On some meshes in three dimensions

Joint work with

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Introduction

Consider the Poisson problem

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

on a bounded polyhedral domain $\Omega \subset \mathbb{R}^3$.

We construct a sequence of tetrahedralizations (*i.e.* meshes) $\mathcal{T}'_k$ of Ω with the property

$$\|u - u_k\|_{H^1(\Omega)} \le C \operatorname{dim}(S_k)^{-m/3} \|f\|_{H^{m-1}(\Omega)},$$

with C independent of k and f. Here:

- S_k ithe associated finite element space of continuous, piecewise polynomials of degree $m \ge 2$

- $u_k \in S_k$ is the finite element approximations of the solution of our Poisson problem above.

u_k has quasi-optimal approximation properties with respect to the dimension of S_k.

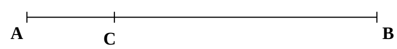
Our method relies on the a priori estimate

$$\|u\|_{\mathcal{D}^{m+1}_{a+1}(\Omega)} \leq C\|f\|_{H^{m-1}(\Omega)}$$

in the anisotropic weighted Sobolev spaces $\mathcal{D}^{m+1}_{a+1}(\Omega)$, with $a > 0$ small and determined by Ω. The weight is the distance to the set of singular boundary points (*i.e.* edges).

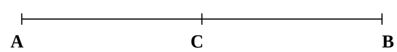
We also need improved interpolation estimates on thin tetrahedra that satisfy the maximum angle condition (but not the minimum angle condition, **Babuška-Aziz**).

The main feature of our refinement procedure is that if AB is a segment in the mesh $\mathcal{T}'_k$, then this segment will be divided into two segments AC and CB in the mesh $\mathcal{T}'_{k+1}$ according to whether A and B are equally singular or not, as follows:



A more singular than B

$$|AC| = \kappa|AB|, \ \kappa = 1/4$$



A and B equally singular

$$|AC| = |AB|$$

Edge decomposition

We can chose $\kappa \leq 2^{-m/a}$. This allows us to use a uniform refinement of the tetrahedra that are away from the edges to construct $\mathcal{T}'_k$.

A priori estimates

Theorem. Let $m \in \mathbb{Z}_+$. Then there exists $\eta > 0$ such that the our Poisson problem ($-\Delta u = f$, $u|_\Omega = 0$) has a unique solution $u \in \mathcal{K}^{m+1}_{a+1}(\Omega)$ for any $f \in \mathcal{K}^{m-1}_{a-1}(\Omega)$ which depends continuously on f:

$$\|u\|_{\mathcal{K}^{m+1}_{a+1}(\Omega)} \leq C_{\Omega,a}\|f\|_{\mathcal{K}^{m-1}_{a-1}(\Omega)}$$

The isotropically weighted Sobolev spaces $\mathcal{K}^m_a(\Omega)$ are defined by

$$\mathcal{K}^m_a(\Omega) := \{u,\ \vartheta^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ \ |\alpha| \leq m\},$$

$m \in \mathbb{Z}_+$, $a \in \mathbb{R}$, and $\vartheta(x) =$ the distance from $x \in \Omega$ to the edges of Ω.

We need more regularity along the edge.

Assume $\Omega = D_\alpha = \{0 < \theta < \alpha\}$, a dihedral angle with edge along the Oz–axis. Then we let

$$\mathcal{D}_a^1(D_\alpha) := \mathcal{K}_1^1(D_\alpha)$$
$$\mathcal{D}_a^m(D_\alpha) := \{u \in \mathcal{K}_a^m(D_\alpha),\ \partial_z u \in \mathcal{D}_a^{m-1}(D_\alpha)\}.$$

Thus our spaces $\mathcal{D}_a^1$ are, in fact, independent of a. We endow the space $\mathcal{D}_a^m(D_\alpha)$ with the norm

$$\|u\|^2_{\mathcal{D}_a^m(D_\alpha)} := \|u\|^2_{\mathcal{K}_a^m(D_\alpha)} + \|\partial_z u\|^2_{\mathcal{D}_a^{m-1}(D_\alpha)}.$$

