Math 220 – **SOLUTIONS to take-home quizzes 7 & 8**

due Thursday, 27, March, 2008

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Collaboration with other Math 220 students is permitted (and encouraged!) However, you have to write and understand all solutions on your own. Fair use of other sources is permitted. Submitting a solution you do not understand (e.g. cannot reproduce within three hours, given a pencil, a supply of paper and the textbook) is not permitted.

Most of the problems are more or less standard and should be solvable within half an hour (if your remember the definitions of rank, dimension, basis etc. with some properties and have reflected about each one for at least 10 minutes.) However, there are a few that may cause difficulties. If you have thought for half an hour and it didn't work, please explain in detail your conclusions, what have you tried, what went wrong, etc. – for substantial progress partial credit will be given, up to *75%*. If you just write "have no idea", you get *25%* just for saving my time and letting me know that this problem is difficult (as opposed to writing nonsense). Thanks!

If you get e.g. *70%*, it will count as *70%* for quiz 7 and *70%* for quiz 8.

1. Problem 2.9.8 — easy.
2. Problem 2.9.11 — easy.
3. Problem 2.9.26.

   Solution: $rank A = 4$ is the dimension of the column space of A. So the column space of A is a 4-dimensional subspace of $\mathbf{R}^n$. The columns 1, 3, 5 and 6 are, in fact, vectors belonging to the column space of A. These four vectors form a linearly independent set and belong to the column space, which is 4-dimensional. By the Basis Theorem, they must therefore be a basis for the column space.

   Comments: A may have have 4 rows, but may also have more. A may have 6 columns, but may also have more. The four mentioned columns may not be the pivot columns.

   For example, consider the following 4×6 matrix: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

   Columns 1, 2, 4, 6 are the pivot columns. However, columns 1, 3, 5, 6 are clearly *the same* as vectors in $\mathbf{R}^4$ and form a basis for the column space (which in this case is equal to the whole of $\mathbf{R}^4$) as well.

   Remember that it does not make sense to ask whether a column is independent or dependent. Only a set of columns (vectors) may be linearly independent or linearly dependent. It is not a property of a single vector, but how different vectors relate to each other.

4. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. These vectors are linearly dependent, because of the linear dependence relation $\vec{u} + \vec{v} = \vec{w}$. Let $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$ be a basis for $\mathbf{R}^2$. Express $\vec{u}, \vec{v}, \vec{w}$ in **B**-coordinates and verify that the same linear dependence relation still holds.

   Solution: $\vec{u} = (1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (0)\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

   $\vec{v} = (-2)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-3)\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

   $\vec{w} = (-1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-3)\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

   Thus, in B-coordinates we have $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_B, \vec{v} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}_B, \vec{w} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}_B$. These numbers can be easily found by solving three linear systems. Finally, the linear dependence relation still holds: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$. In other words, $\vec{u} + \vec{v} = \vec{w}$, and it doesn't matter, whether we are writing the coordinates relative to the standard basis or to some other basis: this linear relation still holds.

5. Let A be an $m \times n$ matrix. Let $\vec{v_1}, \ldots, \vec{v_p}$ be vectors in $\mathbf{R}^n$. Assume also that $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_p\vec{v_p}$ is in $NulA$ only if $c_1 = c_2 = \cdots = c_p = 0$. Explain why $\{A\vec{v_1}, \ldots, A\vec{v_p}\}$ must be linearly independent.

   Sidenote: this implies one half of the Rank Theorem, that $rankA \geq n - dimNulA$.

   Solution: It is given that $c_1\vec{v_1}+c_2\vec{v_2}+\cdots+c_p\vec{v_p}$ is in $NulA$ only if $c_1 = c_2 = \cdots = c_p = 0$. In other words, $A(c_1\vec{v_1}+c_2\vec{v_2}+\cdots+c_p\vec{v_p}) = \vec{0}$ only if $c_1 = c_2 = \cdots = c_p = 0$. Equivalently, $c_1(A\vec{v_1}) + c_2(A\vec{v_2}) + \cdots + c_p(A\vec{v_p}) = \vec{0}$ only if $c_1 = c_2 = \cdots = c_p = 0$. So $\{A\vec{v_1}, \ldots, A\vec{v_p}\}$ is linearly independent, by definition.

