

# Answers to Quiz 11

Math 230. Friday, 12/1/6

There were two problems, both graded out of 4 points. The best score was then chosen.

**Problem 1 (4 points)** Given a vector field  $\vec{F} = \langle \frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \rangle$  (or  $\vec{F} = \langle \frac{-x}{x^2+y^2}, \frac{y}{x^2+y^2} \rangle$ ; here solved for the former vector field). Is it conservative? Give reasons for your answers.

While solving the problem, your reasoning should be approximately as follows: Do I know what is a conservative field *by definition*? By definition, a vector field is conservative if it is the gradient of some scalar field,  $\vec{F}(x, y) = \nabla g(x, y)$ . What is a scalar field? A scalar field, say,  $g(x, y)$ , is just a function of two variables. So one can try to find such  $g(x, y)$  directly. Is it necessary? There is a theorem, that if the component functions  $P(x, y)$  and  $Q(x, y)$  (which are, in fact, scalar fields) of  $F(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$  have continuous first order partial derivatives and  $Q_x = P_y$  on some open, connected, simply connected region  $D$ , then  $\vec{F}$  is conservative on this region. Unfortunately, the domain of  $F(x, y)$  is not simply connected, so this theorem is of no use here.

What if  $F$  is not conservative? How can it be established? If you can't find an appropriate  $g(x, y)$ , it doesn't mean that there is none (unless you prove it directly). However, proving that some good property is absent (such as being conservative) is often simpler: it is enough to show that  $\vec{F}$  lacks any of the features necessary for the conservative fields. For instance, it is enough to find one closed path  $C$  (simple, closed, piecewise smooth curve) such that  $\int_C \vec{F} \cdot d\vec{r} \neq 0$  to prove that  $F$  is not conservative. Another way is to find one point  $(x, y)$  at which  $Q_x(x, y) \neq P_y(x, y)$ , then  $F$  cannot be conservative as well. (Of course, both statements are true only if  $F$  has continuous first order partial derivatives at all points, where  $F$  itself is defined, which is indeed the case).

It is, probably, a good idea to try to prove that  $F$  is not conservative, and if it doesn't work, look for the evidence that it is. Find the partial derivatives of the component functions.  $P(x, y) = \frac{x}{x^2+y^2}$ ,  $Q(x, y) = \frac{-y}{x^2+y^2}$ . The partial derivatives we are interested in are  $Q_x(x, y) = \frac{(0)(x^2+y^2) - (-2x)(-y)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$  and  $P_y(x, y) = \frac{(0)(x^2+y^2) - (2y)(x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$ . Obviously, in general  $Q_x \neq P_y$ , so the vector field is not conservative.

Typical mistakes: make sure you find the partial derivatives correctly. Compare  $Q_x$  with  $P_y$  (not  $P_x$  with  $Q_y$ .  $\vec{F}(x, y) = x\vec{i}$  is conservative, but  $\vec{F}(x, y) = x\vec{j}$  is not). We had no notion of partial differentiation of vector fields, so the expression  $\vec{F}_x$  makes no sense. Partial differentiation should be applied to component functions, comprising the vector field.

**Problem 2 (4 points)** Same question.  $\vec{F}(x, y, z) = \langle yz, xz + z + 1, xy + y \rangle$  (or  $\vec{F}(x, y, z) = \langle yz, xz + z, xy + y + 1 \rangle$ ; here solve for the former vector field).

Notice that  $\vec{F}$  is defined on  $\mathbf{R}^3$ , not on  $\mathbf{R}^2$ . Therefore the theorem about  $Q_x$  and  $P_y$  cannot be applied. You may know, how to generalize it for the three-dimensional case. If  $F(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$  is conservative (and its component functions have continuous first order partial derivatives), then  $P_y = Q_x$ ,  $P_z = R_x$ ,  $Q_z = R_y$ . On the contrary, if these three equalities hold and, for instance,  $\vec{F}$  is defined on the whole  $\mathbf{R}^3$  (in analogy with the simple connectedness requirement), then  $F$  is conservative.

If you do not know this generalization and, presumably, are not supposed to be able to apply it, then there are two options. Either try integrating along some closed path or along two different paths sharing the initial and the terminal point (trying to show  $\vec{F}$  is not conservative) or try to find  $g(x, y, z)$  such that  $\vec{F} = \nabla g(x, y, z)$  directly.

Let us illustrate the second approach. We are looking for  $g(x, y, z)$  such that  $g_x = yz, g_y = xz + z + 1, g_z = xy + y$ . It is simpler to start from  $g_z$ , but for the demonstration purposes let us start from  $g_x$ . In order to satisfy  $g_x = yz$ , we need  $g(x, y, z) = xyz + h(y, z)$  for some function  $h(y, z)$ . (Of course, not  $g(x, y, z) = xyz + h_1(y) + h_2(z)$ ). Now do not do the same thing with  $g_y$  and  $g_z$ . If you try, you will have to introduce two more functions  $h_{another}(x, z)$  and  $h_{yetanother}(x, y)$ , and then you will get two more unknown functions and no more information.

Instead, use what you've got. We already found out that  $g$ , if exists at all, must look like  $g(x, y, z) = xyz + h(y, z)$ . It is a meaningful statement, for instance  $g_{wrong}(x, y, z) = 0$  does not work (there is no  $h(y, z)$  to make  $xyz + h(y, z) = 0$ ). Now use two other conditions involving  $g_y$  and  $g_z$ . We know that  $g_y = xz + z + 1$ . Substitute:  $\frac{\partial}{\partial y}xyz + h(y, z) = xz + h_y(y, z) = xz + z + 1$ . Therefore,  $h_y(y, z) = z + 1$ . Also, we know that  $g_z = xy + y$ . Substitute:  $\frac{\partial}{\partial z}xyz + h(y, z) = xy + h_z(y, z) = xy + y$ . Therefore,  $h_z(y, z) = y$ .

Now it is sufficient to find any  $h(y, z)$  satisfying  $h_y(y, z) = z + 1$  and  $h_z(y, z) = y$ . How to do that? Either guess or use the same strategy. (Indeed, it was possible to guess  $g(x, y, z)$  in the very beginning). How to use the same strategy? Well, we were looking for  $g(x, y, z)$  with a given gradient. Now we are looking for  $h(y, z)$  with a given gradient  $\nabla h(y, z) = \langle z + 1, y \rangle$ , so we have two variables instead of three. Same strategy: if  $h_y(y, z) = z + 1$ , then necessarily  $h(y, z) = yz + y + q(z)$  for some function  $q(z)$ . Also,  $h_z(y, z) = y$ . Substituting,  $\frac{\partial}{\partial z}yz + y + q(z) = y + q'(z) = y$ , so that  $q'(z) = 0$ , and  $q(z) = C$ , any constant. Thus,  $h(y, z) = yz + y + C$  for any constant, and  $g(x, y, z) = xyz + h(y, z) =$

$xyz + yz + y + C$  for any constant. For instance, letting  $C = 5$  (of course, better take zero) we see that  $\nabla(xyz + yz + y + 5) = \langle yz, xz + z + 1, xy + y \rangle = \vec{F}(x, y, z)$ , so  $F$  is conservative.

Self-check: take the curve  $C$  given by the vector function  $\vec{r}(t) = \vec{F}(t, t, t)$ ,  $-1 \leq t \leq 1$ . (Recall: a vector function is not the same as a vector field). Find  $\int_C \vec{F}(x, y, z) \cdot d\vec{r} = ?$ . (The answer is not  $((1)(1)(1) + (1)(1)(1)) - ((-1)(-1)(-1) + (-1)(-1) + (-1)) = 4$ ).