Assume next that $\Omega = \mathcal{C}$, a cone centered at the origin. Denote by $\rho(x) = |x|$ the distance from x to the origin. Then we let.

$$\mathcal{D}_a^1(\mathcal{C}) := \rho^{a-1}\mathcal{K}_1^1(\mathcal{C}) = \{\rho^{a-1}v,\ v \in \mathcal{K}_1^1(\mathcal{C})\},$$

with norm $\|u\|_{\mathcal{D}_a^1(\mathcal{C})} := \|u/\rho^{a-1}\|_{\mathcal{K}_a^1(\mathcal{C})}$.

For $m \geq 2$, let $\rho\partial_\rho = x\partial_x + y\partial_y + z\partial_z$ be the infinitesimal generator of dilations. Then, for $m \geq 2$, we define by induction

$$\mathcal{D}_a^m(\mathcal{C}) := \{u \in \mathcal{K}_a^m(\mathcal{C}),\ \rho\partial_\rho(u) \in \mathcal{D}_a^{m-1}(\mathcal{C})\}.$$

with a similarly defined norm.

For a general bounded polyhedral domain Ω, we define the *anisotropic weighted Sobolev spaces* $\mathcal{D}_a^m(\Omega)$ by localization around vertices, edges, such that in away from the edges we have the usual Sobolev spaces H^m.

Theorem. Let $m \geq 1$ and $f \in H^{m-1}(\Omega)$. Then there exists $\eta \in (0, 1]$ such that our Poisson problem has a unique solution $u \in \mathcal{D}_{a+1}^{m+1}(\Omega)$. This solution depends continuously on f, for any $0 \leq a < \eta$ and any m:

$$\|u\|_{\mathcal{D}_{a+1}^{m+1}(\Omega)} \leq C_{\Omega,a}\|f\|_{H^{m-1}(\Omega)}.$$

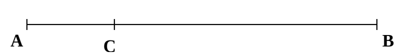
(**Arnold-Falk, Apel99, ApelNicaise, Babuška-Guo, Bacuta-Bramble, Buffo-Costabel-Dauge03, Kellogg-Osborn, ...**)

Refinement

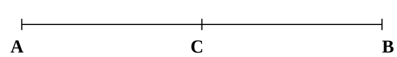
• We shall construct a sequence of divisions (partitions) of Ω into polyhedral domains (mostly tetrahedra and prisms).

• Points of type **V, E, S**. **V** is more singular than **E**, which in turn is more singular than **S**. Edges of type **VE, VS, ES, SS**, triangles of type **VES, ...**

