

# CLOSED-FORM ASYMPTOTICS FOR LOCAL VOLATILITY MODELS

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ABSTRACT. We obtain new closed-form pricing formulas for contingent claims when the asset follows a Dupire-type local volatility model. Using our recently developed framework of [7, 8, 10] for short time asymptotic expansions of heat kernels, we obtain a family of general explicit closed form approximate solutions for both the pricing kernel and derivative price. We also perform analytic as well as a numerical error analysis, and compare our results to other known methods.

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## 1. INTRODUCTION

Financial derivatives (also known as contingent claims) are now a ubiquitous tool in risk management with more than 500 trillion dollars worth of such contracts currently in the market. The pricing of such derivatives is therefore an active area of research in both Mathematics and Finance (see for example [16, 23, 24, 31, 44] and the references therein). The earliest

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model used in pricing derivatives is the Black-Scholes-Merton model [4, 38], for which the risky asset  $X_t$  is modeled by geometric Brownian motion with drift:

$$(1.1) \quad dX_t = rX_t dt + \sigma X_t dW_t.$$

A European option  $U$  on  $X_t$  with payoff  $h(X_T, K)$  at maturity  $T$  and strike  $K$  is then valued by taking the discounted expectation of the payoff function over all possible paths  $X_t(\cdot)$  of (1.1) so that <sup>1</sup>

$$(1.2) \quad \begin{aligned} U_{BSM}(t, x) &= \mathbb{E}^{\mathcal{Q}}[e^{-r(T-t)} h(X_T, K) \mid X_t = x] \\ &= \int_0^\infty \mathcal{G}_t^{BSM}(x, y) e^{-r(T-t)} h(y, K) dy, \end{aligned}$$

where in this context  $\mathcal{G}_t^{BSM}(x, y)$  is the risk-neutral transition density kernel of the process (1.1). For a call option with payoff  $h(X_T, K) = \max\{0, X_T - K\}$ , explicit evaluation of the above integral leads to the well known Black-Scholes-Merton formula

$$(1.3) \quad U_{BSM}(t, x) = x\mathcal{N}(d_-) + Ke^{-r(T-t)}\mathcal{N}(d_+),$$

where  $\mathcal{N}(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$  is the cumulative normal distribution, and

$$(1.4) \quad d_{\pm} = \frac{\ln(x/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

The Black-Scholes-Merton model assumes that the return,  $\frac{dX_t}{X_t}$ , on the underlying security is normally distributed with constant variance  $\sigma^2$ . However, since as early as the 60s with the work of Fama [21] and Mandelbrot [37], a large amount of empirical evidence was building that returns of price-processes in the market exhibited excesses kurtosis. Essentially, this means that the distribution of market returns is not Gaussian with constant variance as predicted by the Black-Scholes-Merton model, but instead exhibits much fatter tails. This results in Black-Scholes-Merton valued options to undervalue far in-the-money calls and overvalue far out-of-the-money calls. Indeed, for market data, the volatility implied by the BSM model is plotted for different strikes  $K$ , the implied volatility is not constant but rather is a curve, typically in the shape of a smile (or skew)[22, 28]. This phenomena is called the volatility smile (or volatility skew). Dupire [18], Derman and Kani [14] and Rubinstein [43] were among the first to try and build models that fit the empirical smile patterns of the implied volatility. The models they proposed are of the form

$$(1.5) \quad dX_t = rX_t dt + \Sigma(t, X_t) dW_t,$$

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<sup>1</sup>Because of its clear connection to a kernel, we prefer to define the option price through the risk-neutral pricing formula rather than constructing a hedging portfolio. However, for all the models we consider here both methods lead to the same partial differential equation for the option price.

which generalize the Black-Scholes-Merton model by allowing a nonconstant volatility in the return of the price-process. This has been fairly successful in replicating the empirical smiles for equity options [17]. However, one drawback of these models compared to the simpler Black-Scholes-Merton model is that there is no closed form solution for general  $\Sigma(t, x)$ . Thus the evaluation of the model must be done by numerical methods, either via Monte-Carlo simulation on the risk-neutral pricing formula or on the resulting partial differential equation (see Section 2 for details). In addition, even if a closed form solution does indeed exist, as for the special case of the Black-Scholes-Merton model, the formula is expressed in terms of error functions and integrals which too have to be evaluated numerically via quadrature and also generate computational errors [20, 45]. For this reason, we shall content ourselves with approximations that are nevertheless easier to compute. To explain our approach by a simple analogy, let us mention the problem of the numerical evaluation of  $\ln(x)$ . This evaluation often relies on identities of the form  $\ln(x) = \sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{n}$  or  $\ln(x) = 1 + \sum_{n=1}^{\infty} (-1)^n e^{-n} \frac{(x-e)^n}{n}$ , where the first identity is obtained by Taylor expanding about  $x = 1$  while the second identity is obtained by Taylor expanding about  $x = e$ . Close to 1 or  $e$ , only a few terms in the sum will give a good approximation of our function.

In this paper, we present an efficient method to compute an approximate closed form solution for the price of an option obeying the local volatility model (1.5). We will evaluate the price of the option by expanding the pricing kernel  $\mathcal{G}_t(x, y)$  as a sort of power series, more precisely an asymptotic time-ordered series in the time to expiry  $\tau = T - t$ , and in analogy with the expansion of  $\ln(x)$  above, different choices of basespoints lead to different expansions. Our starting point for the analysis is that if the underlying asset is modeled by a local volatility model (1.5), then the price of European-style option is given as the solution of a backward parabolic equation of the form

$$(1.6) \quad \partial_t U + LU = 0$$

with  $L$  the (typically degenerate) elliptic operator

$$(1.7) \quad L = \frac{1}{2} \Sigma(t, x)^2 \partial_x^2 + \mu(t, x) \partial_x - r$$

and terminal condition  $h(x; K)$  at time  $t = T$  (see Section 2 for details). The solution to (1.6) formally has the representation

$$(1.8) \quad U(t, x) = \int_0^{\infty} \mathcal{G}_t(x, y) h(y, K) dy,$$

where  $\mathcal{G}_t(x, y)$  is the backwards heat kernel of the elliptic operator  $L$ . Typically, there is no closed form expression for  $\Gamma_t(x, y)$  or  $U(t, x)$ . However, if the time to expiry  $\tau = T - t$  is small, one approach to an analytical study of such problems is to use the beautiful results on short-time expansions of diffusions [3, 36, 39, 40, 48, 49] to obtain an expansion of  $\mathcal{G}_t(x, y)$  in  $\tau$ . This

approach combines deep ideas in probability and Riemannian geometry and has been used successfully on several multi-factor stochastic volatility models, including the SABR model [6, 34, 42] originally introduced in [26]. One drawback of this method is that it is formulated in terms of the geodesic distance  $d(x, y)$  associated to the coefficients of the highest-order derivatives in  $L$  (via a so-called metric tensor, called the Varadhan metric)". See [29] or [50] for a review. Typically, it is difficult to compute the geodesic distance  $d(x, y)$  unless the geometry is particularly simple.

Furthermore, the method fails for diffusions that are very degenerate [19]. A related approach is to use oscillatory-type integrals, as in [47]. In this approach, the coefficients of the expansion are determined through computations involving the symbol of associated pseudo-differential operators. Finally, another related short-time method can be found in [1], which has also been applied to stochastic volatility models [2], in which the coefficients of the expansion are given in terms of Hermite polynomials. It would be informative to compare the expansions from these different methods, but we do not pursue this direction here.

In [7, 8, 10] we developed a computationally simpler framework for small-time asymptotic expansions to that of [39, 40, 48, 49]. In particular it leads to analytically tractable formulas, and circumvents the need to know  $d(x, y)$ , which typically cannot be computed analytically by hand. The method leads to *approximate closed form solutions*, that is, closed form time-asymptotic expansions, which contain an error that depends on the number of terms in the expansion and it is small if time to expiry is sufficiently small. The situation is similar to, say, the error in computing only the first few terms in the expansion of  $\ln(x)$  above. While the procedure in [7, 8, 10] to compute the expansion is general and can be used for any multi-factor stochastic volatility model (see [9]), here we apply the method to the simplest case of local volatility models in order to give full details of the expansions as well perform a full comparison of the different expansions. As in the illustrative expansion of  $\ln(x)$ , we have a choice of different basepoints for which to do the expansion leading to different expansions. However, while in certain cases one expansion may be better than another, the actual choice of basepoint poses other interesting issues. For example, in the example of expanding  $\ln(x)$  we could choose to expand about  $x = 1$  or  $x = e$  but certainly the first is easier to compute by hand since it doesn't involve powers of the transcendental number  $e$ . A similar issue arises in our expansion as well. Namely, we are able to chose different basepoints but certain choices lead to more tractable (by hand) expansions. In particular, in analogy to geometric-based short-time asymptotic expansions, we will be able to chose the geodesic midpoint between  $x$  and  $y$  in the distance induced by the Varadhan metric. We address this issue in Section 4. On the other hand, in this paper, we consider operators whose coefficients do not always satisfy the assumptions of [7, 8, 10] needed to estimate the remainder terms in the

expansion, which is needed to carry out the analytical error analysis. Nevertheless, in Section 6 we perform a numerical error analysis by comparing our approximate solutions with the numerically generated exact solution, (or exact closed form solutions as in the case of the Black-Scholes-Merton model).

