Solutions to Quiz 12

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Problem 1 *Let* $W = Span\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. *Find* $proj_W \vec{u}$ *and* $proj_{(W^\perp)} \vec{u}$.

1 Short solution

Clearly, the three vectors are linearly independent. Therefore, $W = \mathbb{R}^3$ and $W^\perp = \{\vec{0}\}$.

$proj_W \vec{u} = u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. $proj_{(W^\perp)} \vec{u} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

2 Long solution

This is how you would solve this problem by applying the algorithms and formulas that you have seen in class and in the homework.

We are going to project u onto the subspace W. For that we need an orthogonal basis for W. The formula in the book will not work if you use it with a basis that is not orthogonal. The basis for W given in the problem statement is *not* orthogonal (e.g. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \neq 0$).

The straightforward way is to apply the Gram-Schmidt process. Formally,

$\vec{x_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$\vec{v_1} = \vec{x_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

$$\vec{v_2} = \vec{x_2} - proj_{v_1}\vec{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - (1/2) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}.$$

Check: $\vec{v_1} \perp \vec{v_2}$. So we are on the right track.

$$\vec{v_3} = \vec{x_3} - proj_{v_1}\vec{x_3} - proj_{v_2}\vec{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}} \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - (2/2) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - (0) \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We have found an orthogonal basis for W (namely, $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$). $W = Span\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Now it is easy to project.

$$proj_W\vec{u} = proj_{v_1}\vec{u} + proj_{v_2}\vec{u} + proj_{v_3}\vec{u} = \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}} \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} +$$

$$\frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (3/2) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + ((1/2)/(1/2)) \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} + (3/1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Finally, $proj_W\vec{u} = \begin{bmatrix} 3/2 \\ 3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

$\vec{u} = proj_W\vec{u} + proj_{(W^\perp)}\vec{u} = \vec{u} + proj_{(W^\perp)}$. So $proj_{(W^\perp)}\vec{u} = \vec{0}$.

3 Discussion

The idea here is that $\vec{u}$ is already in W. Therefore, the projection of $\vec{u}$ onto W is just $\vec{u}$ itself. Similarly, if you are projecting points in $\mathbb{R}^2$ onto the x-axis, then points that are already on that axis (e.g. $(1,0)$) will stay where they are.

Alternatively, think about orthogonal projection as *the best approximation.* What is the best approximation to $\vec{u}$ by elements of W? Well, there is no need to approximate anything; we can just take the exact value. The best approximation to $\vec{u}$ by elements of W is (in this case) just $\vec{u}$ itself, because $\vec{u}$ *is* in W.

$W^\perp$ contains only one vector (the zero vector). So no matter what you project onto $W^\perp$, there is only one vector you can get: $\vec{0}$.

Why is it that $W^\perp$ contains only the zero vector? Vectors in $W^\perp$ have to be orthogonal to all vectors in W. If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in $W^\perp$, then *in particular* it has to be orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, since W is given as their span.

But this is too restrictive.

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \perp \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ yields $x + y = 0$.

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \perp \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ yields $y = 0$.

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \perp \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ yields $x + y + z = 0$.

The only solution to this system of three linear homogeneous equations is $x = y = z = 0$. Therefore, the only vector in $W^\perp$ is $\vec{0}$. Its orthogonal complement, W, contains all vectors orthogonal to the zero vector, i.e. all vector with three entries. $W = \mathbb{R}^3$.