Representations as the sum of five products of squares

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1 Introduction

We will use the Hardy-Littlewood method to give an asymptotic formula for the number of ways to represent a natural number \( n \) as

\[
n = a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 + a_4^2 b_4^2 + a_5^2 b_5^2.
\]

This is essentially a special case of Waring’s problem, with the added complication of weighting the representations by the divisor function of each part. We denote by \( r(X) \) the number of ways to represent the number \( X \) as a sum of five products of squares as above. We let

\[
\phi = \frac{1}{100}, \quad Q = X^\phi, \quad P = X^{1/2}, \quad \delta = X^{\phi-2}.
\]

We have from Chapter 1 of [4] that

\[
 r(X) = \int_0^1 f(\alpha)^5 e(-X\alpha) \, d\alpha,
\]

where

\[
 f(\alpha) = \sum_{n \leq P} d(n) e(\alpha n^2).
\]

To avoid difficulties at the endpoints, it is more convenient to integrate over the interval \( \mathcal{U} = (\delta, 1 + \delta] \), rather than the interval \((0, 1]\). We define the major arcs to be

\[
 \mathcal{M}(q,a) = \{\alpha \in \mathcal{U} : |\alpha - a/q| \leq \delta\},
\]

for \( 1 \leq a \leq q \leq Q, (a,q) = 1 \). We let

\[
 \mathcal{M} = \bigcup_{1 \leq a \leq q \leq Q \atop (a,q) = 1} \mathcal{M}(q,a),
\]

and we denote the minor arcs by \( \mathcal{m} = \mathcal{U} \setminus \mathcal{M} \). We now have that

\[
 r(X) = \int_{\mathcal{U}} f(\alpha)^5 e(-X\alpha) \, d\alpha = \int_{\mathcal{M}} f(\alpha)^5 e(-X\alpha) \, d\alpha + \int_{\mathcal{m}} f(\alpha)^5 e(-X\alpha) \, d\alpha.
\]
We will treat the major and the minor arcs separately to derive our asymptotic formula. Our goal is to prove the following result:

**Theorem 1.1** Let \( r(X) \) denote the number of ways to represent a number \( X \) in the form

\[
X = a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + a_4b_4^2 + a_5b_5^2.
\]

Then \( r(X) \) satisfies

\[
r(X) = \sum_{q \leq Q} \sum_{a=1}^{q} e\left(-\frac{Xa}{q}\right) G(X; q, a) + O\left(X^{3/2}(\log X)^4\right),
\]

where

\[
G(X; q, a) = \int \cdots \int C(v_1; q, a) \cdots C(v_5; q, a) \, dv_1 \cdots dv_4
\]

and

\[
C(v; q, a) = \frac{1}{2\sqrt{v}} \sum_{r=1}^{q} e\left(\frac{ar^2}{q}\right) \frac{1}{q} \sum_{t|q} c_t(r) \left(\log \left(\frac{\sqrt{v}}{t^2}\right) + 2\gamma\right).
\]

Here \( c_t(r) \) denotes the Ramanujan sum,

\[
c_t(r) = \sum_{k=1}^{t} e\left(\frac{kr}{t}\right).
\]

The methods used in this paper can be extended to show that

\[
r(X) = CX^{3/2}(\log X)^5 \mathcal{S}(X) + O(X^{3/2}(\log X)^4),
\]

where \( C \) is a constant and

\[
\mathcal{S}(X) = \sum_{q=1}^{\infty} \frac{1}{q} \sum_{a=1}^{q} S(q, a)^5 e\left(-\frac{aX}{q}\right)
\]

with

\[
S(q, a) = \sum_{r=1}^{q} e\left(\frac{ar^2}{q}\right).
\]
2 The Minor Arcs

In this section we will find a bound for the integral over the minor arcs. We start with a few auxiliary lemmas.