• We divide the edges as explained

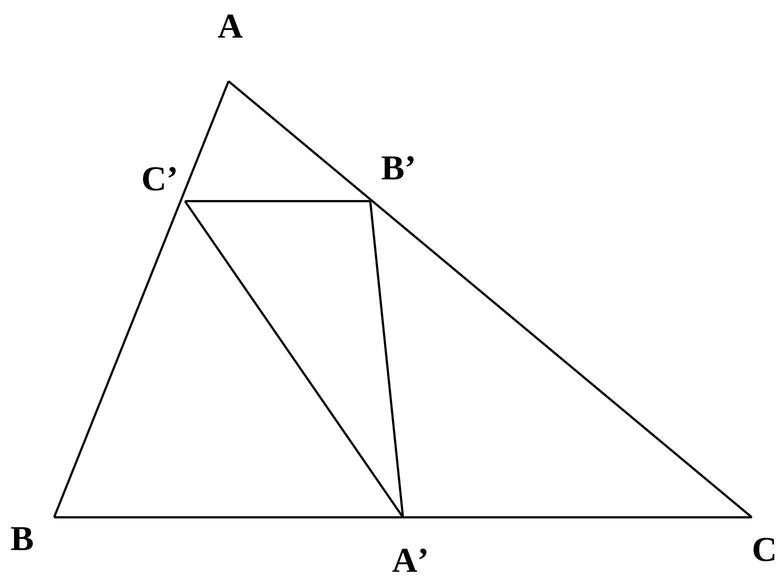


A more singular than B



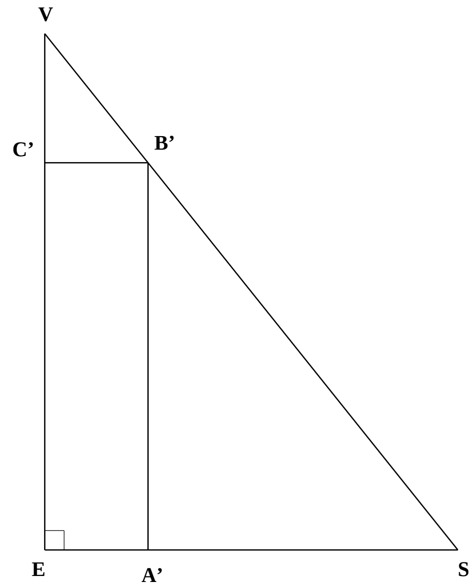
A and B equally singular

• This leads to the following decompositions of triangles



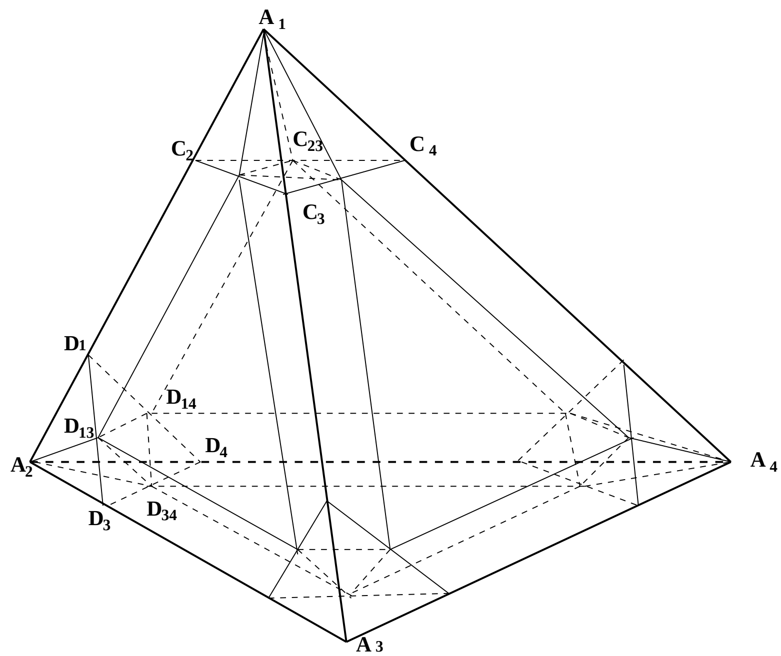
Face decomposition:
A of type **V** or **E**,
B and C of type **S**,
$|AC'| = \kappa|AB|,\ |AB'| = \kappa|AC|,$
$|A'B| = |A'C|,\ \kappa = 1/4$

The decomposition of a triangle of type $\mathbf{S}^3$ is obtained by setting $\kappa = 1/2$ in the above picture. This gives four congruent triangles. Finally,



VES decomposition: $|VC'| = \kappa|VE|$, $|VB'| = \kappa|VS|$, $|EA'| = \kappa|ES|$, $A'C'$ *was removed*, $\angle E = 90^o$

