Answers to Quiz 4

Math 230. Friday, 9/29/6

Problem 1 (75%) *Find arc length.* $x = \sin(e^t), y = \cos(e^t), z = 1$, $-1 \leq t \leq 2$.

"Find arc length" means to find a particular positive real number, not the *arc length function.* This vector function is merely a different parametrization of an arc of the unit circle. However, this observation is not necessary for solving the problem.

$$L = \int_{-1}^{2} |\overrightarrow{r}'(t)| dt$$

$$\overrightarrow{r}' = \frac{d}{dt}\langle \sin(e^t), \cos(e^t), 1\rangle = \langle e^t \cos(e^t), e^t \cdot (-\sin(e^t)), 0\rangle$$

$$|\overrightarrow{r}'| = \sqrt{(e^t \cos(e^t))^2 + (e^t \cdot (-\sin(e^t))^2 + (0)^2} =$$

$$\sqrt{(e^t)^2 \cdot (\cos^2(e^t) + \sin^2(e^t))} = \sqrt{e^{2t}} = e^t$$

$$L = \int_{-1}^{2} |\overrightarrow{r}'| dt = \int_{-1}^{2} e^t dt = e^t \Big|_{-1}^{2} = e^2 - e^{-1}$$

Note that $e^{-1} = \frac{1}{e} \neq 0$. Note that if we replace the vector function $\overrightarrow{r} = \langle \sin(e^t), \cos(e^t), 1\rangle$ with $\overrightarrow{r} = \langle 1000+\sin(e^t), 1000+\cos(e^t), 1000+1\rangle$, the answer (arc length) will not change. Of course, the derivative $\overrightarrow{r}'$ will not change either. However, $\int_{-1}^{2} |\overrightarrow{r}(t)| dt$ will change a lot. Therefore, this integral has nothing to do with the arc length of this curve.

Problem 2 (25%) *Find the osculating plane of the curve* $x = y^2$, $z = 0$ *at the point* $(1, 1, 0)$.

This curve is a plane curve, a parabola in particular. The osculating plane of a plane curve at any point is the plane containing this curve — in this case, the xy-coordinate plane (the plane $z = 0$).

To find formally, the parabola should be parametrized by t. There are two natural parametrizations: $\overrightarrow{r_1}(t) = \langle t, \sqrt{t}, 0\rangle$ and $\overrightarrow{r_2}(t) = \langle t^2, t, 0\rangle$. The latter is simpler, so let $\overrightarrow{r}(t) = \overrightarrow{r_2}(t) = \langle t^2, t, 0\rangle$. The point $(1, 1, 0)$ corresponds to the value $t = 1$.

Taking the derivative, $\overrightarrow{r}'(t) = \langle 2t, 1, 0\rangle$. In particular, $\overrightarrow{r}'(1) = \langle 2, 1, 0\rangle$. $|\overrightarrow{r}'(t)| = \sqrt{4t^2+1}$. In particular, $|\overrightarrow{r}'(1)| = \sqrt{5}$. Normalizing, obtain the unit normal vector. $\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|} = \frac{1}{\sqrt{4t^2+1}}\langle 2t, 1, 0\rangle = \langle \frac{2t}{\sqrt{4t^2+1}}, \frac{1}{\sqrt{4t^2+1}}, 0\rangle$. In particular, $T(1) = \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\rangle$ — is a unit vector, as expected.

The next step is to find $\overrightarrow{T}'(t)$. Note that $T'(t)$ **IS NOT** parallel to $\overrightarrow{r}''(t)$, and one cannot use the latter instead. It is crucial that $\overrightarrow{T}(t)$ is always a unit vector, and therefore $\overrightarrow{T}'(t)$ is always orthogonal to $\overrightarrow{T}(t)$. On the contrary, $\overrightarrow{r}''(t)$ (which can be viewed as the acceleration of a particle) may or may not be orthogonal to $\overrightarrow{r}'(t)$ (which may be viewed as the velocity of a particle). Of course, $\overrightarrow{r}'(t)$ is parallel to $\overrightarrow{T}(t)$, but $\overrightarrow{T}'(t)$ still has to be found.

$\overrightarrow{T}'(t) = \frac{d}{dt}\frac{1}{\sqrt{4t^2+1}}\langle 2t, 1, 0\rangle \neq \frac{1}{\sqrt{5}}\langle 2, 0, 0\rangle$. The whole thing has to be differentiated. Namely, $\overrightarrow{T}'(t) = \frac{d}{dt}\langle \frac{2t}{\sqrt{4t^2+1}}, \frac{1}{\sqrt{4t^2+1}}, 0\rangle = \langle \frac{2\sqrt{4t^2+1} - 2t(\frac{8t}{2\sqrt{4t^2+1}})}{4t^2+1}, \frac{-\frac{8t}{2\sqrt{4t^2+1}}}{4t^2+1}, 0\rangle$. $\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{|\overrightarrow{T}'(t)|}$, but there is no need to express it formally. It suffices to substitute $t = 1$: $\overrightarrow{T}'(1) = \langle \frac{2}{5\sqrt{5}}, \frac{-4}{5\sqrt{5}}, 0\rangle$. So $|\overrightarrow{T}'(1)| = \frac{2}{5}$, and $\overrightarrow{N}(1) = \frac{5}{2}\langle \frac{2}{5\sqrt{5}}, \frac{-4}{5\sqrt{5}}, 0\rangle = \langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0\rangle$. This is a unit vector and it is orthogonal to $\overrightarrow{T}(1) = \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\rangle$, as expected (one can easily check the dot product).

One can finally find $\overrightarrow{B}(1) = \overrightarrow{T}(1) \times \overrightarrow{N}(1) = \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\rangle \times \langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0\rangle = \langle 0, 0, -1\rangle$. Thus, the equation of the osculating plane at $(1, 1, 0)$ has the form $0(x-1) + 0(y-1) - 1(z-0) = 0$, which is essentially $z = 0$.

How to avoid excessive computations? There was no need to find $\overrightarrow{N}(1)$. It was enough to use $\overrightarrow{T}'(1)$, because it has the same direction. Thus, $\overrightarrow{T}(1) \times \overrightarrow{t}'(1)$ is also a normal vector to the osculating plane at $(1, 1, 0)$, and can as well be used to find the equation of the plane.

Moreover, before starting to differentiate $\overrightarrow{T}(t)$ to find $\overrightarrow{T}'(t)$ one could have observed that the z-coordinate of both vectors is zero. Therefore, both $\overrightarrow{T}(t)$ and $\overrightarrow{T}'(t)$ are parallel to the xy-coordinate plane for all t, and thus so is $\overrightarrow{N}(t)$. Hence, the cross product of $\overrightarrow{T}$ and $\overrightarrow{N}$ must be perpendicular to the xy-plane and so be either $(0, 0, 1)$ or $(0, 0, -1)$ (because $\overrightarrow{B} = \overrightarrow{T} \times \overrightarrow{N}$ is also a unit vector). Both yield the same equation of a plane.

Finally, it is also possible to find $\overrightarrow{N}(1)$ without much computations. This should be a horizontal vector with the z-coordinate equal to zero, as discussed above. It is known to be a unit vector, orthogonal to $\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\rangle$. There are only two such vectors, $\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0\rangle$ and $\langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\rangle$. It is easy to graph a parabola and to see, how is directed the normal vector $(1, 1)$. (When graphing, note that this is the parabola $x = y^2$, not $y = x^2$.