## 2. FORMULATION OF THE PROBLEM AND MAIN RESULTS

In this section we formulate the derivative pricing problem that we study and the parabolic partial differential equations that arise.

Let  $\{X_t\}_{0 \leq t \leq T}$  denote the value of a risky asset at time  $t$ , modeled by a local stochastic volatility model

$$(2.1) \quad dX_t = \mu(t, X_t)dt + \Sigma(t, X_t)dW_t,$$

where  $\mu : [0, T] \times [0, \infty) \rightarrow \mathbb{R}$  is the expected rate of return and  $\Sigma : [0, T] \times [0, \infty) \rightarrow [0, \infty)$  is the volatility, which is allowed to depend explicitly on both the asset value  $X_t$  and time  $t$ . This framework, first introduced in [18, 14, 43], includes as special cases both the Black-Scholes-Merton model [4, 38],

$$(2.2) \quad dX_t = rX_tdt + \sigma X_t dW_t,$$

and the CEV model [12],

$$(2.3) \quad dX_t = rX_tdt + \sigma X_t^\alpha dW_t,$$

where  $r, \sigma, \alpha > 0$ . Notice that in both (2.2) and (2.3) the volatility doesn't depend explicitly on time.

The simplest way to introduce explicit time dependence in these models is to make  $\sigma = \sigma(t)$ ,

$$(2.4) \quad dX_t = rX_tdt + \sigma(t)X_t dW_t$$

$$(2.5) \quad dX_t = rX_tdt + \sigma(t)X_t^\alpha dW_t,$$

which we will call the time-dependent Black-Scholes-Merton model and time-dependent CEV model, respectively.

For a given expiry date  $T$  and strike price  $K$ , let  $h(X_T, K)$  denote the payoff of the European-style contingent claim  $U$  on  $X_t$ . For example,

$$(2.6) \quad h_{\text{call}}(X_T; K) = |X_T - K|^+ := \max(0, X_T - K)$$

for a call option and

$$(2.7) \quad h_{\text{put}}(X_T; K) = |K - X_T|^+ := \max(0, K - X_T)$$

for a put option.

By the Fundamental Theorem of Asset Pricing [44], we assume the option price  $U$  is given by the risk-neutral pricing formula

$$\begin{aligned}
(2.8) \quad U(t, x) &:= \mathbb{E}^{\mathcal{Q}} \left[ e^{-\int_t^T \rho(s, X_s) ds} h(X_T, K) | X_t = x \right] \\
&= \int_0^\infty \mathcal{G}_t(x, y) h(y; K) dy,
\end{aligned}$$

where  $\mathcal{Q}$  is the *risk-neutral measure* and  $\mathcal{G}_t(x, y)$  is the *pricing kernel* (see also Remark 2.3).

*Remark 2.1.* Typically, the discount rate  $\rho$  is taken to be a constant, but we include this level of generality since it doesn't present any additional complications with the theory.

Consider now the backwards parabolic equation

$$\begin{aligned}
(2.9) \quad \partial_t U(t, x) + L(t, x)U(t, x) &= 0 \\
U(T, x) &= h(x, K)
\end{aligned}$$

with

$$(2.10) \quad L(t, x) := \frac{1}{2} \Sigma(t, x)^2 \partial_x^2 + \mu(t, x) \partial_x - \rho(t, x).$$

**Example 2.2.** For the time-dependent Black-Scholes-Merton model (2.4) we have

$$(2.11) \quad L(t, x) = L_{BS}(t, x) = \frac{1}{2} \sigma(t)^2 x^2 \partial_x^2 + r x \partial_x - r$$

while for the time-dependent CEV model (2.5)

$$(2.12) \quad L(t, x) = L_{CEV}(t, x) = \frac{1}{2} \sigma(t)^2 x^{2\alpha} \partial_x^2 + r x \partial_x - r.$$

Under certain conditions on the operator  $L$  and payoff function  $h$  there exists a function  $\mathcal{G}_t(x, y)$  such that (2.9) has the representation

$$(2.13) \quad U(t, x) = \int_0^\infty \mathcal{G}_t(x, y) h(y; K) dy.$$

The function  $\mathcal{G}_t(x, y)$  in (2.13) is the pricing kernel in (2.8). The question as to under what conditions  $U$  in (2.8) and  $U$  in (2.9) are equivalent is very subtle and we do not discuss this point here. We mention however the results in [27, 32] giving conditions under which the two equations for  $U$  are equivalent. Nevertheless, at least formally, one can go from one representation to the other by applying Ito's lemma or the Feynman-Kac formula.

As mentioned in the Introduction, except for some very special cases no closed form solutions for either the pricing kernel or option price exist. One special case is that of the Black-Scholes-Merton model which has the well known closed form solution for the kernel  $\mathcal{G}^{BSM}$  and option price  $U^{BSM}$ ,

(2.14)

$$\mathcal{G}_t^{BSM}(x, y) = \frac{\exp(-r(T-t))}{y\sqrt{2\pi\sigma^2(T-t)}} \exp\left(-\frac{|\ln(x/y) + (r - \sigma^2/2)(T-t)|^2}{2\sigma^2(T-t)}\right)$$

$$U^{BSM}(t, x) = \int_0^\infty \mathcal{G}_t^{BSM}(x, y) dy = x\mathcal{N}(d_-) + Ke^{-r(T-t)}\mathcal{N}(d_+),$$

where  $\mathcal{N}(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$  is the error function and

$$(2.15) \quad d_\pm = \frac{\ln(x/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

However, for the time-dependent Black-Scholes-Merton model, or local volatility models in general, closed form solutions are either very complicated or do not exist (c.f. [35, 15]). Our goal in this paper is to give an approximate closed form solution for the European option  $U$  on an asset modeled by a general local stochastic volatility model (1.5) by giving an approximate closed form expansion for the pricing kernel  $\mathcal{G}_t(x, y)$  when an exact closed form solution does not exist. Indeed, using our recently developed framework of [7, 8, 10] we will show that the backwards heat kernel  $\mathcal{G}_t(x, y)$  of  $L$  (or equivalently the transition density kernel of (1.5)) can be expanded in the form

(2.16)

$$\mathcal{G}_t(x, y) = \mathcal{G}_\tau^{[n]}(x, y; z) + \tau^{(n+1)/2} \mathcal{E}_\tau^{[n]}(x, y; z)$$

$$\mathcal{G}_\tau^{[n]}(x, y; z) = G_\tau^{[0]}(x, y; z) + \tau^{1/2} G_\tau^{[1]}(x, y; z) + \dots + \tau^{n/2} G_\tau^{[n]}(x, y; z),$$

where  $\mathcal{G}_t^{[n]}$  is the sum on  $n$  terms of the expansion, it is explicitly computable, and fast if  $n = 1, 2$ ,  $\mathcal{E}_\tau^{[n]}$  is the remainder (or error) in the expansion of the Greens function and we know how to control it, and  $\tau$  is the time to expiry

$$(2.17) \quad \tau = T - t.$$

For each term  $G^{[k]}$  in the expansion of the Green function, let  $\mathcal{U}^{[k]}$  denote the corresponding term in the expansion of the solution,

$$(2.18) \quad \mathcal{U}^{[k]}(t, x) = \int_0^\infty G_\tau^{[k]}(x, y) h(y; K) dy.$$

Then using (2.13) and we arrive at the expansion of the value of the derivative,

(2.19)

$$U(t, x) = U^{[n]}(t, x) + \tau^{(n+1)/2} \mathfrak{E}^{[n]}(t, x)$$

$$U^{[n]}(t, x) = \mathcal{U}^{[0]}(t, x) + \tau^{1/2} \mathcal{U}^{[1]}(t, x) + \tau \mathcal{U}^{[2]}(t, x) + \dots + \tau^{n/2} \mathcal{U}^{[n]}(t, x)$$

where

$$(2.20) \quad \tau^{(n+1)/2} \mathfrak{E}^{[n]}(t, x) := \tau^{(n+1)/2} \int_0^\infty \mathcal{E}_\tau^{[n]} h(y; K) = U(t, x) - U^{[n]}(t, x)$$

is the remainder term (or error) in the expansion of the solution.

For pedagogical purposes and error analysis we will list all the details for the time-dependent Black-Scholes and CEV models, although our results hold for more general functions  $\mu(t, x)$  and  $\Sigma(t, x)$ .

*Remark 2.3.* Given that  $\mathcal{G}_t(x, y)$  arises in several different contexts, we will call the function  $\mathcal{G}_t(x, y)$  the transition density kernel, pricing kernel, heat kernel or Green function interchangeably, depending on the context in which the object arises.