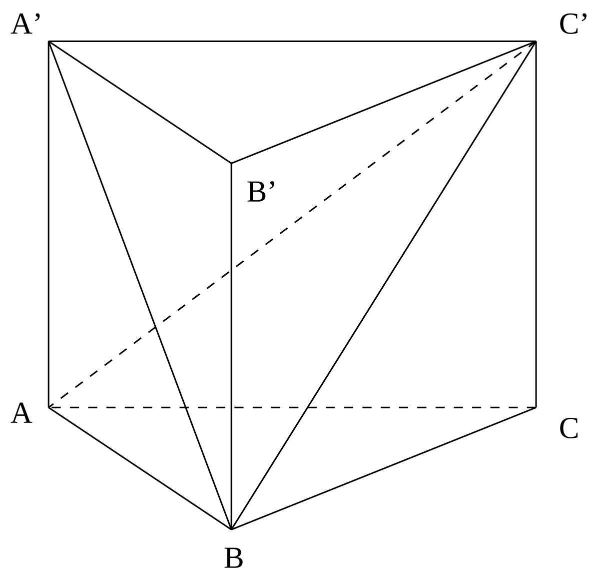
• $\mathcal{T}_0$ initial division of our polyhedral domains in straight triangular prisms, tetrahedra of types **VESS** and **VS**^{3} (thus having a vertex in common with Ω), and an interior region Λ_0. The prisms will be divided in tetrahedra in a canonical way determined by some additional initial choices (marks on the prisms).



Initial decomposition.

We shall deform our edge points so that the prisms become straight triangular prisms (*i.e.*the edges are perpendicular to the bases).

• We tetrahedralize Λ_0 without introducing additional edges on the boundary of Λ (but allowing additional internal edges and vertices) and dividing each prism into three tetrahedra.

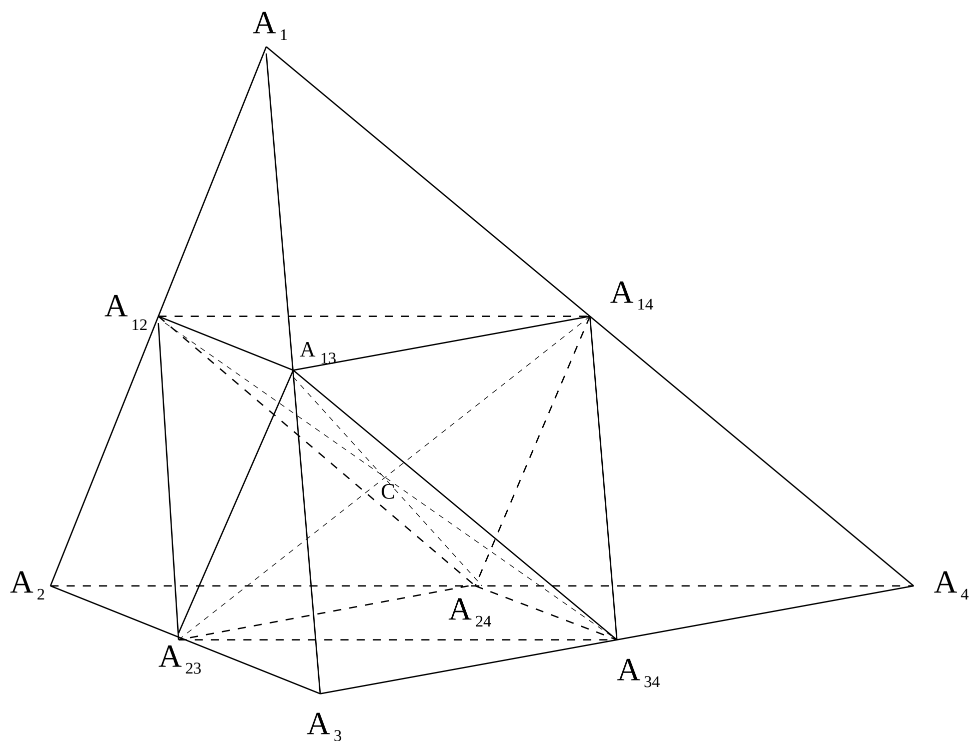


Marking a prism: $BC' = mark$,
$AA' \,||\, BB' \,||\, CC' \perp ABC$ and $A'B'C'$

