

## MATH 251 SOLUTIONS

1.

$$y'' - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = -1$$

**Solution:**

First of all, apply Laplace transform for both sides to get:

$$[s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 6Y(s) = 0$$

Then substitute in initial value  $y(0) = 1$ ,  $y'(0) = -1$  to get:

$$(s^2 - s - 6)Y(s) - s + 1 - (-1) = 0$$

Solve for  $Y(s)$ :

$$Y(s) = \frac{s - 2}{s^2 - s - 6} = \frac{s - 2}{(s - 3)(s + 2)}$$

Next we want to find the inverse of Laplace transform of  $Y(s)$ , which will be our solution. The idea is to break the fraction into two other fractions, then each of them is in  $L\{e^{at}\} = \frac{1}{s-a}$  pattern. Pretend that we have achieved this:

$$\frac{s - 2}{(s - 3)(s + 2)} = \frac{A}{s - 3} + \frac{B}{s + 2} = \frac{A(s + 2) + B(s - 3)}{(s - 3)(s + 2)} = \frac{(A + B)s + (2A - 3B)}{(s - 3)(s + 2)}$$

Then correspond the coefficient for numerator to get:

$$s : \quad A + B = 1$$

$$1 : \quad 2A - 3B = -2$$

Solve it, we have:  $A = \frac{1}{5}$ ,  $B = \frac{4}{5}$ . then substitute back, we get:

$$Y(s) = \frac{1}{5} \frac{1}{s - 3} + \frac{4}{5} \frac{1}{s + 2}$$

Thus we can now write down the solution directly:

$$y(t) = L^{-1}\{Y(s)\} = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$$

2.

$$y'' + 9y = \cos(2t), \quad y(0) = 1 \quad y'(0) = 0$$

**Solution:**

First apply Laplace transform for both sides to get:

$$[s^2Y(s) - sy(0) - y'(0)] + 9Y(s) = \frac{s}{s^2 + 2^2}$$

Then substitute in initial value  $y(0) = 1$ ,  $y'(0) = 0$  to get:

$$(s^2 + 9)Y(s) - s = \frac{s}{s^2 + 4}$$

Solve for  $Y(s)$ :

$$Y(s) = \frac{s(s^2 + 4) + s}{(s^2 + 9)(s^2 + 4)} = \frac{s^3 + 5s}{(s^2 + 9)(s^2 + 4)}$$

Next we want to find the inverse of Laplace transform of  $Y(s)$ , which will be our solution. The idea is to break the fraction into two other fractions, then each of them will be in some pattern. Pretend that we have achieved this:

$$\frac{s^3 + 5s}{(s^2 + 9)(s^2 + 4)} = \frac{As + B}{s^2 + 9} + \frac{Cs + D}{s^2 + 4} = \frac{(As + B)(s^2 + 4) + (Cs + D)(s^2 + 9)}{(s^2 + 9)(s^2 + 4)}$$

Then correspond the coefficient for numerator to get:

$$s^3 : A + C = 1$$

$$s^2 : B + D = 0$$

$$s : 4A + 9C = 5$$

$$1 : 4B + 9D = 0$$

Solve it, we have:  $A = \frac{4}{5}$ ,  $C = \frac{1}{5}$   $B = D = 0$ . then substitute back, we get:

$$Y(s) = \frac{\frac{4}{5}s}{(s^2 + 9)} + \frac{\frac{1}{5}s}{(s^2 + 4)} = \frac{4}{5} \frac{s}{(s^2 + 3^2)} + \frac{1}{5} \frac{s}{(s^2 + 2^2)}$$

Thus we can now write down the solution directly:

$$y(t) = L^{-1}\{Y(s)\} = \frac{4}{5} \cos(3t) + \frac{1}{5} \cos(2t)$$

3.

$$y'' + y = 1 - u_{\pi/2}(t), \quad y(0) = 0, \quad y'(0) = 1$$

**Solution:**

First apply Laplace transform for both sides to get:

$$[s^2 Y(s) - sy(0) - y'(0)] + Y(s) = \frac{1}{s} - \frac{e^{-\frac{\pi}{2}s}}{s}$$

Then substitute in initial value  $y(0) = 0$ ,  $y'(0) = 1$  to get:

$$(s^2 + 1)Y(s) - 1 = \frac{1}{s} - \frac{e^{-\frac{\pi}{2}s}}{s}$$

Solve for  $Y(s)$ :

$$Y(s) = \frac{1}{s^2 + 1} + \frac{1}{s(s^2 + 1)} - \frac{e^{-\frac{\pi}{2}s}}{s(s^2 + 1)} = \frac{1}{s^2 + 1} + H(s) - e^{-\frac{\pi}{2}s}H(s)$$

Obviously  $L^{-1}\{\frac{1}{s^2+1}\} = \sin t$ , so we only need to take care of  $H(s) = \frac{1}{s(s^2+1)}$ , i.e. to find  $h(t) = L^{-1}\{H(s)\}$ . The idea is also to break the fraction into two other fractions, then each of them is in some pattern that we can easily identify. Pretend that we have achieved this:

$$H(s) = \frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1} = \frac{A(s^2 + 1) + (Bs + C)s}{s(s^2 + 1)}$$

