Answers to Quiz 7

Math 230. Friday, 10/27/6

There were two problems, both graded out of 4 points. The best score was then chosen. **The quiz was slightly different for sections 3 and 6, thus here are the solutions for both variants**

Problem 1 (4 points) *Find the equations of the normal plane and the tangent line to* $-x + y^2 - z^2 = 0$ *(respectively,* $-x^2 + y^2 - z = 0$*) at the origin.*

Find the gradient: $\nabla f(x, y, z) = \langle -1, 2y, -2z\rangle$. $\nabla f(0, 0, 0) = \langle -1, 0, 0\rangle$. The tangent plane in question passes through the origin, its normal vector is $\langle -1, 0, 0\rangle$ — the equation of this plane is $-1(x - 0) + 0(y - 0) + 0(z - 0) = 0$, i.e. $x = 0$. The normal line passes through the origin, its direction vector is $\langle -1, 0, 0\rangle$. The parametric equations of this line are: $x = -t, y = 0, z = 0$. The symmetric equations of this line are: $y = z = 0$.

The second version of the problem statement: $-x^2 + y^2 - z = 0$. Then the gradient $\nabla f(x, y, z) = \langle -2x, 2y, -1\rangle$, in particular $\nabla f(0, 0, 0) = \langle 0, 0, 1\rangle$. The tangent plane in question passes through the origin, its normal vector is $\langle 0, 0, -1\rangle$ — the equation of this plane is $0(x - 0) + 0(y - 0) - 1(z - 0) = 0$, i.e. $z = 0$. The normal line passes through the origin, its direction vector is $\langle 0, 0, -1\rangle$. The paramteric equations of this line are: $x = 0, y = 0, z = -t$. The symmetric equations of this line are: $x = y = 0$.

Note that division by zero is not allowed, thus, for instance, the normal line cannot be given by symmetric equations $\frac{x}{-1} = \frac{y}{0} = \frac{z}{0}$. Also note that expression of the form x, $\frac{-1}{x}$, $-z$ are terms, but neither statements nor equations; they do not describe any geometrical objects, hence cannot be the answer to the question 'find the normal line'. Also note that, for instance, $0(x-0)+0(y-0)-1(z-0) = -z$ can never be an equation of the tangent plane, because this expression merely states $z = z$, which is true for all points (x, y, z), not just for points in some plane.

Problem 2 (4 points) *Given* $f(x, y) = e^{xy}$. *Find all directions* $\overrightarrow{u}$ *in which the directional derivative* $D_{\overrightarrow{u}} f(1, 1) = e$ *(respectively,* $-e$*).*

First of all, find the gradient (correctly). $\nabla f(x, y) = \langle ye^{xy}, xe^{xy}\rangle$. In particular, $\nabla f(1, 1) = \langle e, e\rangle$. Then recall that, **if** $\overrightarrow{u}$ **is a unit vector**, then the directional derivative can be expressed as $D_{\overrightarrow{u}} f(x, y) = \nabla f(x, y) \cdot \overrightarrow{u}$. In particular,

$D_{\overrightarrow{u}} f(1,1) = \nabla f(1,1) \cdot \overrightarrow{u} = \langle e, e\rangle \cdot \overrightarrow{u}$. $\overrightarrow{u}$ is a vector, consisting of two components, say, $\overrightarrow{u} = \langle a, b\rangle$. Then $D_{\overrightarrow{u}} f(1,1) = \langle e, e\rangle \cdot \langle a, b\rangle = ea + eb = e(a+b)$. By the problem statement we know that $D_{\overrightarrow{u}} f(1,1) = e$ (respectively, $-e$), hence so is $e(a+b)$. Now we need to find all such $\langle a, b, t\rangle$hat are unit vectors ($a^2+b^2 = 1$), also satisfying $a + b = 1$ (respectively, $a + b = -1$). There are exactly two such vectors: $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$ (respectively, $\langle 0, -1\rangle$ and $\langle -1, 0\rangle$). There are no more such unit vectors — it can be shown from geometrical considerations. All pairs (a, b) satisfying $a^2 + b^2 = 1$ can be thought as points on the unit circle. And condition of the form $a + b = 1$ (or $a + b = -.3$, or $a + b = \pi$ - whatever) corresponds geometrically to a line. But a line and a circle can have no more than two points of intersection.

Of course, it can also be shown algebraically. For instance, if $a + b = 1$ and $a^2 + b^2 = 1$, then it implies that $b = 1 - a$, hence $a^2 + (1 - a)^2 = 1$, so $2a^2 - 2a + 1 = 1$. It implies $2a^2 - 2a = 0$, i.e. $2a(a - 1) = 0$, which is the case only when $a = 0$ or $a = 1$ and, therefore, $b = 1$ or $b = 0$ respectively, as required.