Given a basepoint  $z = z(x, y)$ , our main result is that there exist *explicitly computable* functions  $\mathfrak{P}^\ell(z, x, y)$  with the following property. Let

$$(2.21) \quad \mathcal{G}_\tau^{[n]}(x, y; z) := \tau^{-1/2} \sum_{\ell=0}^n \tau^{\ell/2} \mathfrak{P}^\ell\left(z, z + \frac{x-z}{\tau^{1/2}}, z + \frac{y-z}{\tau^{1/2}}\right) G^{[0]}(z; \frac{x-y}{\tau^{1/2}}),$$

That is, each term  $G^{[k]}(x, y)$  in (2) is given by the product of a function  $\mathfrak{P}^k(z, z + \frac{x-z}{\tau^{1/2}}, z + \frac{y-z}{\tau^{1/2}})$  times a rescaled zeroth order Green function  $G^{[0]}$  (3.22). Indeed, in Section 3 we explicitly carry out the computation of each  $\mathfrak{P}^k$ , for  $k = 0, 1, 2$ . Our method leads to *new* closed form asymptotic expansions of the Kernel for local volatility models. In particular, the first order asymptotic expansion is given by

$$(2.22) \quad \mathcal{G}_t^{[1]}(x, y; z) = \frac{1}{\sqrt{2\pi(T-t)\Sigma(T, z)^2}} e^{-\frac{|x-y|^2}{2(T-t)\Sigma(T, z)^2}} \left[ \left( 1 + \frac{3\Sigma(T, z)\Sigma'(T, z) - 2\mu(T, z)}{2\Sigma(T, z)^2} \right) (x-y) - \frac{\Sigma'(T, z)}{2(T-t)\Sigma(T, z)^3} (x-y)^3 + (x-z) \left( \frac{(x-y)^2 - (T-t)\Sigma(0, z)^2}{(T-t)\Sigma(0, z)^3} \right) \right].$$

For future reference we also record the second order expansion of the Kernel at the end of Section 3. Notice that (2.22) depends on the basepoint  $z = z(x, y)$  of the expansion, and in Section 4 we will investigate the behavior of this expansion as a function of the basepoint. In particular, we will introduce three different basepoints for the expansions. The standard approximation  $z(x, y) = x$  in Section 4.1, the geodesic midpoint approximation  $z = \delta_\Sigma(x, y)$  in Section 4.2 and the the Euclidean midpoint approximation  $z(x, y) = (x+y)/2$  in Section 4.3. While the local volatility operators considered in this paper do not necessarily satisfy the assumptions on the coefficients of  $L$  needed to establish the analytic error estimates performed in [7, 8, 10], in Section 6 we nevertheless perform a numerical error analysis by computing both the numerical solution  $U$  and expansion  $U^{[n]}$  and estimating the error

$$(2.23) \quad |U(t, x) - U^{[1]}(t, x)|$$

pointwise for different basepoints  $z = z(x, y)$ . And the error analysis is in good agreement with the theoretical results.

We now proceed to compute the first few terms in the expansion (2), (2.19) by explicitly computing the polynomials  $\mathfrak{P}^\ell$  in (2.21).

### 3. THEORETICAL FRAMEWORK

We now follow the method in [7, 8, 10] and give the second order expansion for the pricing kernel  $\mathcal{G}_t^L$  and option  $U$  for underlying dynamics of the form (2.1) with  $\mu, \Sigma$  arbitrary smooth functions on  $(0, T) \times (0, \infty)$ .

We first reformulate (2.9) to a standard *forward* initial value problem by rewriting the equations in terms of the time to expiry  $\tau$  (2.17). In this case (2.9) becomes the initial value problem

$$(3.1) \quad \begin{aligned} \partial_\tau U(\tau, x) - L(\tau, x)U(\tau, x) &= 0 \\ U(0, x) &= h(x, K) \end{aligned}$$

which is to be solved *forward* in time.

*Remark 3.1.* Note that we have abused notation in that the function  $U$  in (2.9) and the function  $U$  in (3.1) are different. However, they differ only in a linear transformation in the first argument, and thus when no confusion arises we denote both functions by  $U$ . In particular, functions evaluated at  $\tau = 0$  in the forward problem correspond to functions evaluated at  $t = T$  in the backward problem.

Let  $\mathcal{G}_\tau^L$  be the solution operator of the parabolic problem (3.1), i.e.  $U(\tau, \cdot) = \mathcal{G}_\tau^L h(\cdot, K)$ . Decompose  $L(\tau, x)$  into

$$(3.2) \quad L(\tau, x) = L_0 + V(\tau, x)$$

where  $L_0$  is a constant coefficient second order operator and  $V(t, x)$  is a time-dependent variable coefficient second order operator. Then by the Duhamel principle we have

$$(3.3) \quad \mathcal{G}_\tau^L = e^{\tau L_0} + \int_0^\tau e^{(\tau-\tau_1)L_0} V(\tau_1, \cdot) \mathcal{G}_{\tau_1}^L d\tau_1$$

Using this representation for  $\mathcal{G}_{\tau_1}^L$  and recursively plugging into the above formula yields,

(3.4)

$$\begin{aligned}
\mathcal{G}_\tau^L &= e^{\tau L_0} + \int_0^\tau e^{(\tau-\tau_1)L_0} V(\tau_1, \cdot) \left( e^{\tau_1 L_0} + \int_0^{\tau_1} e^{(\tau_1-\tau_2)L_0} V(\tau_2, \cdot) \mathcal{G}_{\tau_2}^L \right) d\tau_2 d\tau_1 \\
&= e^{\tau L_0} + \int_0^\tau e^{(\tau-\tau_1)L_0} V(\tau_1, \cdot) e^{\tau_1 L_0} d\tau_1 + \int_0^\tau \int_0^{\tau_1} e^{(\tau-\tau_1)L_0} V(\tau_1, \cdot) e^{(\tau_1-\tau_2)L_0} V(\tau_2, \cdot) \mathcal{G}_{\tau_2}^L d\tau_2 d\tau_1.
\end{aligned}$$

Notice that we have "pushed" the term with  $\mathcal{G}_{\tau_2}^L$  to the last integral. Repeating this procedure  $d$  times yields the time-ordered formula,

(3.5)

$$\begin{aligned}
\mathcal{G}_\tau^L &= e^{\tau L_0} + \int_0^\tau e^{(\tau-\tau_1)L_0} V(\tau_1, \cdot) e^{\tau_1 L_0} d\tau_1 + \int_0^\tau \int_0^{\tau_1} e^{(\tau-\tau_1)L_0} V(\tau_1, \cdot) e^{(\tau_1-\tau_2)L_0} V(\tau_2, \cdot) e^{\tau_2 L_0} d\tau_2 d\tau_1 \\
&\quad + \cdots + \int_0^\tau \int_0^{\tau_1} \cdots \int_0^{\tau_{d-1}} e^{(\tau-\tau_1)L_0} V(\tau_1, \cdot) e^{(\tau_1-\tau_2)L_0} V(\tau_2, \cdot) \cdots V(\tau_d, \cdot) e^{\tau_d L_0} d\tau_d \cdots d\tau_2 d\tau_1 \\
&\quad + \int_0^\tau \int_0^{\tau_1} \cdots \int_0^{\tau_{d+1}} e^{(\tau-\tau_1)L_0} V(\tau_1, \cdot) e^{(\tau_1-\tau_2)L_0} V(\tau_2, \cdot) \cdots V(\tau_{d+1}, \cdot) \mathcal{G}_{\tau_{d+1}}^L d\tau_{d+1} \cdots d\tau_2 d\tau_1
\end{aligned}$$

Fix  $z \in \mathbb{R}$ . For any function  $f(\tau, x)$  denote by  $f^{s,z}(\tau, x) := f(s^2\tau, z + s(x-z))$  the parabolic rescaling by  $s$  of the function  $f$  about the basepoint  $z$ . We now define a rescaled operator  $L^{s,z}$  by

$$(3.6) \quad L^{s,z}(\tau, x) := \frac{1}{2} \Sigma^s(\tau, x)^2 \partial_x^2 + s\mu^s(\tau, x) \partial_x - s^2 \rho^s(\tau, x).$$

so that in this scaling the problem (3.1) becomes,

$$(3.7) \quad \begin{aligned} \partial_\tau U^{s,z}(\tau, x) - L^{s,z}(\tau, x) U^{s,z}(\tau, x) &= 0 \\ U^{s,z}(0, x) &= h^{s,z}(x; K) \end{aligned}$$

For any  $z \in \mathbb{R}$  fixed, and for any  $s > 0$ , the relationship between the Green kernel of  $L$  and the Green kernel of the rescaled operator  $L^{s,z}$  is

$$(3.8) \quad \begin{aligned} \mathcal{G}_\tau^L(x, y) &= s^{-1} \mathcal{G}_1^{L^{s,z}}(z + s^{-1}(x-z), z + s^{-1}(y-z)) \\ &= \tau^{-\frac{1}{2}} \mathcal{G}_1^{L^{\sqrt{\tau}}}(z + \tau^{-\frac{1}{2}}(x-z), z + \tau^{-\frac{1}{2}}(y-z)), \text{ if } s = \tau^{-\frac{1}{2}}. \end{aligned}$$

We now proceed to compute the Green kernel  $G_\tau^{L^{s,z}}$  of the rescaled problem (3.7) when  $\tau = 1$ .