Lemma 2.1 Suppose that \(a_m, b_n \in \mathbb{C}\) and
\[
\sum_{m \leq M} |a_m|^2 \ll M^{2+\epsilon}, \quad \sum_{n \leq N} |b_n|^2 \ll N^{2+\epsilon}.
\]
Suppose also that \(\alpha \in \mathbb{R}\) and that there are \(a\) and \(q\) with \((a, q) = 1\) such that \(|\alpha - a/q| \leq q^{-2}\) and \((a, q) = 1\). Finally suppose that \(M \leq M' \leq 2M, N \leq N' \leq 2N, MN \ll P\). Then
\[
\sum_{M < m \leq M'} \sum_{N < n \leq N'} a_m b_n e(\alpha m^2 n^2) \ll P^{1+\epsilon} q^\epsilon \left( M^{1/2} q^{-1/2} + M P^{-1/2} + q^{1/2} M^{1/2} P^{-1} \right) \quad (1)
\]
and
\[
\sum_{M < m \leq M'} \sum_{N < n \leq N'} a_m b_n e(\alpha m^2 n^2) \ll P^{1+\epsilon} q^\epsilon \left( q^{-1/4} + M^{-1/2} + M^{1/4} P^{-1/4} + q^{1/4} P^{-1/2} \right). \quad (2)
\]
These bounds also hold when the summation condition \(mn \leq P\) is removed.

See [1] for a proof of this lemma. We use the above result to prove the following lemma, which is more specific to the particular sums we are dealing with on the minor arcs.

Lemma 2.2 Suppose that \(\alpha \in \mathbb{R}\) and there are \(a\) and \(q\) such that \(|\alpha - a/q| \leq q^{-2}\) and \((a, q) = 1\). Then
\[
\sum_{n \leq P} d(n) e(\alpha n^2) \ll P^{1+\epsilon} q^\epsilon \left( q^{-1/4} + P^{-1/8} + q^{1/4} P^{-1/2} \right).
\]

Proof. We have
\[
\sum_{n \leq P} d(n) e(\alpha n^2) = \sum_{mn \leq P} e(\alpha m^2 n^2) = 2 \sum_{m \leq \sqrt{P}} \sum_{n \leq P/m} e(\alpha m^2 n^2) - \sum_{m \leq \sqrt{P}} \sum_{n \leq \sqrt{P}} e(\alpha m^2 n^2).
\]
We denote the first double sum by \(T_1\) and the second by \(T_2\). We will handle \(T_1\) first. We let \(\mathcal{M} = \{2^k : k \geq 0, 2^k \leq \sqrt{P}\}\) and \(\mathcal{N}(M) = \{2^k : k \geq 0, 2^k \leq P/M\}\). For a typical \(M \in \mathcal{M}\) we let \(M' = \min(2M, \sqrt{P})\), and for \(N \in \mathcal{N}(M)\), let \(N' = \min(2N, P/M)\). We also
let \( Y = \min(P^{1/4}, q^{1/2}, Pq^{-1/2}) \). We then have

\[
T_1 = \sum_{M \leq Y} \sum_{N \in \mathcal{N}(M)} \sum_{M' \leq M} \sum_{N' \leq N} e(\alpha m^2 n^2)
\]

\[
= \sum_{M \leq Y} \sum_{N \in \mathcal{N}(M)} \sum_{M' \leq M} \sum_{N' \leq N} e(\alpha m^2 n^2) + \sum_{M > Y} \sum_{N \in \mathcal{N}(M)} \sum_{M' \leq M} \sum_{N' \leq N} e(\alpha m^2 n^2).
\]

We can now apply Lemma 2.1, using (1) for \( M \leq Y \) and (2) for \( M > Y \), to obtain

\[
T_1 < \sum_{M \leq Y} \sum_{N \in \mathcal{N}(M)} P^{1+\epsilon} q^{\epsilon} \left( M^{1/2} q^{-1/2} + MP^{-1/2} + q^{1/2} M^{1/2} P^{-1} \right)
\]

\[
+ \sum_{M > Y} \sum_{N \in \mathcal{N}(M)} P^{1+\epsilon} q^{\epsilon} \left( q^{-1/4} + M^{-1/2} + M^{1/4} P^{-1/4} + q^{1/4} P^{-1/2} \right)
\]

Observe that in the first sum we have \( M \leq Y = \min(P^{1/4}, q^{1/2}, Pq^{-1/2}) \), while in the second sum we have \( M \leq P^{1/2} \) and \( M^{-1/2} < Y^{-1/2} = \max(P^{-1/8}, q^{-1/4}, P^{-1/2} q^{1/4}) \).