   Typical mistakes:

   $NulA = \{\vec{0}\}$ — no, not necessarily! For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Let $p = 1$ and $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $c_1\vec{v_1}$ is in $NulA$ only if $c_1 = 0$. Indeed, $A(c_1\vec{v_1}) = A\begin{bmatrix} c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ unless $c_1 = 0$. However, $NulA = Span\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\} \neq \{\vec{0}\}$. It can only be inferred that $Span\{\vec{v_1}, \ldots, \vec{v_p}\}$ and $NulA$ have only $\vec{0}$ in their intersection, but both can also include some nonzero vectors.

   We only need to prove that the columns of A are linearly independent — no, not necessarily! $A\vec{v_1}, \ldots, A\vec{v_p}$ are *not* columns of A. What if $\vec{v_1} = \vec{v_2}$? Then $A\vec{v_1} = A\vec{v_2}$, even if the columns of A are linearly independent.

   Neither can one hope to prove that the columns of A are linearly independent, for this is the same as $NulA = \{\vec{0}\}$, which is not necessarily true (see above).

6. Let H be the plane $x + y + z = 0$, a 2-dimensional subspace of $\mathbf{R}^3$.

   (a) Find a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ that lies in H. (Hint: this is easier than you think.)

   (b) Let $\vec{u}, \vec{v}$ be any two vectors in $\mathbf{R}^3$. Is it always possible to find a linear combination of these two vectors, $\alpha\vec{u} + \beta\vec{v}$, that lies in H? (Hint: make a sketch of H and $Span\{\vec{u}, \vec{v}\}$.) (Hint: $H = Span\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$. Note that the set $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{u}, \vec{v} \right\}$ is linearly dependent.)

Solution: many students noticed that there is a very easy answer: $0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is clearly in H. I counted this answer is correct. However, this is not what was meant: I simply forgot to write *nontrivial*. Is there a nontrivial linear combination (not with both weights equal to zero) of these two vectors that lies in H? Clearly, yes. $1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-6)\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -11 \\ 2 \\ 9 \end{bmatrix}$ is one option. It is in H, since $(-11) + 2 + 9 = 0$.

What about the second question? Few students noticed that the same trick works: $0\vec{u} + 0\vec{v}$ is again in H. But here, too, one can find a nontrivial linear combination. If $\{\vec{u}, \vec{v}\}$ happens to be linearly dependent, then one can express $\vec{0}$ as a nontrivial linear combination of $\vec{u}$ and $\vec{v}$. Otherwise, $Span\{\vec{u}, \vec{v}\}$ is a plane in $\mathbf{R}^3$, passing through the origin. H is another plane in $\mathbf{R}^3$, also passing through the origin. If these two planes do not coincide, then their intersection is a line. Any nonzero point on this line works. (The hint about linear dependence helps to formalize this argument — basically, to prove that the intersection of two planes containing the origin is at least a line).

There is yet an easier solution. The idea is to try understand, how the weights 1 and -6 in the first half of the problem were found. Let α be the sum of entries of $\vec{u}$ and β be the sum of entries of $\vec{v}$. For example, for $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ these would be $\alpha = 6$ and $\beta = -1$. Now, if $\alpha = 0$ (or $\beta = 0$), then $\vec{u}$ (respectively, $\vec{v}$) itself is already a linear combination of $\vec{u}$ and $\vec{v}$ that is in H, so we are done. Otherwise take $\beta\vec{u} - \alpha\vec{v}$. This is in H (why?)

7. (a) Find any 2×2 matrix A such that $NulA = ColA$. (Hint: in this case $2 = dimNulA + rankA = 2rank(A)$, as $rankA = dimColA$ by definition.)

(b) Explain why such a matrix must satisfy $A^2 = \mathbf{0}$. (Hint: think about $ColA$ as the range of a linear transformation.)