Taylor expanding (3.6) in  $s$  we obtain

$$(3.9) \quad L^{s,z}(\tau, x) = L_0^z + sL_1^z(x) + s^2L_2^z(\tau, x) + s^3L_3^{s,z}(\tau, x)$$

where

(3.10)

$$\begin{aligned}
L_0^z &:= \frac{1}{2}\Sigma(0, z)^2\partial_x^2 \\
L_1^z(x) &:= \Sigma(0, z)\Sigma'(0, z)(x - z)\partial_x^2 + \mu(0, z)\partial_x \\
L_2^z(\tau, x) &:= \left( \Sigma(0, z)\dot{\Sigma}(0, z)\tau + (\Sigma'(0, z)^2 + \Sigma(0, z)\Sigma''(0, z))(x - z)^2 \right) \partial_x^2 \\
&\quad + \mu'(0, z)(x - z)\partial_x - \rho(0, z)
\end{aligned}$$

and  $L_3^{s,z}(\tau, x)$  is the remainder term which depends on  $s$  and  $z$ . Furthermore, it will be convenient in subsequent calculations to split the operator  $L_2^z(\tau, x)$  into a time dependent part  $\tau\mathbb{L}_2^z$  and time independent part  $\widetilde{L}_2^z(x)$ ,

$$(3.11) \quad L_2^z(\tau, x) = \tau\mathbb{L}_2^z + \widetilde{L}_2^z(x)$$

where

(3.12)

$$\begin{aligned}
\mathbb{L}_2^z &= \Sigma(0, z)\dot{\Sigma}(0, z)\partial_x^2 \\
\widetilde{L}_2^z(x) &= (\Sigma'(0, z)^2 + \Sigma(0, z)\Sigma''(0, z))(x - z)^2\partial_x^2 + \mu'(0, z)(x - z)\partial_x - \rho(0, z)
\end{aligned}$$

*Remark 3.2.* Note that  $L_0^z$  is a constant coefficient operator, for which the Green's function is exactly (3.22).

*Remark 3.3.* Note that each  $L_k^z$  in (3.10) has coefficients of order  $k$  in  $(x - z)$  and order  $k - 1$  in  $\tau$  (where we use the convention that the order is zero if  $k, k - 1 < 0$ ). Thus, in order for the expansion to capture the time dependence of the coefficients, the coefficient must be expanded at least to second order in  $s$ . This means that the correction occurs at order  $s^2 = \tau$  in the expansion of  $\mathcal{G}_\tau^L(x, y)$ .

Setting

$$V^{s,z}(\tau, x) = sL_1^z(x) + s^2L_2^z(\tau, x) + s^3L_3^{s,z}(\tau, x)$$

and using (3.5) with  $d = 2$  and  $\tau = 1$  yields

$$(3.13) \quad \mathcal{G}_1^{L^{s,z}} = e^{L_0^z} + s\mathcal{I}_1^z + s^2\mathcal{I}_{1,1}^z + s^2\mathcal{I}_2^z + \mathcal{R}^{s,z},$$

where

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^1 e^{(1-\tau)L_0^z} L_1^z e^{\tau L_0^z} d\tau \\
\mathcal{I}_{1,1} &= \int_0^1 \int_0^{\tau_1} e^{(1-\tau_1)L_0^z} L_1^z e^{(\tau_1-\tau_2)L_0^z} L_1^z e^{\tau_2 L_0^z} d\tau_2 d\tau_1 \\
\mathcal{I}_2 &= \int_0^1 e^{(1-\tau)L_0^z} L_2^z(\tau) e^{\tau L_0^z} d\tau
\end{aligned}
\tag{3.14}$$

and  $\mathcal{R}^{s,z}$  contains all the higher order terms and thus satisfies  $\lim_{s \rightarrow 0} \mathcal{R}^{s,z}/s = 0$ .

Now we need to calculate the above three integrals  $\mathcal{I}_1, \mathcal{I}_{1,1}, \mathcal{I}_2$ . Before we proceed to do the computation, we first give two definitions

**Definition 3.4.** *The commutator of two operators  $T_1$  and  $T_2$  is*

$$[T_1, T_2] := T_1 T_2 - T_2 T_1.$$

*We say two operators  $T_1, T_2$  commute if  $[T_1, T_2] = [T_2, T_1] = 0$ .*

**Definition 3.5.** *For any two operators  $T_1$  and  $T_2$ , we define  $ad_{T_1}(T_2)$  by*

$$ad_{T_1}(T_2) := [T_1, T_2]$$

*and for any integer  $j$ , we define  $ad_{T_1}^j(T_2)$  recursively by*

$$ad_{T_1}^j(T_2) := ad_{T_1}(ad_{T_1}^{j-1}(T_2))$$

Using the notation of Definitions 3.4 and 3.5 we have

$$\begin{aligned} \mathcal{I}_1 &= \int_0^1 e^{(1-\tau_1)L_0^z} L_1^z e^{\tau_1 L_0^z} d\tau_1 \\ &= \int_0^1 (L_1^z + (1-\tau_1)[L_0^z, L_1^z]) e^{L_0^z} d\tau_1 \\ &= (L_1^z + \frac{1}{2}[L_0^z, L_1^z]) e^{L_0^z} \\ \mathcal{I}_2 &= \int_0^1 e^{(1-\tau_1)L_0^z} L_2^z(\tau_1) e^{\tau_1 L_0^z} d\tau_1 \\ &= \int_0^1 (\tau_1 \mathbb{L}_2^z + \widetilde{L}_2^z + (1-\tau_1)[L_0^z, L_2^z] + \frac{(1-\tau_1)^2}{2}[L_0^z, [L_0^z, L_2^z]]) e^{L_0^z} d\tau_1 \\ &= \left( \frac{1}{2} \mathbb{L}_2^z + \widetilde{L}_2^z + \frac{1}{2}[L_0^z, L_2^z] + \frac{1}{6}[L_0^z, [L_0^z, L_2^z]] \right) e^{L_0^z} \\ \mathcal{I}_{1,1} &= \int_0^1 \int_0^{\tau_1} e^{(1-\tau_1)L_0^z} L_1^z(\tau_1) e^{(\tau_1-\tau_2)L_0^z} L_1^z(\tau_2) e^{\tau_2 L_0^z} d\tau_2 d\tau_1 \\ &= \int_0^1 \int_0^{\tau_1} (L_1^z + (1-\tau_1)[L_0^z, L_1^z])(L_1^z + (1-\tau_1)[L_0^z, L_1^z]) e^{L_0^z} d\tau_2 d\tau_1 \\ &= ((L_1^z)^2 + \frac{1}{2}L_1^z[L_0^z, L_1^z] + \frac{1}{6}[L_0^z, L_1^z]L_1^z + \frac{1}{8}[L_0^z, L_1^z]^2) e^{L_0^z} \end{aligned}$$

Hence (3.13) becomes

$$\begin{aligned} (3.15) \quad e^{L^{s,z}} &= e^{L_0^z} + s \left( L_1^z + \frac{1}{2}[L_0^z, L_1^z] \right) + s^2 \left( \frac{1}{2} \mathbb{L}_2^z + \widetilde{L}_2^z + \frac{1}{2}[L_0^z, L_2^z] + \frac{1}{6}[L_0^z, [L_0^z, L_2^z]] \right) e^{L_0^z} \\ &\quad + s^2 \left( (L_1^z)^2 + \frac{1}{2}L_1^z[L_0^z, L_1^z] + \frac{1}{6}[L_0^z, L_1^z]L_1^z + \frac{1}{8}[L_0^z, L_1^z]^2 \right) e^{L_0^z} + \mathcal{R}^{s,z} \end{aligned}$$

For any  $\theta \in (0, 1)$  we recall the identity [7]

$$(3.16) \quad e^{\theta L_0^z} L_m^z = P_m(\theta, x, z, \partial) e^{\theta L_0^z}$$

where

$$(3.17) \quad P_m(\theta, x, z, \partial) = L_m^z + \sum_{i=1}^m \frac{\theta^i}{i!} \text{ad}_{L_0^z}^i(L_m^z)$$

The usefulness of this identity is that it allows one to "move"  $e^{L_0^{s,z}}$  from the left side of the equation to the right side of the equation by commutating.