Hence we have

\[
T_1 < \sum_{M \leq Y} \sum_{N \in \mathcal{N}(M)} P^{1+\epsilon} q^{\epsilon} \left( q^{-1/4} + P^{-1/4} + q^{1/4} P^{-1/2} \right)
\]

\[
+ \sum_{M > Y} \sum_{N \in \mathcal{N}(M)} P^{1+\epsilon} q^{\epsilon} \left( q^{-1/4} + \max(P^{-1/8}, q^{-1/4}, P^{-1/2} q^{1/4}) + P^{1/8} P^{-1/4} + q^{1/4} P^{-1/2} \right)
\]

\[
< P^{1+\epsilon} q^{\epsilon} \left( q^{-1/4} + P^{-1/8} + q^{1/4} P^{-1/2} \right).
\]

The treatment of \( T_2 \) is similar.

We are now ready to prove a bound on the minor arcs.

**Theorem 2.3** We have

\[
\left| \int_m f(\alpha)^5 e(-X\alpha) \, d\alpha \right| < X^{3/2}.
\]

**Proof.** By Lemma 2.2, we have that

\[
|f(\alpha)| = \left| \sum_{n \leq P} d(n)e(\alpha(n)^2) \right| < P^{1+\epsilon} q^{\epsilon} \left( q^{-1/4} + P^{-1/8} + q^{1/4} P^{-1/2} \right).
\]
provided that $|\alpha - a/q| \leq P^\phi/(qX)$, and $q \leq X/P^\phi = P^{2-\phi}$. Since we are on the minor arcs, we also have that $q > P^\phi$. Thus we have

$$\sup_{\alpha \in m} |f(\alpha)| \ll P^{1+\epsilon} \left( P^{-\phi/4} + P^{-1/8} + P^{-\phi/4} \right).$$

Since $\phi < 1/2$ this is

$$\ll P^{1+\epsilon-\phi/4}q^\epsilon \ll P^{1+\epsilon-\phi/4}(P^{2-\phi})^\epsilon \ll P^{1+\epsilon-\phi/8}$$

for $\epsilon$ sufficiently small. Therefore we see that

$$\left| \int_{\mathbb{R}} f(\alpha)^5 e(-X\alpha) \, d\alpha \right| \leq \sup_{\alpha \in m} |f(\alpha)| \int_0^1 |f(\alpha)|^4 \, d\alpha \ll P^{1-\phi/4+\epsilon} \int_0^1 |f(\alpha)|^4 \, d\alpha.$$

Now, the last integral is given by

$$\int_0^1 |f(\alpha)|^4 \, d\alpha = \sum_{n, m, k, \ell \leq P} d(n)d(m)d(k)d(\ell) \int_0^1 e(\alpha(n^2 + m^2 - k^2 - \ell^2)) \, d\alpha,$$

which is the weighted number of solutions to $n^2 - k^2 = m^2 - \ell^2$ with $0 \leq n, m, k, \ell \leq P$. If $n = k$ then we must have $\ell = m$. This case gives $P$ choices for $n$ and $P$ choices for $\ell$ for a total of $P^2$ solutions. If $n \neq k$ then we have approximately $P^2$ ways to determine $n$ and $k$, each of which gives relatively few choices for $\ell$ and $m$. Thus the number of solutions in this case is $\ll P^{2+\epsilon}$. The weights $d(n)d(m)d(k)d(\ell)$ contribute powers of $\log P$, which can be absorbed by $P^\epsilon$. Thus we have

$$\left| \int_{\mathbb{R}} f(\alpha)^5 e(-X\alpha) \, d\alpha \right| \ll P^{3-\phi/8+2\epsilon} \ll X^{3/2}.$$