Solution:

There are lots of such matrices. One is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. In this case, both the column space and the null space are $Span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$. Another one is $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. In this case, both the column space and the null space are $Span\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$. Yet another one is $\begin{bmatrix} -2 & 7 \\ -4/7 & 2 \end{bmatrix}$. In this case, both the column space and the null space are $Span\{\begin{bmatrix} 7 \\ 2 \end{bmatrix}\}$. The hint only explained that such a matrix has to have rank one.

All these matrices square to zero. Check it. Now notice, what is it that you are doing. You are multiplying rows by columns and you keep getting zeroes. When you multiply any of these matrices by its first or second column, you get the zero vector. This is not surprising: in all these cases, the column space (containing both columns) is also the null space. This is precisely the reason all these matrices square to zero.

Alternatively, $A\vec{x}$ is in $ColA$ for any $\vec{x}$ in $\mathbf{R}^2$, essentially by definition of $ColA$. Now, $ColA = NulA$, so $A(A\vec{x}) = \vec{0}$. So $A^2\vec{x} = \vec{0}$ for any $\vec{x}$. Hence, $A^2 = 0$.

In terms of linear transformations: if A is the standard matrix of a linear transformation T, then A^2 is the standard matrix of a linear transformation, that consists in applying T twice. The range of T is $ColA$. So after the first application of T, any vector will end up in $ColA = NulA$. After the second application, any vector from $NulA$ will be sent to $\vec{0}$. So overall, two applications of T send any vector to $\vec{0}$. Thus, $A^2 = 0$ (here 0 is a 2×2 matrix).

8. (a) If A is a 2×3 matrix of rank 1, what is the geometric form of $NulA$? (point/line/plane/3d space) Why?

(b) Find a 3×3 matrix B with $ColB = Span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}, NulB = Span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

Verify your answer.

(c) Find a 3×3 matrix C with $ColC =$ the plane $x+y+z = 0$, $NulC = Span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

Verify your answer.

Solution:

The first question is trivial. By the rank theorem (or because there are two free variables, since there are three columns and one column is a pivot column), $NulA$ is 2-dimensional — a plane.

The second question. We are looking for a 3×3 matrix B. In particular, B satisfies

$B\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ and $B\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. (We know that $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$ must be in $NulB$). But $B\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $B\begin{bmatrix}0\\1\\0\end{bmatrix}$ are just the first and the second columns of B! Hence, inevitably $B=\begin{bmatrix}0&0&?\\0&0&?\\0&0&?\end{bmatrix}$. We also know $ColB=Span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$. At least we have to make sure that the three numbers in the last column are equal to each other, so that the last column is a multiple of $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

It is now easy to verify that any such matrix works. For example, $B=\begin{bmatrix}0&0&2\\0&0&2\\0&0&2\end{bmatrix}$ is an answer. It is standard to find $ColB$ and $NulB$ and verify that these are as required.

Finally, the last question. We are looking for a 3×3 matrix C. We know that $C\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. So the sum of the three columns of C has to be $\begin{bmatrix}0\\0\\0\end{bmatrix}$. This means that we can find the first two columns of C, and this will uniquely determine the third column. Now simply choose any two vectors that form a basis for $x+y+z=0$ and make them the first two columns of C. Compute the third column so that the three columns add up to $\vec{0}$. This matrix will have two pivots. This will guarantee that the column space is exactly the plane $x+y+z=0$, and not the whole of $\mathbf{R}^3$ (which may have happened with some other third column), and the null space is just $Span\{\begin{bmatrix}1\\1\\1\end{bmatrix}\}$ and not more than that.

One possible answer is $C=\begin{bmatrix}1&20&-21\\3&50&-53\\-4&-70&74\end{bmatrix}$. The first two columns are in the plane $x+y+z=0$, yet not proportional to each other. So they form a basis for this plane. The columns add up to $\vec{0}$, so $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is in the null space. Since the column space is at least the plane $x+y+z=0$ and the null space is at least $Span\{\begin{bmatrix}1\\1\\1\end{bmatrix}\}$, they are equal to this plane and this line (as the dimensions have to add up to 3) and we are done.

Alternatively, one can simply find $ColC$ and $NulC$ by the standard procedure.