Projects “Automorphic forms and applications”

**Project 1.** Give detailed proofs for Theorem 3 and three other results stated in class without proof.

**Project 2.** State and prove a selection of analytic properties of the Bessel function $J_k(x)$, in particular bounds, asymptotic formulae (ideally uniformly in $k$), integral representations etc.

**Project 3.** We showed in class
\[
\sum_{n \leq X} a(n)e(\alpha n) \ll X^{k/2}\log X
\]
for Fourier coefficients of a weight $k$ cusp form and any $\alpha \in \mathbb{R}$. Now assume that $\alpha = a/q$ is a rational number in lowest terms and $f$ has level 1, i.e., is a cusp form of $SL_2(\mathbb{Z})$. Use the transformation properties of the cusp form $f$ (in particular by the matrix $\begin{pmatrix} \bar{a} & (1 - a\bar{a})/q \\ -q & a \end{pmatrix}$ for a fixed representative $\bar{a}$) to prove stronger estimates (ideally uniformly in $q$ for $q$ not too big in terms of $X$).

**Project 4.** [Character sums:] a) While Kloosterman sums to prime modulus cannot be evaluated (in general), a twisted version is easier to handle: Let $p$ be an odd prime, $m,n \in \mathbb{Z}$ with $n$ coprime to $p$. Define the Salié sum
\[
T(n,m;p) := \sum_{d|p} \left( \frac{d}{p} \right) e\left(\frac{m\bar{d} + nd}{p}\right).
\]
These sums play a crucial role in the theory of half-integral weight modular forms. Show
\[
T(n,m;p) = \left( \frac{n}{p} \right) \varepsilon_p \sqrt{p} \sum_{y^2 \equiv nm (p)} e\left(\frac{2y}{p}\right)
\]
where as usual $\varepsilon_p = 1$ if $p \equiv 1 \pmod{4}$ and $\varepsilon_p = i$ otherwise. 

b) Fix $r,n,k \in \mathbb{N}$, $r \geq 2$. Let $p$ be an odd prime, $q = p^r$. As usual let $\tau(\chi)$ denote the Gauß sum associated to a character $\chi$ mod $q$. Show
\[
\sum_{\chi (q)} \bar{\chi}(k)\tau(\chi)^n \ll q^{(n+1)/2}.
\]
Conclude that
\[
\sum_{\chi (q)} \tau(\chi)^n \ll q^{(n+1)/2}
\]
where the sum is only over primitive characters.

**Project 5.** Come up with your own interesting question and solve it.