Solutions to Quiz 2

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Problem 1 *Come up with an example of four vectors in* $\mathbf{R}^3$*:* $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}, \overrightarrow{t}$ *that are all different (*$\overrightarrow{u} \neq \overrightarrow{v} \neq \overrightarrow{w} \neq \overrightarrow{t}, \overrightarrow{u} \neq \overrightarrow{w}, \overrightarrow{v} \neq \overrightarrow{t}, \overrightarrow{u} \neq \overrightarrow{t}$*) and yet* $Span\{\overrightarrow{u}, \overrightarrow{v}\} = Span\{\overrightarrow{w}, \overrightarrow{t}\}$*. Justify your answer.*

There are two approaches to solving this problem. One is to choose the set $Span\{\overrightarrow{u}, \overrightarrow{v}\}$ in advance and then to pick four vectors appropriately. One easy choice is to use the xy-plane. The equation of the xy-plane is $z = 0$, it consists precisely of the vectors with the third coordinate equal to zero. Then there are many choices, such as $\overrightarrow{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \overrightarrow{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \overrightarrow{t} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Why is it that $Span\{\overrightarrow{u}, \overrightarrow{v}\} = Span\{\overrightarrow{w}, \overrightarrow{t}\} = xy$-plane?

Clearly, all linear combinations of these vectors have the third coordinate equal to zero. So we just need to ensure we can obtain any such vector, e.g. $\begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}$. Indeed, $\begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} = b_1\overrightarrow{u} + b_2\overrightarrow{v} = (b_1 + b_2)/2\overrightarrow{w} + (b_1 - b_2)/2\overrightarrow{t}$.

That was one solution. In fact, it would be easier to choose a line instead of a plane. For instance, the x-axis $y = z = 0$. Then again there are lots of choices, such as $\overrightarrow{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{w} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{t} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$. Clearly, $Span\{\overrightarrow{u}, \overrightarrow{v}\} = Span\{\overrightarrow{u}\} = Span\{\overrightarrow{v}\} = Span\{\overrightarrow{w}, \overrightarrow{t}\} = Span\{\overrightarrow{w}\} = Span\{\overrightarrow{t}\} = x$-axis. Indeed, to be a multiple of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the same as to be a multiple of $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$. More formally, anything of the form $c\overrightarrow{u} + d\overrightarrow{v}$ can be written as $c\overrightarrow{u} + d\overrightarrow{v} = (c + 2d)\overrightarrow{u} = (c + 2d)/4\overrightarrow{w}$.

Another approach is not to choose the span in advance, but to make the four vectors related in some way. Perhaps the easiest is to make $\overrightarrow{u}$ to be a multiple of $\overrightarrow{w}$ and $\overrightarrow{v}$ to be a multiple of $\overrightarrow{t}$. (This is not the same as to make $\overrightarrow{u}$ a multiple of $\overrightarrow{v}$ and to make $\overrightarrow{w}$ a multiple of $\overrightarrow{t}$!) For instance, let $\overrightarrow{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\overrightarrow{t} = \begin{bmatrix} 3 \\ 8 \\ 14 \end{bmatrix}$ and let $\overrightarrow{u} = 2\overrightarrow{w}, \overrightarrow{v} = 3\overrightarrow{t}$. Now we need to show that the two spans are indeed equal. How can we show that two sets are equal? We need to show that if a vector is in the first set then it is in the second, and vice versa. So take some vector in $Span\{\overrightarrow{u}, \overrightarrow{v}\}$: $c\overrightarrow{u} + d\overrightarrow{v}$. Then it is also in $Span\{\overrightarrow{w}, \overrightarrow{t}\}$, since $c\overrightarrow{u} + d\overrightarrow{v} = (2c)\overrightarrow{w} + (3d)\overrightarrow{t}$. Conversely, any vector in the latter span, $p\overrightarrow{w} + q\overrightarrow{t}$ is automatically in the former: $p\overrightarrow{w} + q\overrightarrow{t} = (p/2)\overrightarrow{u} + (q/3)\overrightarrow{t}$.

It is not necessary to take multiples. One can choose any $\overrightarrow{u}$ and $\overrightarrow{v}$ and then make $\overrightarrow{w} = 2\overrightarrow{u} + 3\overrightarrow{v}$ and $\overrightarrow{t} = \overrightarrow{u} - 7\overrightarrow{v}$. Just make sure that the resulting four vectors are all distinct; spans will coincide no matter what $\overrightarrow{u}$ and $\overrightarrow{v}$ are. One can show that it happens precisely because the square matrix $\begin{bmatrix} 2 & 3 \\ 1 & -7 \end{bmatrix}$ has 2 pivots.

One typical mistake: the sole fact that $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{w} + \overrightarrow{t}$ is not sufficient. For example, let $\overrightarrow{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\overrightarrow{w} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\overrightarrow{v} = -\overrightarrow{u}, \overrightarrow{t} = -\overrightarrow{w}$. Then $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{w} + \overrightarrow{t} = \overrightarrow{0}$, yet the spans are not equal.

Many other typical mistakes come from the abuse of terminology. For example, it does not make any sense to say that $Span\{\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}\}$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The reason is that this span is a line, that is, an infinite set of vectors in $\mathbf{R}^3$, specifically those satisfying $x = y = z$. On the other hand, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is just a single vector. The correct statement would be $Span\{\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}\} = Span\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$.

In some solutions it was only proved that $Span\{\overrightarrow{u}, \overrightarrow{v}\} \subset Span\{\overrightarrow{w}, \overrightarrow{t}\}$, but not conversely. In other words, it was demonstrated that all vectors in the former span are also in the latter span, but there was the remining possibility that the latter span is strictly larger. This has to be ruled out to show that the two are equal.

Exercise 1 *If you want to practice, find an example of three vectors in* $\mathbf{R}^3$, $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ *such that* $Span\{\overrightarrow{u}, \overrightarrow{v}\} = Span\{\overrightarrow{v}, \overrightarrow{w}\} = Span\{\overrightarrow{u}, \overrightarrow{w}\}$. *Additional constraints:* $Span\{\overrightarrow{u}\} \neq Span\{\overrightarrow{v}\} \neq Span\{\overrightarrow{w}\} \neq Span\{\overrightarrow{u}\}$.

Exercise 2 *Further practice: can there be four vectors in* $\mathbf{R}^3$, $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}, \overrightarrow{t}$ *such that the four sets* $Span\{\overrightarrow{u}\}, Span\{\overrightarrow{v}\}, Span\{\overrightarrow{w}\}, Span\{\overrightarrow{t}\}$ *are all different and, moreover, the sets* $Span\{\overrightarrow{u}, \overrightarrow{v}\}$ *and* $Span\{\overrightarrow{w}, \overrightarrow{t}\}$ *have only one vector in common (namely,* $\overrightarrow{0}$*)?*

Once you become accustomed to a certain kind of geometrical thinking and also become comfortable with sets and set-theoretic operations such as the intersection of two sets, these questions become trivial.