△

### 3 The Major Arcs

The results of the previous section imply that the contribution from the minor arcs is negligible. In this section, we turn our focus to the major arcs. We define

$$M(X) := \int_{\mathbb{R}} f(\alpha)^5 e(-X\alpha) \, d\alpha.$$

Then from Theorem 2.3, we see that

$$r(X) = M(X) + O(X^{3/2}).$$

Thus in proving Theorem 1.1, it suffices to evaluate $M(X)$. 5
We first consider \( f(\alpha) \) when \( \alpha = a/q, 1 \leq a \leq q \leq Q, (a, q) = 1 \). We can split the sum in terms of the residue class of \( n \mod q \). Then we have
\[
f(a/q) = \sum_{n \leq P} d(n)e\left(\frac{an^2}{q}\right) = \sum_{r=1}^{q} e\left(\frac{ar^2}{q}\right) \sum_{n \leq P \mod q \equiv r \mod q} d(n).
\]

By Theorem 3.3 of [2], the last sum satisfies
\[
\sum_{n \leq P \mod q \equiv r \mod q} d(n) = \frac{P}{q} \sum_{t|q} c_t(r) \left(\log \left(\frac{P}{t^2}\right) + 2\gamma - 1\right) + O\left((\sqrt{P} + q) \log 2q\right).
\]

Therefore we have
\[
f(a/q) = \sum_{r=1}^{q} e\left(\frac{ar^2}{q}\right) \left(\frac{P}{q} \sum_{t|q} c_t(r) \left(\log \left(\frac{P}{t^2}\right) + 2\gamma - 1\right) + O\left((\sqrt{P} + q) \log 2q\right)\right),
\]
for \( 1 \leq a < q \leq Q, (a, q) = 1 \).

For a general \( \alpha \in \mathfrak{M}(q, a) \), we can write \( \alpha = a/q + \beta \) with \(-\delta < \beta < \delta\). Now, let
\[
S(x) = \sum_{n \leq x} d(n)e\left(\frac{an^2}{q}\right),
\]
and
\[
F(x) = \sum_{r=1}^{q} e\left(\frac{ar^2}{q}\right) \left(\frac{x}{q} \sum_{t|q} c_t(r) \left(\log \left(\frac{x}{t^2}\right) + 2\gamma - 1\right)\right).
\]

Note that
\[
S(x) = F(x) + O\left(q(\sqrt{x} + q) \log 2q\right).
\]

Hence we have
\[
f\left(\frac{a}{q} + \beta\right) = \sum_{n \leq P} d(n)e\left(\left(\frac{a}{q} + \beta\right)n^2\right)
= \sum_{n \leq P} d(n)e\left(\frac{an^2}{q}\right) \left(e(\beta P^2) - 2\pi i\beta \int_{n}^{P} 2u e(\beta u^2) du\right)
= e(\beta P^2)S(P) - 2\pi i\beta \int_{0}^{P} S(u) 2u e(\beta u^2) du
= e(\beta P^2)F(P) - \left[F(u)e(\beta u^2)\right]_{u=0}^{P} + \int_{0}^{P} F'(u) e(\beta u^2) du
+ O\left((1 + 2\pi|\beta|P^2)q(\sqrt{P} + q) \log 2q\right)
= \int_{0}^{P} F'(u) e(\beta u^2) du + O\left((1 + |\beta|P^2)q(\sqrt{P} + q) \log 2q\right).
\]
If we now define
\[
R(\beta; q, a) := \int_0^P F'(u) e(\beta u^2) \, du
\]
and observe that
\[
F'(u) = \sum_{r=1}^q e\left(\frac{ar^2}{q}\right) \left(\frac{1}{q} \sum_{t|q} c_t(r) \left(\log\left(\frac{u}{t^2}\right) + 2\gamma\right)\right),
\]
then we see that
\[
f(a/q + \beta) = R(\beta; q, a) + E(\beta; q, a),
\]
where
\[
E(\beta; q, a) = O \left((1 + |\beta|P^2)q(\sqrt{P} + q)\log 2\right).
\]

We can now write
\[
M(x) = \sum_{q \leq Q} \sum_{a=1}^q \int_{\mathfrak{M}(q,a)} f(\alpha)^5 e(-X\alpha) \, d\alpha
\]
\[
= \sum_{q \leq Q} \sum_{a=1}^q \int_{-\delta}^\delta (R(\beta; q, a) + E(\beta; q, a))^5 e(-X(a/q + \beta)) \, d\beta.
\]
In order to deal with the major arcs thoroughly, we will need a rough bound on \(R(\beta; q, a)\).

