# Philosophy of Teaching Mathematics

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This essay describes my teaching philosophy, specifically concerning my background in teaching mathematics at Penn State where I taught three undergraduate courses: Differential Equations, Vector Calculus and Linear Algebra (in Spring 2006, Fall 2006 and Fall 2007, respectively). I find the last one (Linear Algebra) to have been the most successful, so most of the examples illustrate my teaching of linear algebra.

## 1 Lecturing

The problems I am facing in the classroom at Penn State are, as I now realize, very unusual for my (Russian) educational background and are very interesting for me, personally. Surprisingly, I found the students here to be highly motivated and hardworking and many of them have a sincere desire to master whatever I am teaching, yet they are literally in need of guidance. I found that it is more important to just teach thinking and the clarity of thinking, to help the students to be able to evaluate their own progress and to learn to individually pursue their own way through mathematics. The matters are complicated, it appears, by a blatantly wrong impression about mathematics. It is fortunate for me that the students have the qualities that I don't like to teach (e.g. being able to spend hours on weekly homework assignments throughout the whole semester — I still find it truly incredible!), yet lack the qualities that I do like to teach them. Specifically, I am striving to convey that

1. Mathematical statements can be 'really' true or false by virtue of their meaning, not just because it is known or written in the book;

2. Understanding the syntax is essential: to find out if a statement is true it is essential to understand, what it says;

3. An indispensable tool to achieving the latter is articulating precise questions, such as "why is step 3 in this explanation correct?", "what does this word in the definition mean?", "what is $m$? is it for any $m$?", etc.

The students, and I had this problem as a student as well, are often misled by the surface features of the mathematics presented and do not even suspect that there is more there to be understood. This is why I ended up asking lots of different question in the classroom. "How many people remember the definition of orthogonality?" "Give me four different examples of $3 \times 3$ matrices that are not invertible." "Can a matrix have exactly seven eigenvectors?" "What is it that I tried to illustrate by this example?" "Why did I write 'in particular'?" I am trying to make them appreciate when my words or the words written in the textbook may often have meaning, while on the surface these words seem weird and are likely to be ignored as unimportant, e.g. when you see two seemingly true but unrelated statements and ignore the 'in particular' between them.

*For example, this is how I would explain that a number $N$ is divisible by 9 if and only if the sum of its digits is divisible by 9. I would ask if an even number always has an even sum of digits. Then the same for divisibility by 7. (Every time waiting for $20-30$ seconds, giving suggestions, checking examples on the blackboard.) Then finally the same question about 9. The rest depends on the situation. I would be trying to convey that this fact is true, yet nontrivially. An example such as $1234 = 999 + 1 + 2 \cdot 99 + 2 + 3 \cdot 9 + 3 + 4$ can usually replace a proof. (This is nearly a proof: we just need to formalize that 1234 is typical enough and there are no exceptions; in this case algebra is the machinery but not the point.)*

This style clearly limits the amount of material you can present in class. Let me address this objection. I am not proposing to teach all math classes in this fashion. As I have already pointed out, I am glad that I can teach the students that, I believe, need and can greatly benefit from that style of teaching. I hope that if I succeed, the students will then be able to learn, understand and appreciate mathematics in different environments, such as listening to a lecture-oriented presentation, discussing the subject with a friend or working in groups. This is why every once in a while I am temporarily switching to a lecture-intensive format, delivering the material at a higher level and asking questions only at the key points. Besides being a great source of motivation, it allows the students to empower their thinking by concentrating and working hard to follow the presentation (even if they are able to do that only sometimes and only for $10-15$ minutes — it will improve). Even then, though, I am trying to give some methodological tips on how to regain attention after being distracted, what to do if you struggle when you see a lot of unfamiliar letters on the blackboard, etc. This is why it makes sense to ask simple questions such as *so (I am writing on the blackboard) . . . what do I write here?* This is not, as is the popular misconception, to check if the class is following: basically, they are not. This is to help them realize, what "really following a math presentation" is or could be (e.g. that one can start from understanding every single word on the blackboard in isolation).

