Z}}(-1) = 1/6$ and $-\zeta_A(-1) =$

$1/(q^2 - 1)$, where $\zeta_{\mathbb{Z}}(s)$ is the Riemann zeta-function of \mathbb{Z} and $\zeta_A(s)$ is the zeta-function of A .

6. Asymptotically optimal series of curves. Let X be a curve of genus $g(X)$ defined over \mathbb{F}_q . (In this section, a *curve* means smooth, projective, geometrically irreducible algebraic curve.) According to a celebrated result of Weil, the number of \mathbb{F}_q -rational points on X is bounded by

$$\#X(\mathbb{F}_q) \leq q + 1 + 2g(X)\sqrt{q}.$$

On the other hand, it turns out that when $g(X)$ is very large compared to q , then Weil’s bound can be significantly improved. Drinfeld and Vladut [18] showed that

$$(6.1) \quad \limsup_X \left(\frac{\#X(\mathbb{F}_q)}{g(X)} \right) \leq \sqrt{q} - 1,$$

where X runs over the set of all curves over \mathbb{F}_q . A series of curves $\{X_i\}_{i \in \mathbb{N}}$ is called *asymptotically optimal* if

$$\lim_{i \rightarrow \infty} \left(\frac{\#X_i(\mathbb{F}_q)}{g(X_i)} \right) = \sqrt{q} - 1,$$

or, in other words, $\{X_i\}_{i \in \mathbb{N}}$ realizes the bound (6.1). If q is not a square then no asymptotically optimal series of curves is known. If q is a square then asymptotically optimal series always exist, but every known such series has the property that for all sufficiently large i the curve X_i is a modular curve of some sort.

In [14], we showed that if one fixes D and considers the modular curves of \mathcal{D} -elliptic sheaves with appropriate level structures, then one obtains asymptotically optimal series of curves over \mathbb{F}_{q^2} as the level varies. Using the genus formula proven in the current paper, we can show that the sequence “perpendicular” to the one mentioned in the previous sentence is also asymptotically optimal. Namely, we *fix* the level (actually $I = \emptyset$) and *vary* D :

Theorem 6.1. *Let $o = (T)$ be fixed. Let R run over all subsets of $|C| - o - \infty$ having even cardinality. Then*

$$\lim_{\deg(\mathfrak{r}) \rightarrow \infty} \left(\frac{\#X_o^R(\mathbb{F}_{q^2})}{g(X_o^R)} \right) = q - 1.$$

Proof. By comparing the expression for the genus in Theorem 5.4 with the expressions for the number of supersingular points in Corollary 4.8, one concludes that \liminf of the sequence in question is greater or equal to $(q - 1)$. (We know that all supersingular points are rational over $\mathbb{F}_{(T)}^{(2)} \cong \mathbb{F}_{q^2}$.) On the other hand, the \limsup of the same sequence is bounded from above by $(q - 1)$ according to (6.1). The claim follows. \square

Even though the series $\{X_o^R\}_R$ is asymptotically optimal over $\mathbb{F}_o^{(2)}$, the individual curves fail to have a particularly large number of rational points when $\deg(\mathfrak{r})$

is small. The reason for this is that the number of supersingular points on X_o^R is roughly $(q_o - 1)g(X_o^R)$ for any R , whereas the Weil bound for a curve over $\mathbb{F}_o^{(2)}$ is roughly $2g(X)q_o$. The Weil bound is known to be quite close to the best possible when the genus is relatively small, hence it is not surprising to find that the known maximal number of points on curves of genus $g(X_o^R)$ is approximately twice as large as the provable number of points on X_o^R . We give a table for comparison when $q = 2, 3$ and genus ≤ 50 .

In the tables below we consider the curves X_o^R when $\#R = 2$ and $o = (T)$. The first column indicates the degrees of the places in R , the second and the third columns give the genus and the number of supersingular points on X_o^R , respectively. The fourth and the fifth columns indicate the known maximum number of \mathbb{F}_{q^2} -rational points on a curve of genus $g(X^R)$ and the best known theoretic upper bound, respectively; these numbers are taken from [16].

$\mathbf{q = 2}$	R	$g(X^R)$	# supersingular points	max # of \mathbb{F}_{q^2} -rational points known	upper bound
	(1, 2)	2	1	10	10
	(1, 3)	2	5	10	10
	(1, 4)	6	5	20	20
	(1, 5)	10	13	27	27
	(1, 6)	22	21	42	48
	(1, 7)	42	45	75	80
	(2, 2)	4	3	15	15
	(2, 3)	8	7	21	24
	(2, 4)	16	15	36	38
	(2, 5)	32	31	57	65
	(3, 3)	16	19	36	38
	(3, 4)	36	35	64	71

$\mathbf{q = 3}$	R	$g(X^R)$	# supersingular points	max # of \mathbb{F}_{q^2} -rational points known	upper bound
	(1, 1)	0	4	4	4
	(1, 2)	3	4	28	28
	(1, 3)	6	16	35	40
	(1, 4)	21	40	88	95
	(2, 2)	9	16	48	50
	(2, 3)	27	52	104	114

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