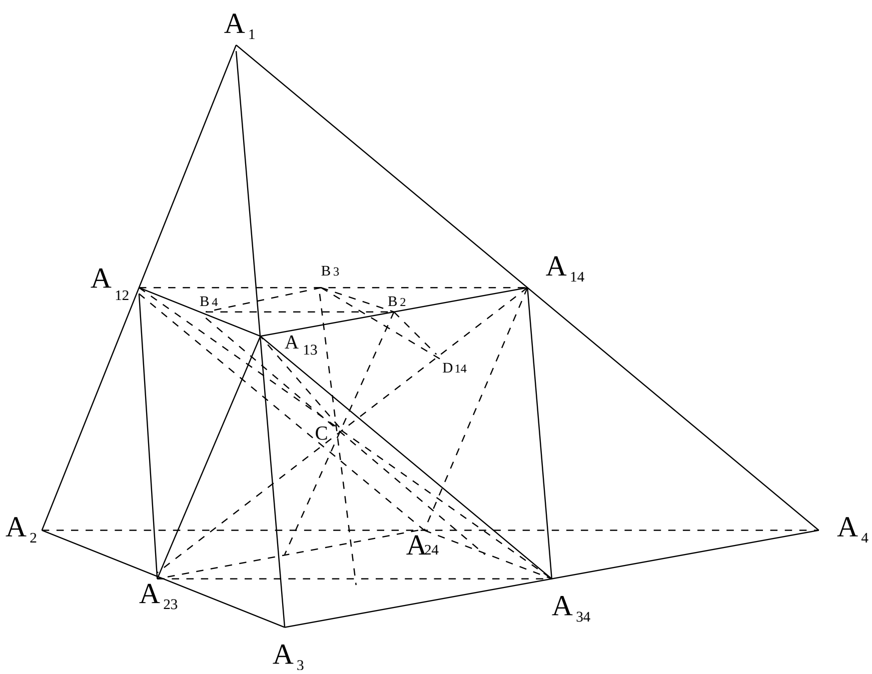
• We then apply uniform, semi-uniform, and non-uniform refinements to obtain the divisions $\mathcal{T}_n$ of Ω into marked prisms and tetrahedra, as explained below.

The meshes $\mathcal{T}_n$ are obtained by dividing each prism into three tetrahedra as determined by the mark.

- Uniform refinement

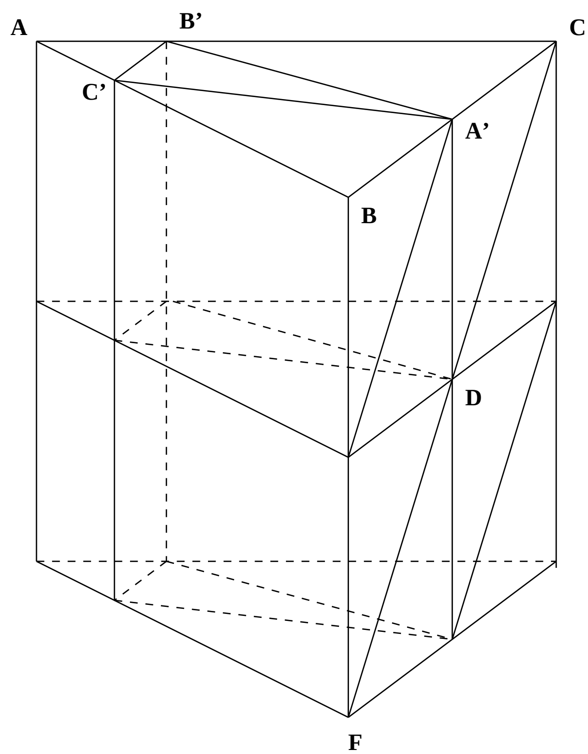


First level uniform barycentric refinement



Refinement of one tetrahedron from $\mathcal{C}_1$.