Therefore, we only need to compute some commutators in the above formula to get an approximation of  $\mathcal{G}_t^{L^{s,z}}$ . We now proceed to calculate the commutators.

$$(3.18) \quad \begin{aligned} [L_0^z, L_1^z] &= \Sigma(0, z)^3 \Sigma'(0, z) \partial_x^3 \\ [L_0^z, L_1^z]^2 &= \Sigma(0, z)^6 \Sigma'(0, z)^2 \partial_x^6 \\ L_1^z [L_0^z, L_1^z] &= \Sigma(0, z)^4 \Sigma'(0, z)^2 (x-z) \partial_x^5 + \mu(0, z) \Sigma(0, z)^3 \Sigma'(0, z) \partial_x^4 \\ [L_0^z, L_1^z] L_1^z &= \Sigma(0, z)^4 \Sigma'(0, z)^2 (x-z) \partial_x^5 + (\mu(0, z) + 3\Sigma(0, z) \Sigma'(0, z)) \Sigma(0, z)^3 \Sigma'(0, z) \partial_x^4 \\ [L_0^z, L_2^z] &= \Sigma(0, z)^2 (\Sigma'(0, z)^2 + \Sigma(0, z) \Sigma''(0, z)) (x-z) \partial_x^3 \\ &\quad + \Sigma(0, z)^2 \left( \mu'(0, z) + \frac{1}{2} (\Sigma'(0, z)^2 + \Sigma(0, z) \Sigma''(0, z)) \right) \partial_x^2 \\ [L_0^z, [L_0^z, L_2^z]] &= \Sigma(0, z)^4 (\Sigma'(0, z)^2 + \Sigma(0, z) \Sigma''(0, z)) \partial_x^4 \\ L_1^{z,2} &= (\Sigma(0, z) \Sigma'(0, z) (x-z))^2 \partial_x^4 + 2(\Sigma(0, z)^2 \Sigma'(0, z)^2 + \Sigma(0, z) \Sigma'(0, z) \mu(0, z)) \\ &\quad \cdot (x-z) \partial_x^3 + (\Sigma(0, z) \Sigma'(0, z) \mu(0, z) + \mu(0, z)^2) \partial_x^2 \end{aligned}$$

By Remark 3.2,

$$(3.19) \quad e^{L_0^z} = \frac{1}{\sqrt{2\pi\Sigma(0, z)^2}} \exp\left(-\frac{|x-y|^2}{2\Sigma(0, z)^2}\right).$$

Now let

$$(3.20) \quad \Theta = \frac{x-y}{\Sigma(0, z)^2}.$$

To aid us in the computations we need the following derivatives of (3.19)

$$\begin{aligned}
(3.21) \quad & \partial_x e^{L\tilde{z}} = -\Theta e^{L\tilde{z}} \\
& \partial_x^2 e^{L\tilde{z}} = \left( \Theta^2 - \frac{1}{\Sigma(0, z)^2} \right) e^{L\tilde{z}} \\
& \partial_x^3 e^{L\tilde{z}} = - \left( \Theta^2 - \frac{3}{\Sigma(0, z)^2} \right) \Theta e^{L\tilde{z}} \\
& \partial_x^4 e^{L\tilde{z}} = \left( \Theta^4 - \frac{3}{\Sigma(z)^2} \left( 2\Theta^2 - \frac{1}{\Sigma(0, z)^2} \right) \right) e^{L\tilde{z}} \\
& \partial_x^5 e^{L\tilde{z}} = - \left( \Theta^5 - \frac{10}{\Sigma(0, z)^2} \Theta^3 + 15\Theta \right) e^{L\tilde{z}} \\
& \partial_x^6 e^{L\tilde{z}} = \left( \Theta^6 - \frac{15}{\Sigma(0, z)^2} \Theta^4 + \frac{45}{\Sigma(0, z)^4} \Theta^2 - \frac{15}{\Sigma(0, z)^6} \right) e^{L\tilde{z}}
\end{aligned}$$

Using (3.21) we have <sup>2</sup>

$$(3.22) \quad G^{[0]}(x, y; z) = e^{L\tilde{z}} = \frac{1}{\sqrt{2\pi\Sigma(0, z)^2}} \exp\left(-\frac{|x-y|^2}{2\Sigma(0, z)^2}\right)$$

$$\begin{aligned}
(3.23) \quad & G^{[1]}(x, y; z) = (L_1 + \frac{1}{2}[L_0, L_1])e^{L_0} \\
& = \left( \Sigma(0, z)\Sigma'(0, z)(x-z)\partial_x^2 + \mu(0, z)\partial_x + \frac{1}{2}\Sigma(0, z)^3\Sigma'(0, z)\partial_x^3 \right) e^{L\tilde{z}} \\
& = \frac{1}{\sqrt{2\pi\Sigma(0, z)^2}} e^{-\frac{|x-y|^2}{2\Sigma(0, z)^2}} \left[ \left( \frac{3\Sigma(0, z)\Sigma'(0, z) - 2\mu(0, z)}{2\Sigma(0, z)^2} \right) (x-y) - \frac{\Sigma'(0, z)}{2\Sigma(0, z)^3} (x-y)^3 \right. \\
& \quad \left. + (x-z) \left( \frac{(x-y)^2 - \Sigma(0, z)^2}{\Sigma(0, z)^3} \right) \right]
\end{aligned}$$

---

<sup>2</sup>All the formulas henceforth are computed with the aid of the mathematical software Maple, Waterloo Maple Inc.

(3.24)

$$\begin{aligned}
G^{[2]}(x, y; z) &= \left( \frac{1}{2} \mathbb{L}_2^z + \widetilde{L}_2^z + \frac{1}{2} [L_0^z, L_2^z] + \frac{1}{6} [L_0^z, [L_0^z, L_2^z]] \right. \\
&\quad \left. + L_1^{z,2} + \frac{1}{2} L_1^z [L_0^z, L_1^z] + \frac{1}{6} [L_0^z, L_1^z] L_1^z + \frac{1}{8} [L_0^z, L_1^z]^2 \right) e^{L_0^z} \\
&= \left\{ \left( \frac{(x-y)^2}{\Sigma^3(0, z)} - \frac{1}{\Sigma(0, z)} \right) \dot{\Sigma}(0, z) + \frac{1}{2} \Sigma(0, z) \Sigma''(0, z) \right. \\
&\quad - \left( \frac{1}{4} \Sigma'(0, z)^2 + \mu'(z) + 2\rho'(0, z) \right) + (-1.5 \Sigma''(0, z)(x-y)^2 \\
&\quad + 3(x-z)(x-y) \Sigma''(0, z) + 3\mu(0, z) \Sigma'(0, z) - (x-z)^2 \Sigma''(0, z)) \frac{1}{\Sigma(0, z)} \\
&\quad + \left( \frac{15}{4} \Sigma'(0, z)^2 + \mu'(0, z) \right) (x-y)^2 - (11(x-z) \Sigma'(0, z)^2 \\
&\quad + 2(x-z) \mu'(0, z)(x-z))(x-y) - \mu(0, z)^2 + 2(x-z)^2 \Sigma'(0, z)^2 \left. \right\} \frac{1}{\Sigma(0, z)^2} \\
&\quad + (\Sigma''(0, z)(x-y)^3/3 - (x-z) \Sigma''(0, z)(x-y)^2 + ((x-z)^2 \Sigma''(0, z) \\
&\quad - 7\mu(z) \Sigma'(0, z))(x-y) + 6 \Sigma'(0, z)(x-z) \mu(z) \left. \right\} \frac{x-y}{\Sigma(0, z)^3} \\
&\quad - (29 \Sigma'(0, z)^2 (x-y)^2/12 + 5(x-z)^2 \Sigma'(z)^2 - 31(x-z)(x-y) \Sigma'(0, z)^2/3 \\
&\quad - \mu(0, z)^2) \frac{(x-y)^2}{\Sigma(z)^4} + \frac{2(x-y)^3((x-z) - 2(x-y)/3) \Sigma'(0, z) \mu(0, z)}{\Sigma(0, z)^5} \\
&\quad + \left. \frac{(x-y)^4 \Sigma'(0, z)^2 [(x-z)^2 - 4(x-z)(x-y)/3 + (x-y)^2/4]}{\Sigma(0, z)^6} \right\} \frac{1}{2} e^{L_0}
\end{aligned}$$

**Example 3.6.** For the time-dependent Black-Scholes-Merton model (2.4) we have,

(3.25)

$$\begin{aligned}
G_{BSM}^{[1]}(x, y; z) &= \frac{1}{\sigma_0 z \sqrt{2\pi}} e^{-\frac{|x-y|^2}{2\sigma_0^2 z^2}} \left[ \left( \frac{3\sigma_0^2 - 2r}{2\sigma_0} \right) \left( \frac{x-y}{\sigma_0 z} \right) - \frac{\sigma_0}{2} \left( \frac{x-y}{\sigma_0 z} \right)^3 \right. \\
&\quad \left. + \left( \frac{x-z}{\sigma_0 z} \right) \left( \left( \frac{x-y}{\sigma_0 z} \right)^2 - 1 \right) \right]
\end{aligned}$$

while for CEV model (2.5) we have

(3.26)

$$G_{CEV}^{[1]}(x, y; z) = \frac{1}{\sigma_0 z^\alpha \sqrt{2\pi}} e^{-\frac{|x-y|^2}{2\sigma_0^2 z^{2\alpha}}} \left[ \left( \frac{3\alpha\sigma_0^2 z^{2(\alpha-1)} - 2r}{2\sigma_0} \right) \left( \frac{x-y}{\sigma_0 z^\alpha} \right) - \frac{\alpha\sigma_0 z^{\alpha-1}}{2} \left( \frac{x-y}{\sigma_0 z^\alpha} \right)^3 + \left( \frac{x-z}{\sigma_0 z^\alpha} \right) \left( \left( \frac{x-y}{\sigma_0 z^\alpha} \right)^2 - 1 \right) \right]$$

where we have set  $\sigma_0 = \sigma(0)$ .

In Section 4 below we compute the expansion for different basepoints  $z$  and compare them in Section 6.

#### 4. KERNEL EXPANSIONS AT DIFFERENT BASEPOINTS $z = z(x, y)$

The goal of this section is to understand the behavior of certain properties of the expansion on the particular choice of basepoint  $z$ . For example, the choice  $z = x$  yields a much simpler expression since certain terms disappear. On the other hand, it seems natural to conjecture that the approximation should perform better if  $z$  is the midpoint in the geodesic distance.