**Lemma 3.1** We have
\[
R(\beta; q, a) \ll q^{-1/2} P \log P \left(1 + |\beta|P^2\right)^{1/2}.
\]

**Proof.** We can rewrite \(R(\beta; q, a)\) as
\[
R(\beta; q, a) = \int_0^P \sum_{r=1}^q e\left(\frac{ar^2}{q}\right) \left(\frac{1}{q} \sum_{t|q} c_t(r) \left(\log\left(\frac{u}{t^2}\right) + 2\gamma\right)\right) e(\beta u^2) \, du
\]
\[
= \int_0^P (C_1(q, a) \log u + C_2(q, a)) e(\beta u^2) \, du,
\]
where
\[
C_1(q, a) = \sum_{r=1}^q e\left(\frac{ar^2}{q}\right) \frac{1}{q} \sum_{t|q} c_t(r) \frac{1}{t}.
\]
and
\[ C_2(q, a) = \sum_{r=1}^{q} e \left( \frac{ar^2}{q} \right) \frac{1}{q} \sum_{t} c_t(r) \left( 2\gamma - \log t^2 \right). \]

It then follows that
\[ R(\beta; q, a) \ll \left| C_1(q, a) \int_0^P \log u e(\beta u^2) \, du \right|. \]

Now,
\[
C_1(q, a) = \sum_{r=1}^{q} e \left( \frac{ar^2}{q} \right) \frac{1}{q} \sum_{t} c_t(r) \\
= \frac{1}{q} \sum_{t} \frac{1}{t} \sum_{r=1}^{q} e \left( \frac{ar^2}{q} \right) e \left( \frac{sr}{t} \right) \\
= \frac{1}{q} \sum_{t} \frac{1}{t} \sum_{s=1}^{q} e \left( \frac{ar^2 + (sq/t)r}{q} \right) \\
= \frac{1}{q} \sum_{t} \frac{1}{t} \sum_{s=1}^{q} S(q, a, (sq/t)),
\]

where
\[ S(q, a, y) = \sum_{r=1}^{q} e \left( \frac{ar^2 + yr}{q} \right). \]

We claim that \(|S(q, a, y)| \leq \sqrt{2q}\). Indeed, we have
\[
|S(q, a, y)|^2 = \sum_{r=1}^{q} \sum_{k=1}^{q} e \left( \frac{a(r^2 - k^2) + y(r - k)}{q} \right).
\]

Letting \(h\) be such that \(r \equiv k + h \mod q\), this is
\[
= \sum_{h=1}^{q} \sum_{k=1}^{q} e \left( \frac{a(2kh + h^2) + yh}{q} \right) = \sum_{h=1}^{q} e \left( \frac{ah^2 + yh}{q} \right) \sum_{k=1}^{q} e \left( \frac{2ahk}{q} \right) \\
= q \sum_{h=1}^{q} e \left( \frac{ah^2 + yh}{q} \right) \leq 2q.
\]
The last inequality follows from the fact that \((a, q) = 1\), so \(q | 2ah\) if and only if \(q | 2h\) and there can be at most two such \(h\) with \(1 \leq h \leq q\). We can now see that

\[
|C_1(q, a)| \leq \frac{\sqrt{2q}}{q} \sum_{t \mid q} \frac{\phi(t)}{t} \ll \frac{d(q)}{\sqrt{q}} \ll q^{\epsilon - \frac{1}{2}}
\]

Using this bound and evaluating the integral gives the result. \(\Box\)

