Math 220 – **Take-home quizzes 7 & 8**

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Collaboration with other Math 220 students is permitted (and encouraged!) However, you have to write and understand all solutions on your own. Fair use of other sources is permitted. Submitting a solution you do not understand (e.g. cannot reproduce within three hours, given a pencil, a supply of paper and the textbook) is not permitted.

Most of the problems are more or less standard and should be solvable within half an hour (if your remember the definitions of rank, dimension, basis etc. with some properties and have reflected about each one for at least 10 minutes.) However, there are a few that may cause difficulties. If you have thought for half an hour and it didn't work, please explain in detail your conclusions, what have you tried, what went wrong, etc. – for substantial progress partial credit will be given, up to *75%*. If you just write "have no idea", you get *25%* just for saving my time and letting me know that this problem is difficult (as opposed to writing nonsense). Thanks!

If you get e.g. *70%*, it will count as *70%* for quiz 7 and *70%* for quiz 8.

1. Problem 2.9.8.
2. Problem 2.9.11.
3. Problem 2.9.26.
4. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. These vectors are linearly dependent, because of the linear dependence relation $\vec{u} + \vec{v} = \vec{w}$. Let $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$ be a basis for $\mathbf{R}^2$. Express $\vec{u}, \vec{v}, \vec{w}$ in $\mathbf{B}$-coordinates and verify that the same linear dependence relation still holds.
5. Let A be an $m \times n$ matrix. Let $\vec{v_1}, \ldots, \vec{v_p}$ be vectors in $\mathbf{R}^n$. Assume also that $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_p\vec{v_p}$ is in $NulA$ only if $c_1 = c_2 = \cdots = c_p = 0$. Explain why $\{A\vec{v_1}, \ldots, A\vec{v_p}\}$ must be linearly independent.

   Sidenote: this implies one half of the Rank Theorem, that $rankA \geq n - dimNulA$.
6. Let H be the plane $x + y + z = 0$, a 2-dimensional subspace of $\mathbf{R}^3$.
   (a) Find a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ that lies in H. (Hint: this is easier than you think.)
   (b) Let $\vec{u}, \vec{v}$ be any two vectors in $\mathbf{R}^3$. Is it always possible to find a linear combination of these two vectors, $\alpha\vec{u} + \beta\vec{v}$, that lies in H? (Hint: make a sketch of

H and $Span\{\vec{u},\vec{v}\}$.) (Hint: $H = Span\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$. Note that the set $\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \vec{u}, \vec{v}\right\}$ is linearly dependent.)

7. (a) Find any 2×2 matrix A such that $NulA = ColA$. (Hint: in this case $2 = dimNulA + rankA = 2rank(A)$, as $rankA = dimColA$ by definition.)
   (b) Explain why such a matrix must satisfy $A^2 = \mathbf{0}$. (Hint: think about $ColA$ as the range of a linear transformation.)
8. (a) If A is a 2×3 matrix of rank 1, what is the geometric form of $NulA$? (point/line/plane/3d space) Why?
   (b) Find a 3×3 matrix B with $ColB = Span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}, NulB = Span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$.

   Verify your answer.
   (c) Find a 3×3 matrix C with $ColC =$ the plane $x+y+z = 0$, $NulC = Span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.

   Verify your answer.