## 2 Quizzes

Let me elaborate on using quizzes as a teaching tool. The usual rationale for having quizzes is to monitor the student progress and remind them not to fall behind too much. I have three more reasons:

1. When I am grading quizzes and distributing the solutions (or posting online), I can augment what I am doing in the classroom by reaching each student individually. I am bringing the attention of each student to his or her personal mistakes as well as to the common mistakes, making sure to explain what I think are the sources of these mistakes and so what are the general guidelines for avoiding such mistakes in the future. For example, when changing the order of integration in a repeated integral they obtain an integral with an expression involving $z$ in the limit, while the integration is with respect to $z$ as well so that the answer, strictly speaking, ends up being a function of $z$. I use it to emphasize that they should use mathematical notation meaningfully and check all the time whether what they write makes sense and that it makes the sense that they intend;

2. For some people the time-constrained atmosphere of a quiz is a problem and they can't do their best. The good thing here is that many students are ready to put more thinking efforts into a quiz and try to concentrate at least for $5-7$ minutes — which may never happen while doing a homework. The students get a chance to struggle on a quiz and so they literally learn during a quiz. Actively struggling with a subject on a regular basis, when taken in combination with other methods, is a very efficient way of learning (this is not dissimilar from physical exercises);

3. This same effect allows me to learn from reading quizzes about the students' current vision of the subject. Homework is not so useful for this purpose, because there the students would often skip a few more conceptually difficult questions while on a quiz they are ready to concentrate for a few minutes, because it is (in points) worth an hour of doing routine exercises. They sincerely try to do their best within this constrained framework, and it really gives me some useful and interesting information.

In order to accomplish these goals I tried a variety of different formats of quizzes. I am still working on the question, how much information to give the students about the next quiz (I tried a range from no specific information to giving a specific problem and explaining that I will just change the numbers.) The only solid conclusion I have so far is that giving very simple quizzes is at best useless. According to the criteria above, it gives me no information, the students do not learn from struggling and they can easily prepare for the quiz by memorizing a certain procedure while understanding nothing of what's going on in the class. In fact, it is harmful and perhaps dishonest, because the students get the impression that they learned something while simply wasting their time.

To give a specific example of a (rather difficult) quiz I had: *It had been announced the quiz would be on eigenvalues, the characteristic equation and diagonalization. On the quiz I asked the following question: $A$ and $B$ are $n \times n$ matrices that have the same characteristic polynomial. What can you say about* *their eigenvalues, determinants, being invertible and being diagonalizable? (I did not require to answer all or even three of the questions completely to get full credit.)*

This is how I graded this quiz. I would count as a completely correct answer about eigenvalues something like "the eigenvalues are the same as they come from the same characteristic polynomial", since this is correct and explains the subject matter. I would count as a completely wrong answer about eigenvalues something like "the eigenvalues are the same, because the characteristic polynomials are the same", since this is just the question and the answer, with no explanation. Similarly, I would regard "the determinants are the same as products of equal eigenvalues" as completely correct, and "the determinants are the same as $det(A - \lambda I)$ are the same" as half-correct. (Of course, nobody gave the answer that $detA = detB$ because we can evaluate $det(A - \lambda I)$ at $\lambda = 0$ — this requires a much deeper understanding that $det(A - \lambda I)$ is just a polynomial and so $\lambda$ does not have to be a (suspected) eigenvalue.) A completely wrong answer would be "the determinants $detA = detB$ (are the same), because $\lambda I$ is the same in both cases". (This answer assumes that $det(A - \lambda I) = det(A) - det(\lambda I)$.)