**4.1. Standard Approximation.** We first choose  $z(x, y) = x$  as the basepoint of the expansion for the kernel  $\mathcal{G}_t(x, y)$ . An advantage is that in this case many terms in our above calculation will disappear and we can even get a closed form solution for the option price itself since in this case the integrals can be evaluated in closed form as well, so we do not need to do numerical quadrature which is computationally more intensive. Precisely, by setting  $z = x$  in (3.23) and using (3.8) the first order Green function for the Standard Approximation  $z(x, y) = x$  reads

(4.1)

$$G^{[1]}(x, y) = \frac{1}{\sqrt{2\pi\Sigma(0, x)^2}} e^{-\frac{(x-y)^2}{2\Sigma(0, x)^2}} \left( 1 + s \left( \frac{3\Sigma(0, x)\Sigma'(0, x) - 2\mu(x)}{2\Sigma(0, x)^2} (x-y) - \frac{\Sigma'(0, x)}{2\Sigma(0, x)^3} (x-y)^3 \right) \right)$$

And similarly, the second order approximated Green function is given by

(4.2)

$$G^{[2]}(x, y) = \frac{1}{8\Sigma(0, x)^6} \cdot \frac{1}{\sqrt{2\pi\Sigma(0, x)^2}} e^{-\frac{(x-y)^2}{2\Sigma(0, x)^2}} \cdot \left\{ \Sigma'(0, x)^2 (x-y)^6 + 4((x-y)^2 - \Sigma^2(0, x)) \Sigma^3(0, x) \dot{\Sigma}(0, x) + [4/3\Sigma''(0, x)\Sigma(0, x)^3 - 29/3\Sigma(0, x)^2\Sigma'(0, x)^2 + 16/3\Sigma(0, x)\Sigma'(0, x)\mu(0, x)](x-y)^4 + [-6\Sigma''(0, x)\Sigma(0, x)^5 + (15\Sigma'(0, x)^2 + 4\mu'(0, x))\Sigma(0, x)^4 - 28\mu(0, x)\Sigma'(0, x)\Sigma(0, x)^3 + 4\mu(0, x)^2\Sigma(0, x)^2](x-y)^2 + 2\Sigma(0, x)^7\Sigma''(0, x) + (-8\mu'(0, x) - \Sigma'(0, x)^2 - 4\mu'(0, x))\Sigma(0, x)^6 + 12\Sigma(0, x)^5\mu(0, x)\Sigma'(0, x) - 4\Sigma(0, x)^4\mu(0, x)^2 \right\}$$

**Example 4.1.** For the time-dependent Black-Scholes-Merton equation, we have  $\mu(\tau, x) = rx$  and  $\Sigma(\tau, x) = \sigma(\tau)x$  with  $\sigma(0) = \sigma_0$  so that

$$(4.3) \quad G_{BSM}^{[1]}(x, y) = \frac{1}{\sqrt{2\pi}\sigma_0 x} e^{-\frac{1}{2}\left(\frac{x-y}{\sigma_0 x}\right)^2} \left[ 1 + s \left( \frac{3\sigma_0^2 - 2r}{2\sigma_0} \left( \frac{x-y}{\sigma_0 x} \right) - \frac{\sigma}{2} \left( \frac{x-y}{\sigma_0 x} \right)^3 \right) \right]$$

We also record the second order approximation here for later reference,

$$(4.4) \quad G_{BSM}^{[2]}(x, y) = \frac{1}{\sqrt{2\pi}\sigma_0 x} e^{-\frac{1}{2}\left(\frac{x-y}{\sigma_0 x}\right)^2} \cdot \left[ -\frac{\dot{\sigma}(0)}{2\sigma_0} - \frac{(12r + \sigma_0^2)\sigma_0^2 + 4r^2 - 12\sigma_0^2 r}{8\sigma_0^2} \right. \\ \left. + \frac{(4\dot{\sigma}(0)\sigma_0 + 15\sigma_0^2 + 4r)\sigma_0^2 - 28r\sigma_0^2 + 4r^2}{8\sigma_0^2} \left( \frac{x-y}{\sigma_0 x} \right)^2 + \frac{16r - 29\sigma_0^2}{24} \left( \frac{x-y}{\sigma_0 x} \right)^4 + \frac{\sigma_0^2}{8} \left( \frac{x-y}{\sigma_0 x} \right)^6 \right]$$

Similarly for the time-dependent CEV model,  $\mu(\tau, x) = rx$  and  $\Sigma(\tau, x) = \sigma(\tau)x^\alpha$  with  $\sigma(0) = \sigma_0$  so that

$$(4.5) \quad G_{CEV}^{[1]}(x, y) = \frac{1}{\sqrt{2\pi}\sigma_0 x^\alpha} e^{-\frac{1}{2}\left(\frac{x-y}{\sigma_0 x^\alpha}\right)^2} \left[ 1 + s \left( \frac{3\alpha\sigma_0^2 x^{\alpha-1} - 2rx^{1-\alpha}}{2\sigma_0} \right) \left( \frac{x-y}{\sigma_0 x^\alpha} \right) - \frac{\alpha\sigma_0 x^{\alpha-1}}{2} \left( \frac{x-y}{\sigma_0 x^\alpha} \right)^3 \right]$$

$$(4.6) \quad G_{CEV}^{[2]}(x, y) = \frac{1}{8\sqrt{2\pi}\sigma_0^5 X^7} \cdot e^{-\frac{(x-y)^2}{2\sigma_0^2 X^2}} \cdot \left\{ \alpha^2 X^2 / x^2 \cdot \left( \sigma_0^6 X^6 + 9\sigma_0^4 (y-x)^2 X^4 - \frac{25}{3}\sigma_0^2 (y-x)^4 X^2 + (y-x)^6 \right) \right. \\ \left. + \frac{16\alpha\sigma_0 r X^2}{3} \cdot (1.5\sigma_0^4 X^4 - 4.5\sigma_0^2 (y-x)^2 X^2 + (y-x)^4) \right. \\ \left. + \frac{8X^2}{3} \cdot [(\alpha-1)\sigma_0^2 \alpha X^2 / x^2 (1.5\sigma_0^4 X^4 - 4.5\sigma_0^2 (y-x)^2 X^2 + (y-x)^4) - 4.5r\sigma_0^4 X^4 - 1.5X^3 \sigma_0^3 \dot{\sigma}(0) \right. \\ \left. + 1.5\sigma_0^2 r X^2 (x^2 - 2rx^2 - 2xy + y^2) + 1.5\dot{\sigma}(0)\sigma_0 (y-x)^2 X + 3r^2 x^2 (y-x)^2 \right\}$$

where  $X = x^\alpha$ .

Notice that  $G_{CEV}^{[j]}$  reduces to  $G_{BSM}^{[j]}$  when  $\alpha = 1$ .

**4.2. Geodesic Midpoint Approximation.** In this section we compute the expansion when the basepoint  $z$  is the geodesic midpoint between the points  $x, y \in [0, \infty)$ .

4.2.1. *Geodesic Distance and Midpoint.* We now introduce some geometry into the problem to better understand the properties of the expansion. Consider the operator  $L$  on  $\mathbb{R}_+ = [0, \infty)$ . To  $L$  we may associate a metric  $g$  through the coefficient  $\Sigma(\tau, x)$  by

$$(4.7) \quad g_\Sigma = \frac{1}{\Sigma(\tau, x)}.$$

The geodesic distance  $d_\Sigma(x, y)$  induced by the metric  $g$  of  $L$  is given by

$$(4.8) \quad d_\Sigma(x, y) := \inf_{\gamma(0)=x, \gamma(1)=y} \int_0^1 \sqrt{g(\gamma(\theta))} \dot{\gamma}^2(\theta) d\theta = \left| \int_x^y \frac{1}{\Sigma(\tau, z)} dz \right|.$$

In particular for the time-dependent CEV model  $\Sigma(t, z) = \sigma(t)z^\alpha$  so we have

$$(4.9) \quad d_\Sigma(x, y) = \begin{cases} \frac{1}{\sigma(\tau)} \left| \log \left( \frac{y}{x} \right) \right|, & \alpha = 1 \\ \frac{1}{\sigma(\tau)(\alpha-1)} \frac{|y^{\alpha-1} - x^{\alpha-1}|}{|xy|^{\alpha-1}}, & \alpha \neq 1 \end{cases}$$

**Definition 4.2** (Geodesic Midpoint). *We define the midpoint between two points in the distance  $d_\Sigma(x, y)$  (4.8) by the point  $\delta = \delta_\Sigma$  satisfying*

$$(4.10) \quad d_\Sigma(x, \delta) = d_\Sigma(y, \delta).$$

*Note that (4.10) uniquely defines  $\delta_\Sigma$  if we assume all the points  $x, \delta_\Sigma, y \in \mathbb{R}$  lie on a line.*

*Remark 4.3.* For the time-dependent CEV model we have

$$(4.11) \quad \delta_\Sigma = \begin{cases} \sqrt{xy} & \alpha = 1 \\ \left( \frac{x^{\alpha-1} + y^{\alpha-1}}{2} \right)^{\frac{1}{\alpha-1}} & \alpha \neq 1 \end{cases}$$

We now compute the expansion using  $z = \delta_\Sigma$ .