Now that we have a bound on \(R(\beta; q, a)\), we can start to evaluate \(M(X)\). We have

\[
M(X) = \sum_{q \leq Q} \sum_{a = 1}^{q} \int_{-\delta}^{\delta} R(\beta; q, a)^5 e(-X(a/q + \beta)) d\beta + E_1,
\]

where

\[
E_1 \ll \sum_{q \leq Q} \sum_{a = 1}^{q} \int_{-\delta}^{\delta} |R(\beta; q, a)|^4 |E(\beta; q, a)| + |E(\beta; q, a)|^5 d\beta
\]

\[
< \ll \sum_{q \leq Q} \sum_{a = 1}^{q} \int_{-\delta}^{\delta} \left( \frac{q^6 P \log P}{(q + q|\beta|P^2)^{1/2}} \right)^4 \left( (1 + |\beta|P^2)q\sqrt{P} \log 2q \right) + \left( (1 + |\beta|P^2)q\sqrt{P} \log 2q \right)^5 d\beta.
\]

The first part of the integral above is

\[
S_1 := \sum_{q \leq Q} \sum_{a = 1}^{q} \int_{-\delta}^{\delta} \left( \frac{q^6 P \log P}{(q + q|\beta|P^2)^{1/2}} \right)^4 \left( (1 + |\beta|P^2)q\sqrt{P} \log 2q \right) d\beta
\]

\[
= 2 \sum_{q \leq Q} \frac{\phi(q) \log 2q}{q^{1-\epsilon}} P^{5/2} (\log P)^4 \int_{0}^{\delta} \frac{P^2}{1 + \beta P^2} d\beta
\]

\[
= 2 \sum_{q \leq Q} \frac{\phi(q) \log 2q}{q^{1-\epsilon}} P^{5/2} (\log P)^4 \int_{0}^{\delta + P^{-2}} \frac{1}{\beta} d\beta
\]

\[
< \ll Q^{1+\epsilon} P^{5/2} (\log P)^4 \log \left( \frac{\delta + P^{-2}}{P^{-2}} \right) < \ll \frac{Q^{1+\epsilon} P^{5/2} (\log P)^4 \log(1 + \delta P^2)}{P^{2-\epsilon}}
\]

\[
< \ll X^{\delta+\epsilon} X^{5/4} (\log X)^4 X^{\delta-1} \ll X^{3/2} (\log X)^4.
\]
Meanwhile the second part of the integral in $E_1$ is

$$S_2 := \sum_{q \leq Q} \sum_{a=1}^{q} \int_{-\delta}^{\delta} \left( (1 + |\beta| P^2)^q \sqrt{P \log 2q} \right)^5 d\beta$$

$$= 2 \sum_{q \leq Q} q^5 \phi(q)(\log 2q)^5 P^{1/2} \int_{0}^{\delta} P^2 \left( 1 + \beta P^2 \right)^5 d\beta$$

$$= 2 \sum_{q \leq Q} q^5 \phi(q)(\log 2q)^5 P^{1/2} \int_{1}^{1 + \delta P^2} \beta^5 d\beta$$

$$<< Q^{6+\epsilon} P^{1/2} \left( (1 + \delta P^2)^6 + 1 \right) << X^{1/2}.$$  

Hence we have that $E_1 << X^{3/2} (\log X)^4$.

If we extend the integral in the main term out to infinity, we obtain

$$M(X) = \sum_{q \leq Q} \sum_{a=1}^{q} e\left( -\frac{Xa}{q} \right) \int_{-\infty}^{\infty} R(\beta; q, a)^5 e(-X\beta) d\beta + E_1 - E_2,$$

where $E_1$ is as above and

$$E_2 = \sum_{q \leq Q} \sum_{a=1}^{q} \left( \int_{-\infty}^{-\delta} R(\beta; q, a)^5 e(-X(a/q + \beta)) \right) d\beta + \int_{\delta}^{\infty} R(\beta; q, a)^5 e(-X(a/q + \beta)) d\beta$$

$$<< \sum_{q \leq Q} \sum_{a=1}^{q} \int_{-\delta}^{\infty} |R(\beta; q, a)|^5 d\beta << \sum_{q \leq Q} \sum_{a=1}^{q} \int_{\delta}^{\infty} \left( \frac{q^5 (1 + \delta P^2 |\beta|)^{1/2}}{P^2} \right)^5 d\beta$$

