Midterm II. Math 230. Spring2005

8, November, 2006

Problem 1 $f(x, y) = 7 - 3x^2 - y^2$. *Find the gradient:* $\nabla f(x, y) = \langle -6x, -2y \rangle$. *The slope of the trail in the direction* $\overrightarrow{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ *at a point* $(0, 1, 6)$ *is the directional derivative of* $f(x, y)$ *in this direction, evaluated at* $(0, 1)$*:* $D_{\overrightarrow{u}} f(0, 1) = \nabla f(0, 1) \cdot \overrightarrow{u} = \langle 0, -2 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = 0 \cdot \frac{1}{\sqrt{2}} + (-2) \cdot \frac{1}{\sqrt{2}} = -\sqrt{2}$.

In order to stay at constant elevation, one needs to start walking in the direction, perpendicular to the gradient at this point. (Since the gradient shows the direction of the steepest ascent and, respectively, the opposite direction is the direction of the steepest descent. $\nabla f(0, 1) = \langle 0, -2 \rangle$. *There are two directions, perpendicular to the gradient,* $\langle 1, 0 \rangle$ *and* $\langle -1, 0 \rangle$. *Both of them satisfy the given requirement. Note that at* $(0, 1)$ *the partial derivative with respect to* x, $f_x(0, 1) = 0$. *However, the interpretation is slightly tricky: in fact, one will have to change the direction almost immediately (after an infinitely small period of time), because* $f(x, 1, 6) < f(0, 1, 6)$ *for any* $x \neq 0$.

So, if you are asked to find, in which direction to walk to stay at a constant elevation, find the direction in which the directional derivative is equal to zero. (Note that it would not work for critical points, where the directional derivative is zero in all directions, but the elevation may not be constant in any direction in case of a local minimum or maximum).

The slope of the trail with the steepest ascent is the directional derivative in the direction of the gradient. Equivalently, it is the length of the gradient. (To ensure these are equivalent, take $\overrightarrow{u} = \frac{\nabla f(x,y)}{|\nabla f(x,y)|}$*). Then,* $|\nabla f(0, 1)| = |\langle 0, -2 \rangle| = 2$.

It may be possible to solve the problem by considering a level surface $F(x, y, z) = 3x^2 + y^2 + z = 7$*, however, it is rather artificial, while resulting in the same solution as above.*

Problem 2 $\nabla f(x, y) = \langle y^2 + x/2 - 4, 2xy \rangle$. *Both directional derivatives exist at all points, so the critical points are exactly those where the gradient vanishes. The condition is* $y^2 + x/2 - 4 = 2xy = 0$. *Let us first analyze* $2xy = 0$. *It is possible when* $x = 0$ *or* $y = 0$.

Assume $x = 0$. *Then* $y^2 + 0/2 - 4 = 0$*, i.e.* $y = \pm 2$. *On the other hand, assume* $y = 0$. *Then* $0 + x/2 - 4 = 0$*, i.e.* $x = 8$. *Thus we get three critical points:* $(0, 2), (0, -2), (8, 0)$. *To classify the critical points, use the second derivative test.*

$f_x(x, y) = y^2 + x/2 - 4$, $f_{xx}(x, y) = 1/2$, $f_{xy}(x, y) = 2y$.

$f_y(x, y) = 2xy$, $f_{yx}(x, y) = 2y$, $f_{yy}(x, y) = 2$.

$D(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{vmatrix} = \begin{vmatrix} 1/2 & 2y \\ 2y & 2x \end{vmatrix} = x - 4y^2$. *Checking the critical points,* $D(0,2) = D(0,-2) = -16, D(8,0) = 8$. *Therefore,* $(0,2) and (0,-2)$ *are saddle points.* $(8,0)$ *is a local extremum. As far as* $f_{xx}(8,0) = 1/2 > 0$, *it is a local minimum.*

Problem 3 *Let us use the method of Lagrange Multipliers with one constraint.* $\nabla f(x,y) = \langle 2xy, x^2\rangle, \nabla g(x,y) = \langle 2x, 2y\rangle$, *there* $f(x,y)$ *is the function to be minimized or maximized, while* $g(x,y) = 1$ *is the constraint. Clearly, the condition* $x^2 + y^2 = 1$ *implies* $\nabla g(x,y) \neq 0$, *otherwise we would have to check manually all the points where this condition is violated.*

Then construct a system. $\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ x^2 + y^2 = 1 \end{cases}$

Coordinatewse, $\begin{cases} 2xy = \lambda 2x \\ x^2 = \lambda 2y \\ x^2 + y^2 = 1 \end{cases}$

The simplest way to solve it is to start from the first equation, $2xy = \lambda 2x$. *It implies that either* $y = \lambda$, *or* $x = 0$. *In the former case, substituting* $y = \lambda$ *in the second equation yields* $x^2 = 2y^2$, *together with* $x^2 + y^2 = 1$, *which gives us four candidate points:* $(\pm\sqrt{2/3}, \sqrt{1/3}), (\pm\sqrt{2/3}, -\sqrt{1/3})$. *The latter case,* $x = 0$, *yields* $y = \pm 1$, $\lambda = 0$, *which gives us two more candidate points,* $(0, \pm 1)$. *Evaluating* $f(x,y)$ *at all six points, we conclude that the maximum value subject to this constraint is* $\frac{2}{3}\sqrt{1/3}$, *achieved at* $(\pm\sqrt{2/3}, \sqrt{1/3})$, *while the minimum value is* $-\frac{2}{3}\sqrt{1/3}$, *achieved at* $(\pm\sqrt{2/3}, -\sqrt{1/3})$.

