Answers to Quiz 9

Math 230. Friday, 11/10/6

There were two problems, both graded out of 4 points. The best score was then chosen.

Problem 1 (4 points) $D = \{(x,y)| -1 \leq x,y \leq 1, x^2+y^2 \geq 1\}$. *Find* $\iint_D (x^2+y^2)dA$

The simplest way is to represent D as the difference of the square $R = [-1,1]\times[-1,1] = \{(x,y)|-1 \leq x,y \leq 1\}$ and the circle $C = \{(x,y)|x^2+y^2 < 1\}$. Then, since D and C intersect only at the border and their union is the whole square R, $\iint_R (x^2+y^2)dA = \iint_C (x^2+y^2)dA + \iint_D (x^2+y^2)dA$. Equivalently, $\iint_D (x^2+y^2)dA = \iint_R (x^2+y^2)dA - \iint_C (x^2+y^2)dA$. Note that this problem has absolutely nothing to do with areas of D, R, C. The area of R is $A(R) = \iint_R dA = 2\cdot 2 = 4$. The area of C is π (using the formula $S = \pi r^2$ with $r = 1$). Hence, the area of D is $4-\pi$, but it is textbfirrelevant. Certainly, it implies $\iint_D dA = 4-\pi$, but the question was to find $\iint_D (x^2+y^2)dA$.

Applying Fubini's theorem (since x^2+y^2 is continuous at all (x,y)) $\iint_R (x^2+y^2)dA = \int_{-1}^1\int_{-1}^1 (x^2+y^2)dydx = \int_{-1}^1 (x^2y + y^3/3|_{y=-1}^{y=1})dx = \int_{-1}^1 (x^2+1/3-(-x^2-1/3))dx = \int_{-1}^1 (2x^2+2/3)dx = 2x^3/3+2x/3|_{x=-1}^{x=1} = 2/3+2/3-(-2/3-2/3) = 8/3$.

Since x^2+y^2 is continuous and C can be viewed as a polar rectangle, $\iint_C (x^2+y^2)dA = \int_0^{2\pi}\int_0^1 (r^2)rdrd\theta = \int_0^{2\pi} (r^4/4|_{r=0}^{r=1})d\theta = \int_0^{2\pi} (1/4)d\theta = \theta/4|_{\theta=0}^{\theta=2\pi} = 2\pi/4 = \pi/2$.

Therefore, $\iint_D (x^2+y^2)dA = \iint_R (x^2+y^2)dA - \iint_C (x^2+y^2)dA = 8/3-\pi/2$.

Problem 2 (4 points) *Find the surface area of the solid, defined as the intersection of the ball* $x^2+y^2+z^2 \leq 2$ *and the (solid) cylinder* $x^2+z^2 \leq 1$.

The simplest way is to observe that this solid has the same surface area as another solid, which is the intersection of $x^2+y^2+z^2 \leq 2$ and $x^2+y^2 \leq 1$ (from the symmetry). **Disclaimer:** it does not mean that $x^2+z^2 \leq 1$ is the same as $x^2+y^2 \leq 1$. It does not mean that the intersection of $x^2+y^2+z^2 \leq 2$ and $x^2+z^2 \leq 1$ is the same as the intersection of $x^2+y^2+z^2 \leq 2$ and $x^2+y^2 \leq 1$ either. It does not mean that y and z can be exchanged in any context preserving all the properties. (Because it will change a right coordinate system to a left coordinate system and vice versa, hence cross products will change their sign).

What it does mean is that the surface area of the intersection of $x^2+y^2+z^2 \leq 2$ and $x^2+y^2 \leq 1$ is equal to the surface area in question.

Note also that it is wrong to replace $x^2+y^2+z^2 \leq 2$ with $x^2+y^2+z^2 = 2$ and $x^2+z^2 \leq 1$ with $x^2+z^2 = 1$, since these are not the same.