$$= \sum_{q \leq Q} \sum_{a=1}^{q} \int_{\delta}^{\infty} \left( \frac{q^{5/2} \log P}{P^{2} + \beta^{1/2}} \right)^5 d\beta = \sum_{q \leq Q} \sum_{a=1}^{q} \int_{\delta + P^{-2}}^{\infty} \left( \frac{q^{5/2} \log P}{\beta^{1/2}} \right)^5 d\beta$$

$$= \sum_{q \leq Q} \sum_{a=1}^{q} \left( q^{5/2} \log P \right)^5 \left( \delta + P^{-2} \right)^{-3/2} = \sum_{q \leq Q} \sum_{a=1}^{q} \left( q^{5/2} \log P \right)^5 \left( \frac{P^2}{1 + \delta P^2} \right)^{3/2}$$

$$<< Q^{3/2} (\log P)^5 P^3 << X^{3/2(1-\phi)+\epsilon} (\log X)^5 << X^{3/2} (\log X)^4.$$

So we see that

$$M(X) = \sum_{q \leq Q} \sum_{a=1}^{q} e\left( -\frac{Xa}{q} \right) \int_{-\infty}^{\infty} R(\beta; q, a)^5 e(-X\beta) d\beta + O(X^{3/2} (\log X)^4).$$
In proving Theorem 1.1, it remains only to show that the main term above is equal to the main term given in the theorem. We will first need an expression for \( R(\beta; q, a) \). Recall that

\[
R(\beta; q, a) = \int_0^P (C_1(q, a) \log u + C_2(q, a)) e(\beta u^2) \, du,
\]

where

\[
C_1(q, a) = \sum_{r=1}^q e\left(\frac{ar^2}{q}\right) \frac{1}{q} \sum_{t|q} \frac{c_t(r)}{t}
\]

and

\[
C_2(q, a) = \sum_{r=1}^q e\left(\frac{ar^2}{q}\right) \frac{1}{q} \sum_{t|q} \frac{c_t(r)}{t} (2\gamma - \log t^2).
\]

Making the substitution \( v = u^2 \) we obtain,

\[
R(\beta; q, a) = \int_0^{P^2} C(v; q, a) e(\beta v) \, dv,
\]

where

\[
C(v; q, a) = \frac{C_1(q, a) \log v + 2C_2(q, a)}{4\sqrt{v}}.
\]

We can now write

\[
R(\beta; q, a)^5 = \left( \int_0^{P^2} C(v; q, a) e(\beta v) \, dv \right)^5
\]

\[
= \int_0^{P^2} \cdots \int_0^{P^2} C(v_1; q, a) \cdots C(v_5; q, a) e(\beta(v_1 + \cdots + v_5)) \, dv_1 \cdots dv_5
\]

\[
= \int_0^{5P^2} \left( \int_0^{P^2} C(v_1; q, a) \cdots C(v_5; q, a) \, dv_1 \cdots dv_4 \right) e(\beta w) \, dw
\]

\[
= \int_0^G(w; q, a) e(\beta w) \, dw,
\]

where

\[
G(w; q, a) = \int_0^{P^2} C(v_1; q, a) \cdots C(v_5; q, a) \, dv_1 \cdots dv_4.
\]

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Clearly the support of $G(w; q, a)$ is confined to the integral $[0, 5P^2]$. Plugging our new expression for $R(\beta; q, a)^5$ into the main term, we see that

$$M(X) = \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q e\left(\frac{-Xa}{q} \right) \int_{-\infty}^{\infty} R(\beta; q, a)^5 e(-X\beta) \, d\beta + E_1 - E_2$$

$$= \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q e\left(\frac{-Xa}{q} \right) \int_{-\infty}^{\infty} \left( \int_0^{5P^2} G(w; q, a)e(\beta w) \, dw \right) e(-X\beta) \, d\beta + E_1 - E_2$$

$$= \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q e\left(\frac{-Xa}{q} \right) \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} G(w; q, a)e(\beta w) \, dw \right) e(-X\beta) \, d\beta + E_1 - E_2$$

$$= \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q e\left(\frac{-Xa}{q} \right) G(X; q, a) + O\left(X^{3/2}(\log X)^4\right),$$

by the Fourier inversion formula. Thus we have our desired result.

4 References


