

Answers to Quiz 10

Math 230. Friday, 11/17/6

There were two problems, both graded out of 4 points. The best score was then chosen.

Problem 1 (4 points) Find the center of mass of the solid tetrahedron $x, y, z \geq 0, x + y + z = 1$ with the density function $\rho(x, y, z) = z$ (or $\rho(x, y, z) = x$; solved for the former).

It is easy to see that the vertices of this tetrahedron have coordinates $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Thus it is symmetric with respect to the line $x = y = z$.

To find its center of mass we need to find some trick or to integrate a lot, or find some trick to simplify the integration. I do not know, how to avoid the integration completely.

Since $\rho(x, y, z) = z$, if the center of mass has coordinates $(\bar{x}, \bar{y}, \bar{z})$, then $\bar{x} = \bar{y}$ from the symmetry. Now we just need to integrate and to find the mass, m , and the three coordinates of the center of mass. The only help is that $\bar{x} = \bar{y}$, so that it is enough to find m , \bar{x} and \bar{z} .

Find the mass. $m = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \rho(x, y, z) dz dy dx = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z dz dy dx$. Indeed, some orders of iterated integration may be more convenient than some other orders (in this case this one is, probably, the most convenient). This observation was intended to direct your attention to the problem second problem. At this moment stop and take a look at the second problem, and think how to solve it (if you don't know yet).

$m = \int_0^1 \int_0^{1-x} (\int_0^{1-x-y} z dz) dy dx = \int_0^1 (\int_0^{1-x} \frac{(1-x-y)^2}{2} dy) dx = \int_0^1 (-\frac{(1-x-y)^3}{6} \Big|_{y=0}^{y=1-x}) dx = \int_0^1 \frac{(1-x)^3}{6} dx = -\frac{(1-x)^4}{24} \Big|_0^1 = 1/24$. If you don't understand this evaluation, you need to refresh in your memory, how do we make a change of variables in the integral. Here two changes of variables are applied implicitly, the first one, for instance, is $u = 1 - x - y$. If you understand this evaluation but cannot do it on your own during a quiz, you need practice: find 10-15 integrals using the change of variables. (Check Chapter 8 in the textbook). (Also, if you are going to take Math 250/251, you need to repeat integration by parts as well).

Find the coordinates of the center of mass. $\bar{z} = \frac{\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \rho(x, y, z) dz dy dx}{m} = \frac{\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z^2 dz dy dx}{1/24} = 24 \int_0^1 \int_0^{1-x} (\frac{(1-x-y)^3}{3}) dy dx = 24 \int_0^1 (-\frac{(1-x-y)^4}{12} \Big|_0^{1-x}) dx = 24 \int_0^1 \frac{(1-x)^4}{12} dx = -24 \frac{(1-x)^5}{60} \Big|_0^1 = 24/60 = 2/5$.

$$\bar{x} = \frac{\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \rho(x,y,z) dz dy dx}{m} = \frac{\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x z dz dy dx}{1/24} = 24 \int_0^1 x \left(\int_0^{1-x} \int_0^{1-x-y} z dz dy \right) dx.$$

But we already know that $\int_0^{1-x} \int_0^{1-x-y} z dz dy = \frac{(1-x)^3}{6}$ from the previous derivation. Thus, $\bar{x} = 24 \int_0^1 x \frac{(1-x)^3}{6} dx = 4 \int_0^1 x(1-x)^3 dx = 4 \int_0^1 (x - 3x^2 + 3x^3 - x^4) dx = 4(x^2/2 - x^3 + 3x^4/4 - x^5/5) \Big|_0^1 = 4(1/2 - 1 + 3/4 - 1/5) = 4(\frac{10-20+15-4}{20}) = 4(\frac{1}{20}) = 1/5.$

Thus, $\bar{x} = \bar{y} = 1/5$, $\bar{z} = 2/5$ and the center of mass is $(1/5, 1/5, 2/5)$.

Note: you were not supposed to solve this problem (although, some people did find m correctly, mostly those who did it as above). You were supposed to solve the other one, which requires less than one minute, after you understand what to do.

Self-check: now try to find this center of mass on your own and to get the correct answer, $(1/5, 1/5, 2/5)$ (or $(2/5, 1/5, 1/5)$ for $\rho(x, y, z) = x$).

Problem 2 (4 points) Write at least two other iterated integrals, equal to the given one. (Change the order of integration). Given one: $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x, y, z) dz dy dx$ (or $\int_0^1 \int_0^{1-y} \int_0^{1-y-x} f(x, y, z) dz dx dy$).

How to change the order of integration? You can try to do it purely algebraically (i.e. $y = 1 - x$, hence $x = 1 - y$), but I always try to imagine the region of integration geometrically. What is the region of integration? It is easy to see that this region is the tetrahedron from the first problem. From the symmetry, all the six integrals are pretty much the same:

$$\begin{aligned} \int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x, y, z) dz dy dx &= \\ \int_0^1 \int_0^{1-y} \int_0^{1-y-x} f(x, y, z) dz dx dy &= \\ \int_0^1 \int_0^{1-x} \int_0^{1-x-z} f(x, y, z) dy dz dx &= \\ \int_0^1 \int_0^{1-z} \int_0^{1-z-x} f(x, y, z) dy dx dz &= \\ \int_0^1 \int_0^{1-y} \int_0^{1-y-z} f(x, y, z) dx dz dy &= \\ \int_0^1 \int_0^{1-z} \int_0^{1-z-y} f(x, y, z) dx dy dz & \end{aligned}$$

A typical wrong answer looks like $\int_0^1 \int_0^{1-z} \int_0^{1-z-x} f(x, y, z) dz dy dx$. The problem here is that the inner integral $\int_0^{1-z-x} f(x, y, z) dz$ is integrated with respect to z and involves z as well. Obviously, $\int_{anything}^{anythingelse} f(x, y, z) dz = \int_{anything}^{anythingelse} f(x, y, t) dt$, thus $\int_0^{1-z-x} f(x, y, z) dz = \int_0^{1-z-x} f(x, y, t) dt$ is a function of x, y and z . Then the whole integral will be a function of z , while it should be a number.