**Example 4.4.** *For the time-dependent Black Scholes equation,  $\Sigma(\tau, z) = \sigma(\tau)z$  and  $\delta_\Sigma = \sqrt{xy}$  so that  $\Sigma(\tau, z) = \sigma(\tau)\sqrt{xy}$  and (3.23) becomes*

$$(4.12) \quad G_{BSM}^{[1]}(x, y, z = \sqrt{xy}) = \frac{1}{\sigma_0 \sqrt{2\pi xy}} e^{-\frac{1}{2} \left( \frac{x-y}{\sigma_0 \sqrt{xy}} \right)^2} \left[ \left( \frac{3\sigma_0^2 - 2r}{2\sigma_0} \right) \left( \frac{x-y}{\sigma_0 \sqrt{xy}} \right) - \frac{\sigma_0}{2} \left( \frac{x-y}{\sigma_0 \sqrt{xy}} \right)^3 + \frac{1}{\sigma_0} \left( \frac{x}{\sqrt{xy}} - 1 \right) \left( \left( \frac{x-y}{\sigma_0 \sqrt{xy}} \right)^2 - 1 \right) \right]$$

*For the time-dependent CEV model,  $\Sigma(t, z) = \sigma(t)z^\alpha$  and  $\delta_\Sigma = \left( \frac{x^{\alpha-1} + y^{\alpha-1}}{2} \right)^{\frac{1}{\alpha-1}}$  so that  $\Sigma(t, z) = \sigma \left( \frac{x^{\alpha-1} + y^{\alpha-1}}{2} \right)^{\frac{\alpha}{\alpha-1}}$  and (3.23) becomes*

(4.13)

$$G_{CEV}^{[1]}(x, y; z = Z^{\frac{1}{\alpha-1}}) = \frac{1}{\sigma_0 \sqrt{2\pi} Z^{\frac{\alpha}{\alpha-1}}} e^{-\frac{|x-y|^2}{2\sigma^2 Z^{\frac{2\alpha}{\alpha-1}}}} \left[ \left( \frac{3\alpha\sigma_0^2 Z^2 - 2r}{2\sigma_0} \right) \left( \frac{x-y}{\sigma_0 Z^{\frac{\alpha}{\alpha-1}}} \right) + \frac{\alpha\sigma_0 Z}{2} \left( \frac{x-y}{\sigma_0 Z^{\frac{\alpha}{\alpha-1}}} \right)^3 + \left( \frac{x}{\sigma Z^{\frac{\alpha}{\alpha-1}}} - \frac{1}{\sigma Z} \right) \left( \left( \frac{x-y}{\sigma_0 Z^{\frac{\alpha}{\alpha-1}}} \right)^2 - 1 \right) \right]$$

where

$$(4.14) \quad Z := \frac{x^{\alpha-1} + y^{\alpha-1}}{2}.$$

*Remark 4.5.* It would be interesting to compare our geodesic midpoint approximations (4.12) and (4.13) with asymptotics in [30][Proposition 4.2].

**4.3. Midpoint Approximation.** While geodesics distances on one dimensional manifolds can be computed explicitly, this is not generically the case for manifolds of higher dimension. Thus the analytical results obtained in Section 4.2 can not in general be extended to multi-factor stochastic volatility models. Hence, here we propose another alternative to both the standard and geodesic midpoint approximation, namely the standard midpoint approximations which is obtained by setting

$$(4.15) \quad z = \frac{x+y}{2}.$$

**Example 4.6.** For the time-dependent Black-Scholes-Merton model we have

(4.16)

$$G_{BSM}^{[1]}(x, y; z = \frac{x+y}{2}) = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{|x-y|^2}{\sigma^2(x+y)^2}}}{\sigma(x+y)} \left[ \frac{3\sigma_0^2 - 2r - \sigma_0}{\sigma_0^2} \left( \frac{x-y}{x+y} \right) + 4 \frac{1 - \sigma_0}{\sigma_0^3} \left( \frac{x-y}{x+y} \right)^3 \right]$$

while for the time-dependent CEV model we have

(4.17)

$$G_{CEV}^{[1]}(x, y; z = \frac{x+y}{2}) = \frac{2^\alpha}{\sigma_0(x+y)^\alpha \sqrt{2\pi}} e^{-\frac{|x-y|^2}{2^{1-2\alpha}\sigma_0^2(x+y)^{2\alpha}}}. \left[ \frac{3\alpha\sigma_0^2 2^{2-2\alpha}(x+y)^{2(\alpha-1)} - 2r - \sigma_0}{2^{1-\alpha}\sigma_0^2} \cdot \frac{x-y}{(x+y)^\alpha} + \frac{1 - \sigma_0 \alpha 2^{1-\alpha}(x+y)^{\alpha-1}}{2^{1-3\alpha}\sigma_0^3} \cdot \left( \frac{x-y}{(x+y)^\alpha} \right)^3 \right]$$

**4.4. Relationship to Taylor expansion of the Kernel.** The asymptotic expansion described above is useful in computing a closed form approximation of the pricing kernel and option price when the pricing kernel for the model is not known in closed form. However, if there indeed is a closed form solution for the pricing kernel (as for the Black-Scholes-Merton model)

one can certainly Taylor expand the pricing kernel directly. The question then arises as to the relationship between the expansion obtained by Taylor expanding the known pricing kernel and the expansion obtained by asymptotic expansions of this paper. In particular, if they are the same. In this section we will show that for the particular case of the Black-Scholes-Merton model, the Taylor expansion of the known pricing kernel and our asymptotic expansions are the same. Indeed, we believe this is true in general, but will leave the verification of this question for future work.

The pricing kernel for the Black-Scholes-Merton model (2.2) is given explicitly by

$$(4.18) \quad \mathcal{G}_\tau^L(x, y) = \frac{\exp(-r\tau)}{y\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{|\ln(x/y) + \gamma\tau|^2}{2\sigma^2\tau}\right)$$

where again  $\tau$  is given by (2.17) and

$$(4.19) \quad \gamma = r - \sigma^2/2$$

Note this is the same as the Green function for the Black-Scholes operator (1.1).

Using (3.8), we have

$$(4.20) \quad \begin{aligned} \mathcal{G}^{L^{s,z}}(x, y) &= s\mathcal{G}_\tau^L(x, x + s(y - x)) \\ &= \frac{(1 - s\Delta)^{-1} \exp(-rs^2)}{\sigma x \sqrt{2\pi}} \exp\left(-\frac{|\ln((1 - s\Delta)^{-1}) + \gamma s^2|}{2\sigma^2 s^2}\right) \end{aligned}$$

where as usual  $s = \sqrt{\tau}$  and

$$(4.21) \quad \Delta = \frac{(x - y)}{x}$$

We anticipate that when we expand  $\mathcal{G}_1^{L^{s,z}}(x, y)$  with respect to  $s$ , we should be able to recover  $\mathcal{G}_{BSM}^{[1]}(x, y)$  in (4.3) and  $\mathcal{G}_{BSM}^{[2]}(x, y)$  in (4.4). To prepare for the calculations we rewrite (4.20) as

$$(4.22) \quad \mathcal{G}^{L^{s,z}}(x, y) = A(s, x, y)B(s, x, y)$$

with

$$(4.23) \quad \begin{aligned} A(s, x, y) &= \frac{(1 - s\Delta)^{-1} \exp(-rs^2)}{\sigma x \sqrt{2\pi}} \\ B(s, x, y) &= \exp\left(-\frac{|\ln((1 - s\Delta)^{-1}) + \gamma s^2|}{2\sigma^2 s^2}\right) \end{aligned}$$

Then a straightforward calculation yields

$$\begin{aligned}
(4.24) \quad A(0, x, y) &= \frac{1}{\sigma x \sqrt{2\pi}} \\
\frac{dA}{ds}(0, x, y) &= \frac{\Delta}{x \sigma \sqrt{2\pi}} \\
B(0, x, y) &= e^{-\frac{\Delta^2}{2\sigma}} \\
\frac{dB}{ds}(0, x, y) &= -e^{-\frac{\Delta^2}{2\sigma}} \left( \frac{\gamma \Delta}{\sigma^2} + \frac{\Delta^3}{\sigma^2} \right)
\end{aligned}$$

Thus to second order in  $s$  we have

$$\begin{aligned}
(4.25) \quad \mathcal{G}^{L^{s,z}}(x, y) &= \mathcal{G}^{L^{s,z}}(x, y)|_{s=0} + s \left( \frac{d}{ds} \mathcal{G}^{L^{s,z}}(x, y) \right) \Big|_{s=0} + \mathcal{O}(s^2) \\
&= A(0, x, y)B(0, x, y) + s \left( \frac{dA}{ds}(0, x, y)B(0, x, y) + A(0, x, y) \frac{dB}{ds}(0, x, y) \right) + \mathcal{O}(s^2) \\
&= \frac{e^{-\frac{\Delta^2}{2\sigma}}}{\sigma x \sqrt{2\pi}} \left[ 1 + s \left( \Delta - \frac{\gamma \Delta}{\sigma^2} + \frac{\Delta^3}{\sigma^2} \right) \right] \\
&= \frac{e^{-\frac{\Delta^2}{2\sigma}}}{\sigma x \sqrt{2\pi}} \left[ 1 + s \left( \left( \frac{3\sigma^2 - 2r}{2\sigma^2} \right) \Delta - \frac{\Delta^3}{\sigma^2} \right) \right]
\end{aligned}$$

Recalling the definition of  $\Delta$  (4.21), this is exactly the same as the Standard Approximation (4.3). A similar calculation shows the equivalence of the second order terms.