Then correspond the coefficient for numerator to get:

$$s^2 : A + B = 0$$

$$s : C = 0$$

$$1 : A = 1$$

Solve it, we have:  $A = 1$ ,  $B = -1$ ,  $C = 0$ . then substitute back, we get:

$$H(s) = \frac{1}{s} - \frac{s}{s^2 + 1}$$

Hence we see:

$$h(t) = L^{-1}\{H(s)\} = 1 - \cos t$$

Finally we get the solution: (use the formula  $L\{u_c(t)h(t-c)\} = e^{-cs}H(s)$ ,  $c = \frac{\pi}{2}$  for the third part)

$$\begin{aligned} y(t) &= L^{-1}\{Y(s)\} = \sin t + h(t) - u_{\pi/2}(t)h(t - \frac{\pi}{2}) \\ &= \sin t + 1 - \cos t - u_{\pi/2}(t)(1 - \cos(t - \frac{\pi}{2})) \\ &= \sin t + 1 - \cos t - u_{\pi/2}(t)(1 - \sin t) \end{aligned}$$

4.

$$y'' + 2y' + 2y = \cos t + \delta(t - \pi/2), \quad y(0) = 0, \quad y'(0) = 0$$

**Solution:**

First apply Laplace transform for both sides to get:

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 2Y(s) = \frac{s}{s^2 + 1} + e^{-\frac{\pi}{2}s}$$

Then substitute in initial value  $y(0) = 0$ ,  $y'(0) = 0$  to get:

$$(s^2 + 2s + 2)Y(s) = \frac{s}{s^2 + 1} + e^{-\frac{\pi}{2}s}$$

Solve for  $Y(s)$ :

$$Y(s) = \frac{s}{(s^2 + 1)(s^2 + 2s + 2)} + \frac{e^{-\frac{\pi}{2}s}}{s^2 + 2s + 2} = \frac{s}{(s^2 + 1)(s^2 + 2s + 2)} + e^{-\frac{\pi}{2}s} \frac{1}{(s + 1)^2 + 1}$$

Applying the formula  $L\{u_c(t)f(t - c)\} = e^{-cs}F(s)$  for  $c = \frac{\pi}{2}$ . It's easy to see:

$$L^{-1}\left\{e^{-\frac{\pi}{2}s} \frac{1}{(s + 1)^2 + 1}\right\} = u_{\pi/2}(t)e^{-(t-\pi/2)} \sin(t - \pi/2)$$

so we only need to take care of  $H(s) = \frac{s}{(s^2+1)(s^2+2s+2)}$ , i.e. to find  $h(t) = L^{-1}\{H(s)\}$ . The idea is also to break the fraction into two other fractions, then hopefully each of them is in some pattern that we can easily identify. Pretend that we have achieved this:

$$H(s) = \frac{s}{(s^2 + 1)(s^2 + 2s + 2)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2s + 2} = \frac{(As + B)(s^2 + 2s + 2) + (Cs + D)(s^2 + 1)}{(s^2 + 1)(s^2 + 2s + 2)}$$

Then correspond the coefficient for numerator to get:

$$s^3 : A + C = 0$$

$$s^2 : 2A + B + D = 0$$

$$s : 2A + 2B + C = 1$$

$$1 : 2B + D = 0$$

Solve it, we have:  $A = \frac{1}{5}$ ,  $B = \frac{2}{5}$ ,  $C = -\frac{1}{5}$ ,  $D = -\frac{4}{5}$ . then substitute back, we get:

$$H(s) = \frac{\frac{1}{5}s + \frac{2}{5}}{s^2 + 1} + \frac{-\frac{1}{5}s - \frac{4}{5}}{s^2 + 2s + 2} = \frac{1}{5} \frac{s}{s^2 + 1} + \frac{2}{5} \frac{1}{s^2 + 1} - \frac{1}{5} \frac{s + 1}{(s + 1)^2 + 1} - \frac{3}{5} \frac{1}{(s + 1)^2 + 1}$$

Hence we see:

$$h(t) = L^{-1}\{H(s)\} = \frac{1}{5} \cos t + \frac{2}{5} \sin t - \frac{1}{5} e^{-t} \cos t - \frac{3}{5} e^{-t} \sin t$$

Finally we get the solution:

$$\begin{aligned} y(t) &= L^{-1}\{Y(s)\} = h(t) + u_{\pi/2}(t)e^{-(t-\pi/2)} \sin(t - \pi/2) \\ &= \frac{1}{5} \cos t + \frac{2}{5} \sin t - \frac{1}{5} e^{-t} \cos t - \frac{3}{5} e^{-t} \sin t + u_{\pi/2}(t)e^{-(t-\pi/2)} \sin(t - \pi/2) \end{aligned}$$