## 3 Homework

Homework is important for sharing with the students the flow of the course. The idea that the course is 'going on' throughout the semester is rather abstract and ephemeral. Homework is giving substance to it. It is important to understand that the students will prepare for the exam anyway, which usually means to practice to solve standard types of problems. Moreover, they will forget most of the mechanical skills they acquired during the semester such as performing the Laplace transform or diagonalizing a matrix, if this was learnt without proper understanding, and they will forget everything of that kind after the exam. This is why I am currently focusing on *assigning meaningful assignments*.

For example, I may assign some simple problems and then several more difficult ones. Those students who just want a 'C' can solve the easy ones and save their time. Other students are somewhat more motivated to think. Unfortunately, I found the usual grading practice at Penn State to be against my objectives. If a C is *70%* and the completely trivial exercises in the homework are worth *75%* of the points, this strongly encourages aiming for a C by making a tacit and implicit qualitative distinction between a C and, say, a B-. So I am trying to fix it by introducing occasional corrections such as giving a quiz with two problems and announcing that it is enough to solve only one, i.e. they get the maximum of two scores. Similarly, I am always moving towards dropping more lowest grades for quizzes/homeworks, but having a higher standard for grading. For example, getting 0 for half of the quizzes and maximum for the other half is much better, in my opinion, than getting *50%* for all quizzes.

Most of the homework I assigned were graded by a grader. I was lucky to have good graders. Yet I did a lot to ensure they will help me to achieve all the goals outlined above and emphasize the principles I find important, rather than promote formalism and mediocrity. For that purpose, so far I have always provided complete solutions to my graders. (That is, I solved all of the homework I assigned.) I have also provided guidelines for grading. The guidelines may include examples of specific mistakes and how many points to take off, or sometimes “this problem is not important and it does not matter, how you grade it”. For an important problem I can quote a particular wrong solution and suggest a grade and a reason. (Still, I mention that the grader doesn’t have to follow these guidelines if she has a well articulated, yet different idea for how to grade and why, and implements it consistently.) I also have some general guidelines for grading, such as *if the solution can be formally viewed as the problem statement and the answer, it is not to be treated differently from just the answer*, which typically qualifies for 0 points (“show work”). I find it working well: the graders can grade meaningfully and in the spirit that I need, I just need to be clear enough on what I need. Occasionally I am looking through the homework and checking the grading, especially in the beginning of the semester. When I find something that I don’t like, I may send a copy of these pages to the grader. (How would they learn to grade better otherwise, if they are not shown the mistakes they make in their grading?)

# 4 Teaching via feedback to individual students

For me getting feedback on the work done is an effective mechanism of learning. It seems, it is also true for many students. One way is distributing solutions to homework and/or quizzes, physically or online. I do that sometimes. I’ve been told that this was very useful. I spend a lot of time writing comments on the quizzes. (I convinced myself that I have a lot of experience and so can share it directly, and that the students had and will have few other opportunities of that kind, if any at all.) Sometimes I would grade one particular problem on the homework and write my comments there, or half of the problems, or even the whole homework, especially if the homework contains the problems that, I think, my grader is not qualified to grade. I am correcting the use of terminology, e.g. *if the matrix is linearly independent* to *if the columns of the matrix are linearly independent*. If the solution contains unusual ideas, I reply if this alternative way is fruitful or not. If there is no explanation for the answer or it is unsatisfactorily vague, I am giving specific reasons: *How do you know that what you are saying here is true?*; *If A and B have the same characteristic polynomial, you cannot say they are similar because of this example... You cannot say a matrix is consistent: only a linear system can be consistent or inconsistent*. It also means that I am spelling out the reasons for taking points off and, hopefully, suggesting a constructive way to do better in the future.

I am certain that all that is where my learning as a teacher actually takes place: when I am grading or looking through homeworks/quizzes, writing comments for the grader, explaining the students why is it that something is graded that way and not the other. The experience I get from all these activities directly affects my teaching, perfects my tools and deepens my vision. Since I have been determining virtually everything in the courses I have been teaching so far, if I cannot explain why something is done that way and not the other, at least on a level accessible to myself, I quickly convince myself that I should change the way of doing it.