- Semi-uniform refinement



First level of semi-uniform refinement of a prism,
$CD, DF =$ marks.

We divide the base in a non-uniform way (as a triangle of type **VSS**) n times, and we divide the edges in 2^n equal segments. $n = 1$ in the picture (level one).

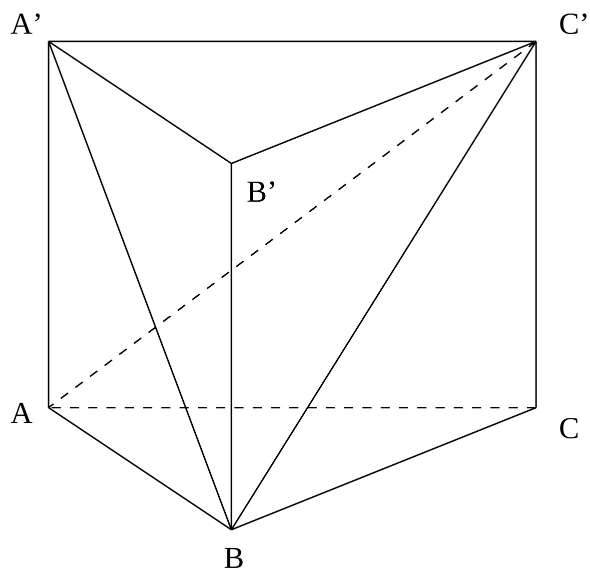
• Non-uniform refinement

If T is a tetrahedron of type **VS**^{3}, we divide it the in 12 tetrahedra like in the uniform strategy, but with the edges through the vertex of type **V** divided in the ration given by κ. These tetrahedra belong to $\mathcal{T}_{n+1}$. There will be a tetrahedron of type **VS**^{3} and 11 tetrahedra of type **S**^{4}. (Then, as explained in the previous subsection, we shall iterate this construction for the tetrahedron of type **VS**^{3}, whereas the tetrahedra of type **S**^{4} are divided according to the uniform strategy.)

If, on the other hand, T is a tetrahedron of type **VESS**, we divide it into 6 tetrahedra of type **S**^{4}, one tetrahedron of type **VS**^{3}, and a prism. The vertex of type **E** of T will belong only to the prism. This division is obtained by first dividing it into 12 pieces like in the uniform strategy. The union of the tetrahedron containing the vertex of type **E** and of the tetrahedra adjacent to it will form the prism.

Interpolation on thin tetrahedra

Let $ABCA'B'C'$ be a straight prism divided into three tetrahedra.



$BC' = mark$, $AA' \parallel BB' \parallel CC' \perp ABC$ and $A'B'C'$

Let $\hat{\sigma}$ be any of these tetrahedra and $m \geq 2$. Let $u \in \mathcal{C}^1(\hat{\sigma})$ and $I(u) = u_I$ be interpolant associated to the linear m-simplex.

Theorem. We have $\|\partial_z(u-u_I)\|_{L^2(\widehat{\sigma})} \leq \widehat{C}|\partial_z u|_{H^m(\widehat{\sigma})}$ and $\|\partial_x(u-u_I)\|_{L^2(\widehat{\sigma})} + \|\partial_y(u-u_I)\|_{L^2(\widehat{\sigma})} \leq \widehat{C}\Big(|\partial_x u|_{H^m(\widehat{\sigma})} + |\partial_y u|_{H^m(\widehat{\sigma})}\Big)$, for a constant $\widehat{C}$ that depends only on ϵ and δ.

This theorem (valid for $m \geq 2$) yields, through affine transformations (dilations), the needed interpolation estimates on the resulting thin tetrahedra.

$\partial_z u = 0$ implies $\partial_z I(u)$, $\partial_x u = 0$ and $\partial_y u = 0$ imply $\partial_x I(u)$ and $\partial_y I(u)$.