In this particular case the method of Lagrange multipliers is not the simplest one. It is simpler to make a change of variables $x = \cos(\phi), y = \sin(\phi)$, *and to look for* ϕ, *maximizing or minimizing* $f(\phi) = \cos(\phi)^2\sin(\phi)$. *Taking the derivative and setting it equal to zero,* $f'(\phi) = 2\cos(\phi)\sin(\phi)(-\sin(\phi)) + \cos(\phi)^3 = \cos(\phi)(\cos(\phi)^2 - 2\sin(\phi)^2) = 0$, *so either* $\cos(\phi) = 0$, *or* $\cos(\phi)^2 = 2\sin(\phi)^2$, *which gives the same six candidate points as above.*

Problem 4 *Geometrically, it is a cylinder (finite cylinder, in the simple geometric sense) minus a hemisphere centered at* $(0,0,2)$ *with radius* 2. *The volume of a sphere of radius* r *is* $\frac{4}{3}\pi r^3$, *so for* $r = 2$ *it is* $32\pi/3$, *and the volume of a hemisphere is then* $16\pi/3$. *The whole cylinder has volume equal to the area of its base times its height, i.e.* $(\pi(2)^2)\cdot(2) = 8\pi$. *Therefore, the answer is* $8\pi - 16\pi/3 = \pi(8 - 16/3) = \pi(24/3 - 16/3) = \frac{8\pi}{3}$. *Of course, it would do no harm to integrate and to ensure that this integral indeed evaluates to* $\frac{8\pi}{3}$.

Problem 5 $\int_0^2\int_{y^2}^4 ye^{x^2}dxdy = \int_0^4\int_0^{\sqrt{x}} ye^{x^2}dydx = \int_0^4 (e^{x^2}\frac{y^2}{2}\big|_{y=0}^{y=\sqrt{x}})dx = \int_0^4 e^{x^2}\frac{x}{2}dx = \frac{e^{x^2}}{4}\big|_0^4 = \frac{e^{16}-1}{4}$.

Of course, it was taken into account, that if $0 \leq x \leq 4$, *then* $(\sqrt{x})^2 = x$. *(For instance, it is not the case for* $x = -1$*).*

Problem 6 *Using Fubini's theorem, express this integral as an iterated integral, and then convert it to polar coordinates. It is possible, because* R *is a polar* *rectangle for* $1 \leq r \leq 2$, $0 \leq \theta \leq \pi/4$. $\iint_R \frac{1}{(x^2+y^2)^{3/2}} dA = \int_0^{\pi/4} \int_1^2 \frac{1}{r^3} r dr d\theta = \int_0^{\pi/4} (-\frac{1}{r}\Big|_{r=1}^{r=2}) d\theta = \int_0^{\pi/4} (-\frac{1}{2} + \frac{1}{1}) d\theta = \pi/4(1/2) = \pi/8$.

Note that the problem statement is somewhat ambiguous, and there are four regions which could have been meant in the problem statement. For example, R *can also be a polar rectangle for* $1 \leq r \leq 2$, $\pi/4 \leq \theta \leq \pi$, *and under this interpretation the answer is* $3\pi/8$. *There is nothing that can be done in this case. The problem statement is not precise enough, thus, strictly speaking, the answer is* $\pi/8$ *or* $3\pi/8$, *depending on the interpretation.*

Problem 7 *Interpreting it in cylindrical coordinates, the cones are* $z = 4 - r$ *and* $z = r$, *thus their intersection is a circle* $z = 4 - r = r$, *i.e.* $z = 2, r = 2$. *So the required part of the cone lies above the circle* $x^2 + y^2 \leq 4$ *or, equivalently,* $r \leq 2$. *Then, since* $f(x, y) = 4 - \sqrt{x^2 + y^2}$, $f_x(x, y) = \frac{-x}{\sqrt{x^2+y^2}}$, $f_y(x, y) = \frac{-y}{\sqrt{x^2+y^2}}$. *Therefore, the surface area is question can be found as* $\iint_D \sqrt{(f_x(x, y))^2 + (f_y(x, y))^2 + 1} dA = \iint_D \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} dA = \iint_D \sqrt{2} dA = \sqrt{2} A(D)$, *where* D *is the circle of radius* 2, $D = \{(x, y) | x^2 + y^2 <= 4\}$, $A(D)$ *is the area of* D. *Here the property* $A(D) = \iint_D dA$ *was used.*

So the surface area in question is equal to $\sqrt{2}A(D)$, $A(D) = \pi(2)^2 = 4\pi$, *hence the answer is* $4\sqrt{2}\pi$.

Problem 8 *We did not cover triple integrals. However, in this case it is a straightforward generalization.* $f_{ave} = \frac{\int_0^{2\pi} \int_0^4 \int_0^1 f(x,y,z) dz dy dx}{Volume(E)}$, *where* $Volume(E) = (2\pi) \cdot 4 \cdot 1 = 8\pi$. $\int_0^{2\pi} \int_0^4 \int_0^1 f(x, y, z) dz dy dx = \int_0^{2\pi} \int_0^4 \int_0^1 y \sin(x) + z dz dy dx = \int_0^{2\pi} \int_0^4 (zy \sin(x) + \frac{z^2}{2}\Big|_{z=0}^{z=1}) dy dx = \int_0^{2\pi} \int_0^4 (y \sin(x) + 1/2) dy dx = \int_0^{2\pi} (\frac{y^2}{2} \sin(x) + \frac{y}{2}\Big|_{y=0}^{y=4}) dx = \int_0^{2\pi} (8 \sin(x) + 2) dx = -8 \cos(x) + 2x\Big|_0^{2\pi} = (-8 + 4\pi) - (-8) = 4\pi$.

Finally, $f_{ave} = \frac{4\pi}{8\pi} = \frac{1}{2}$.

Problem 9 *We did not cover this material.*

Problem 10 *We did not cover this material.*