Solutions to Quiz 11

Instructor: Sergey Orshanskiy

PSU Math 220. Thursday, 4/24/8

Problem 1 *Let W be the plane $3x+2y+z=0$ (or $x+2y+3z=0$) in $\mathbb{R}^3$. Find $W^\perp$, its orthogonal complement. (The solution given here is for $3x+2y+z=0$, but it is absolutely the same for e.g. $7x-y+15z=0$.)*

Before reading the solution sketch the line $x+2y=0$ in the plane and the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (an arrow from $(0,0)$ to $(1,2)$). Then continue reading.

1 First solution: absolutely straightforward

If you just want to solve this problem as quickly and as reliably as possible, choose the most straightforward way. $W^\perp$ consists precisely of vectors $\vec{v}$ satisfying $\vec{v} \perp W$. By definition, $\vec{v}$ is orthogonal to W if $\vec{v}$ is orthogonal to any vector in W. Equivalently, $\vec{v}$ is orthogonal to all vectors in some basis for W.

How to find a basis for W? There is a standard technique for finding a basis for the null space of a matrix or the column space of a matrix. The latter is not very useful (because to find such a matrix you probably need to find a basis for W in the first place). But the former is extremely useful. Let $A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$. Then $W = NulA$. *The subspace of all vectors in $\mathbb{R}^3$ satisfying $3x+2y+z=0$ is the same as (equal to) the null space of the matrix A.*

Please, stop here until you completely understand the last sentence. It is *trivially* true that $W = NulA$. For example, $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. This vector satisfies $3x+2y+z=0$ (indeed, $3(1)+2(-1)+(-1)=0$), hence this vector is in W. Also, $A\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 3(1)+2(-1)+1(-1)=0$, hence this vector is in $NulA$.

Now find $NulA$. $x=-2/3y-1/3z$, y and z are free. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2/3y-1/3z \\ y \\ z \end{bmatrix} =$

$y\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}$. Indeed, $W = NulA = Span\left\{\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Check your answer. $\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}$ *clearly* satisfy $3x + 2y + z = 0$. These two vectors form a linearly independent set, hence span a 2-dimensional subspace of $\mathbb{R}^3$, viz. W.

The rest is easy. $\vec{v}$ is in $W^\perp$ precisely if $v \perp \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}$ and $v \perp \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}$. Why? Well, if $\vec{v}$ is in $W^\perp$, then it is orthogonal to all vectors in W and to the two given ones in particular. Conversely, if v is orthogonal to these two vectors, it is also orthogonal to all their linear combinations and thus to their span, but that is W.

Rewriting the last two conditions for $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, one obtains: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} = 0$. Equivalently, $\begin{cases} -2/3x + y = 0 \\ -1/3x + z = 0 \end{cases}$

The solution set to this homogeneous system is $Span\left\{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right\}$, as is easily verified. (*DO verify it*, writing down everything. Otherwise you risk missing the point after reading all that...) So this is the answer: $Span\left\{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right\}$.

2 Second solution

The orthogonal complement to a plane in $\mathbb{R}^3$ is clearly a line (of course, passing through the origin). *Which line?*

This about the equation $3x + 2y + z = 0$. W contains all such vectors. Any vector in W satisfied $3x + 2y + z = 0$. So for any $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in W it is true that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 0$. Hence, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is in $W^\perp$ and spans it (because the complement is one-dimensional).

$\vec{v} \cdot \vec{n} = 0$ is called *the normal equation of a plane.* Here $\vec{n}$ is a vector ($\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ in the above example) normal (i.e. orthogonal) to a plane. Then being in the plane is precisely being orthogonal to this vector.

If this doesn't make any sense, notice that the vectors on the line $x + 2y = 0$ in $\mathbb{R}^2$ are precisely the vectors that are orthogonal to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Sketch $x + 2y = 0$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, if you haven't done it yet.

3 Common mistakes

Again, W is the subspace $3x + 2y + z = 0$.

3.1 One mistake

W is not a vector. It is a subspace of $\mathbb{R}^3$. So $W \neq \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. It is not spanned by $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ either. This is for two reasons. Firstly, one vector cannot span a two-dimensional subspace. Secondly, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is not in W because it does not satisfy the equation $3x + 2y + z = 0$. Indeed, $3(3) + 2(2) + 1(1) = 14 \neq 0$.

3.2 Another mistake

$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is in W. However, not everything orthogonal to this vector is in $W^\perp$. More specifically, $Span\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}^\perp$ is a 2-dimensional subspace of $\mathbb{R}^3$ that contains inside it a smaller (1-dimensional) subspace of $\mathbb{R}^3$, $W^\perp$. Vectors in $Span\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}^\perp$ are orthogonal to $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Vectors in $W^\perp$ are orthogonal to any vector in W, not just $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. As an example, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \perp \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, yet $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is not in $W^\perp$: it is not orthogonal to $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, another vector in W.