Math 220 – **Homework 9**

due Thursday, 1, November, 2007

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Collaboration with other Math 220 students is permitted (and encouraged!) However, you have to write and understand all solutions on your own. Fair use of other sources is permitted. Submitting a solution you do not understand (e.g. cannot reproduce within three hours, given a pencil, a supply of paper and the textbook) is not permitted.

Most of the problems are more or less standard and should be solvable within half an hour (if your remember the definitions of rank, dimension, basis etc. with some properties and have reflected about each one for at least 10 minutes.) However, there are a few that may cause difficulties. If you have thought for half an hour and it didn't work, please explain in detail your conclusions, what have you tried, what went wrong, etc. – for substantial progress partial credit will be given, up to *75%*. If you just write "have no idea", you get *15%* just for saving my time and letting me know that this problem is difficult (as opposed to writing nonsense)!

I will try to grade everything on my own (well, at least the second half of the problems), since this is a difficult topic and I want to see, how is it going.

1. Problem 2.9.9.
2. Problem 2.9.13.
3. Problem 2.9.23.
4. Problem 2.9.25.
5. Problem 2.9.29.
6. Let $\mathbf{B} = \{\vec{e_1}, \vec{e_2}\}$ be the standard basis for $\mathbf{R}^2$. That is, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   Let $\tilde{\mathbf{B}} = \{\vec{v_1}, \vec{v_2}\}$ be another basis for $\mathbf{R}^2$. Here $\vec{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Construct the matrix $A = [\vec{v_1}, \vec{v_2}] = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$.

   (a) Express $\vec{e_1}, \vec{e_2}$ using $\vec{v_1}, \vec{v_2}$. That is, find four real numbers $\alpha, \beta, \gamma, \delta$ such that $\vec{e_1} = \alpha\vec{v_1} + \beta\vec{v_2}$ and $\vec{e_2} = \gamma\vec{v_1} + \delta\vec{v_2}$. (Hint: this is standard. Just solve two linear systems.) Form the matrix $B = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$ — the columns contain the coordinates of $\vec{e_1}$ and $\vec{e_2}$ relative to the new basis $\tilde{\mathbf{B}}$ (as opposed to A which contains the coordinates of $\vec{v_1}, \vec{v_2}$ relative to the standard basis $\mathbf{B}$.)

   (b) Compute AB.

   (c) Explain the result. Use the fact that $\vec{v_1} = 1\vec{e_1} - 2\vec{e_2}$ and $\vec{v_2} = 2\vec{e_1} + 2\vec{e_2}$. (Hint: think about linear transformations.)

7. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Find $rank(A)$, $rank(A^2)$ and $rank(A^3)$.

8. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
   (a) Compute AB. Find $Nul(B)$ and $Nul(AB)$.
   (b) To see that $Nul(B) \neq Nul(AB)$, find a vector $\vec{x}$ that is in $Nul(AB)$, but not in $Nul(B)$. (There are many possible answers.)
   (c) Find $\dim Nul(B)$ and $\dim Nul(AB)$. (Hint: dim means 'dimension'. Read the definition. $Nul(B)$ and $Nul(AB)$ are subspaces of $\mathbf{R}^3$.)
   (d) Find three vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ such that simultaneously $\{\vec{v_1}\}$ is a basis for $Nul(B)$, $\{\vec{v_1}, \vec{v_2}\}$ is a basis for $Nul(AB)$ and $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a basis for $\mathbf{R}^3$. (Hint: pick any nonzero $\vec{v_1}$ that is in $Nul(B)$, since you have hopefully concluded that $Nul(B)$ is one-dimensional. After you find $\vec{v_1}$ and $\vec{v_2}$, pick any $\vec{v_3}$ so that the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent.)

9. Let $\vec{v_1}, \vec{v_2}, \vec{v_3}$ be vectors in $\mathbf{R}^3$.
   Assume that $\dim Span\{\vec{v_1}, \vec{v_2}\} = 1$ and $\dim Span\{\vec{v_2}, \vec{v_3}\} = 1$.
   What can be $\dim Span\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$? (Hint: try $v_1 = v_2 = v_3$. Hint: think geometrically.)

10. Let A be an $m \times n$ matrix. Let $\vec{e_1}, \ldots, \vec{e_n}$ be the standard basis for $\mathbf{R}^n$. Assume that $\{A\vec{e_1}, \ldots, A\vec{e_n}\}$ is a basis for $\mathbf{R}^m$.
    (a) What does it say about m and n?
    (b) What does it say about A?

    (Hint: try an example. Choose some particular, small m and n, a matrix A and see if it works.)

11. Let A be a 2×3 matrix. Let B be a 3×3 matrix. $ColA = \mathbf{R}^2, ColB = \mathbf{R}^3$.
    (a) Find $\dim Nul(A)$ and $\dim Nul(B)$.
    (b) One is greater than the other. Why? Explain geometrically.

12. Definition: if $A = A^T$, then matrix A is called *symmetric*.
    (a) Check if $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is symmetric.
    (b) Check if $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & -7 \end{bmatrix}$ is symmetric.
    (c) Explain, why $A^T A$ is symmetric for any A. ($A^T A$ means "A transposed, multiplied by A".)
    (d) Choose any matrix A that is itself not symmetric and check that $A^T A$ **is** indeed symmetric.