However, let us solve the problem without this simplification. Firstly, find the intersection of the surface of the ball (i.e. the sphere $x^2+y^2+z^2 = 2$, which is not the same as the ball itself) and the surface of the solid cylinder (i.e. the cylinder $x^2+z^2 = 1$, which is not the same as the solid cylinder itself). If simultaneously $x^2+y^2+z^2 = 2$ and $x^2+z^2 = 1$, then $y^2 = 1$, $y = \pm 1$. It does not mean that the planes $y = 1$ and $y = -1$ constitute the intersection of these two surfaces, but that this intersection is contained in these two planes $y = 1$ and $y = -1$. In fact, the intersection of these two surfaces consists of two circles: $\{(x,y,z)|x^2+z^2 = 1, y = 1\}$ and $\{(x,y,z)|x^2+z^2 = 1, y = -1\}$.

It helps to understand that the surface of the given solid (whose surface area we are to find) can be represented as consisting of three pieces. $S_1 = \{(x,y,z)|x^2+z^2 \leq 1, y \geq 1\}$, $S_2 = \{(x,y,z)|x^2+z^2 \leq 1, y \leq -1\}$, $S_3 = \{(x,y,z)|x^2+z^2 = 1, -1 \leq y \leq 1$. Then to find the answer we need to add up the areas of these three surfaces, $A(S_1) + A(S_2) + A(S_3)$.

From the obvious symmetry, $A(S_1) = A(S_2)$. Each of these two areas can be found by the standard procedure. Namely, view S_1 as the graph of a function $y = f(x,z) = \sqrt{2-x^2-z^2}$. Then, if $D = \{(x,z)|x^2+z^2 \leq 1\}$, $A(S_1) = \iint_D \sqrt{(f_x(x,z))^2 + (f_z(x,z))^2 + 1)}dA$. $f_x(x,z) = \frac{-x}{\sqrt{2-x^2-z^2}}$, $f_z(x,z) = \frac{-z}{\sqrt{2-x^2-z^2}}$. Hence, $\sqrt{(f_x(x,z))^2 + (f_z(x,z))^2 + 1)} = \sqrt{(\frac{-x}{\sqrt{2-x^2-z^2}})^2 + (\frac{-z}{\sqrt{2-x^2-z^2}})^2 + 1} = \sqrt{\frac{x^2+z^2}{2-x^2-z^2} + 1} = \sqrt{\frac{x^2+z^2+(2-x^2-z^2)}{2-x^2-z^2}} = \sqrt{\frac{2}{2-x^2-z^2}}$. Therefore, $A(S_1) = \iint_D \sqrt{\frac{2}{2-x^2-z^2}}dA$. Then the function under the integral sign is continuous on D, so the integral can be converted to polar coordinates: $A(S_1) = \sqrt{\frac{2}{2-r^2}}rdrd\theta = \sqrt{2}\int_0^{2\pi}(\int_0^1 \frac{r}{\sqrt{2-r^2}}dr)d\theta = \frac{-\sqrt{2}}{2}\int_0^{2\pi}(\int_0^1 \frac{-2r}{\sqrt{2-r^2}}dr)d\theta$.

However, $\int_0^1 \frac{-2r}{\sqrt{2-r^2}}dr$ can be found by a simple change of variables (or guessed). $u = 2-r^2$, $du = -2rdr$. When r changes from 0 to 1, $u = 2-r^2$ monotonously changes from $2-0^2$ to $2-1^2$, i.e. from 2 to 1. Hence, $\int_0^1 \frac{-2r}{\sqrt{2-r^2}}dr = \int_2^1 \frac{du}{\sqrt{u}} = 2\sqrt{u}\Big|_{u=2}^{u=1} = 2(1-\sqrt{2})$. Therefore, $A(S_1) = \frac{-\sqrt{2}}{2}\int_0^{2\pi}(\int_0^1 \frac{-2r}{\sqrt{2-r^2}}dr)d\theta = \frac{-\sqrt{2}}{2} \cdot 2(1-\sqrt{2}) = -(1-\sqrt{2}) = \sqrt{2}-1$.

$A(S_2) = A(S_1)$. Finally, $A(S_3) = (2\pi \cdot 1) \cdot 2$, since it is the side surface area of a finite cylinder, which can be "unwrapped". (Direct integration gives the same result). Therefore, the answer is $A(S_1)+A(S_2)+A(S_3) = 2(\sqrt{2}-1)+4\pi$.