## 5. CLOSED FORM APPROXIMATE SOLUTIONS

In this section, we shall consider European call options. For European put options similar results can also be obtained, either directly from the definition or by using put-call parity. In what follows, we will work with the expansion obtained by setting  $z(x, y) = x$  as the basepoint. The main reason for this is that many terms in our calculation will disappear and we can even get a closed form solution, i.e. we do not need to do numerical quadrature which is computationally more intensive. Precisely, by (4.1) and (3.8) the first order Green function reads

$$(5.1) \quad \mathcal{G}_\tau^{[1]}(x, y) = \frac{1}{\sqrt{2\pi\tau}\Sigma(0, x)} e^{-\frac{(x-y)^2}{2\Sigma(0, x)^2\tau}} \left( 1 + \frac{3\Sigma(0, x)\Sigma'(0, x) - 2\mu(0, x)}{2\Sigma(0, x)^2} (x-y) - \frac{\Sigma'(0, x)}{2\Sigma(0, x)^3\tau} (x-y)^3 \right)$$

And similarly, the second order approximated Green function is given by

$$\begin{aligned}
\mathcal{G}_\tau^{[2]}(x, y) = & \mathcal{G}_\tau^{[1]}(x, y) + \frac{\tau}{8\Sigma(0, x)^6} \cdot \frac{1}{\sqrt{2\pi\Sigma(0, x)^2\tau}} e^{-\frac{(x-y)^2}{2\Sigma(0, x)^2\tau}} \cdot \left\{ \Sigma'(0, x)^2 \frac{(x-y)^6}{\tau^3} \right. \\
& + [4/3\Sigma''(0, x)\Sigma(0, x)^3 - 29/3\Sigma(0, x)^2\Sigma'(0, x)^2 + 16/3\Sigma(0, x)\Sigma'(0, x)\mu(0, x)] \frac{(x-y)^4}{\tau^2} \\
& + [-6\Sigma''(0, x)\Sigma(0, x)^5 + (15\Sigma'(0, x)^2 + 4\mu'(0, x))\Sigma(0, x)^4 - 28\mu(0, x)\Sigma'(0, x)\Sigma(0, x)^3 \\
& + 4\mu(0, x)^2\Sigma(0, x)^2] \frac{(x-y)^2}{\tau} + 2\Sigma(0, x)^7\Sigma''(0, x) + (-8r - \Sigma'(0, x)^2 - 4\mu'(0, x))\Sigma(0, x)^6 \\
& \left. + 12\Sigma(0, x)^5\mu(0, x)\Sigma'(0, x) - 4\Sigma(0, x)^4\mu(0, x)^2 + 4 \left( \frac{(x-y)^2}{\tau} - \Sigma(0, x)^2 \right) \Sigma(0, x)^3 \dot{\Sigma}(0, x) \right\}
\end{aligned}$$

Then for European type options with strike price  $K$ , by (2.8) we know that the  $i^{th}$  order approximated option price is

$$(5.2) \quad U^{[i]}(\tau, x) = \int_0^\infty \mathcal{G}_\tau^{[i]}(x, y)(y - K)^+ dy, i = 1, 2$$

After examine the form of the heat kernels  $\mathcal{G}_\tau^{[i]}(x, y)$ , we find that after a change of variable the (5.2) is just an integral of polynomials against Gaussians. Therefore, we are able to obtain the closed form solutions involving error functions. Explicitly,

$$\begin{aligned}
U^{[1]}(\tau, x) &= \frac{\sqrt{\tau}}{2\sqrt{2\pi}} e^{-\frac{(x-K)^2}{2\Sigma(0, x)^2\tau}} (2\Sigma(0, x) - \Sigma'(0, x)(x - K)) \\
&\quad + \frac{1}{2} \cdot \left( \operatorname{erf} \left( \frac{x - K}{\sqrt{2\tau\Sigma(0, x)}} \right) + 1 \right) (\mu(0, x)\tau + x - K) \\
U^{[2]}(\tau, x) &= U^{[1]}(\tau, x) + \frac{\tau r}{2} \left( \operatorname{erf} \left( \frac{x - K}{\sqrt{2\tau\Sigma(0, x)}} \right) + 1 \right) (x - K) + \frac{1}{3\sqrt{2\pi\tau\Sigma^3(0, x)}} \cdot e^{-\frac{(x-K)^2}{2\Sigma^2(0, x)\tau}} \cdot \{ \\
&\quad 0.5\Sigma^5(0, x)\tau^2\Sigma''(0, x) + 1.5\tau^2\Sigma^4(0, x)(-2r + \frac{1}{12}\Sigma'(0, x)^2 + \mu'(0, x)) \\
&\quad + \tau\Sigma^3(0, x)((1.5\dot{\Sigma}(0, x) + \Sigma'(0, x)\mu(0, x))\tau + \Sigma''(0, x)(x - K)^2) \\
&\quad + 0.25\tau\Sigma^2(0, x)(12\tau\mu^2(0, x) + \Sigma'(0, x)^2(x - K)^2) + 2\tau\mu(0, x)\Sigma'(0, x)\Sigma(0, x)(x - K)^2 \\
&\quad \left. + \frac{3}{8}\Sigma'(0, x)^2(x - K)^4 \right\}
\end{aligned}$$

**Example 5.1.** For the Black-Scholes-Merton model we have

$$\begin{aligned}
(5.3) \quad U_{BSM}^{[1]}(\tau, x) &= \frac{\sigma\sqrt{\tau}}{2\sqrt{2\pi}} e^{-\frac{(x-K)^2}{2\sigma^2\alpha\tau}} (x + K) \\
&\quad + \frac{1}{2} \cdot \left( \operatorname{erf} \left( \frac{x - K}{\sqrt{2\tau\sigma x}} \right) + 1 \right) ((1 + r\tau)x - K)
\end{aligned}$$

whereas for the CEV model we have

| $\tau \backslash x$ | 12     | 13     | 14     | 15     | 16     | 17     | 18     |
|---------------------|--------|--------|--------|--------|--------|--------|--------|
| 0.01                | 0.0000 | 0.0000 | 0.0313 | 0.3266 | 0.0387 | 0.0019 | 0.0000 |
| 0.05                | 0.0461 | 0.3385 | 0.0179 | 0.3915 | 0.0179 | 0.4068 | 0.3957 |
| 0.1                 | 0.7    | 0.7    | 0.2    | 0.5    | 0.2    | 0.4    | 1.2    |
| 0.2                 | 2.2    | 0.3    | 0.7    | 0.9    | 0.7    | 0.3    | 1.3    |
| 0.5                 | 1.2    | 2.1    | 2.5    | 2.7    | 2.7    | 2.9    | 1.9    |

TABLE 1. Error of the first order approximation,  $K = 15, \sigma = 0.3, r = 0$ , error scale= $10^{-3}$

| $\tau \backslash x$ | 12     | 13     | 14     | 15     | 16     | 17     | 18     |
|---------------------|--------|--------|--------|--------|--------|--------|--------|
| 0.01                | 0.0000 | 0.0000 | 0.1000 | 0.0000 | 0.9000 | 2.0000 | 3.0000 |
| 0.05                | 0.1    | 0.9    | 1.4    | 0.1    | 3.6    | 8.7    | 14.5   |
| 0.1                 | 1.7    | 3.8    | 3.3    | 0.3    | 7.0    | 15.9   | 26.4   |
| 0.2                 | 9.3    | 10.7   | 7.1    | 1.2    | 14.0   | 30.2   | 48.8   |
| 0.5                 | 39.0   | 31.4   | 15.1   | 8.4    | 39.4   | 76.0   | 116.8  |

TABLE 2. Error of the first order approximation,  $K = 15, \sigma = 0.3, r = 0.1$ , error scale= $10^{-3}$

$$\begin{aligned}
(5.4) \quad U_{CEV}^{[1]}(\tau, x) &= \frac{\sigma x^{\alpha-1} \sqrt{\tau}}{2\sqrt{2\pi}} e^{-\frac{(x-K)^2}{2\sigma^2\alpha\tau}} ((2-\alpha)x - \alpha K) \\
&\quad + \frac{1}{2} \cdot \left( \operatorname{erf} \left( \frac{x-K}{\sqrt{2\tau\sigma x^\alpha}} \right) + 1 \right) ((1+r\tau)x - K)
\end{aligned}$$

## 6. COMPARISON AND PERFORMANCE OF THE METHOD

**6.1. Performance of the method.** Even if our approximated heat kernel is a short time expansion, numerical results show that the closed form solutions we derived above are accurate for relatively large time  $\tau$ . First we take Black Schole's model for example, and consider the case when the base-point  $z = x$  for simplicity. We choose the parameters  $K = 15, \sigma = 0.3$  and  $r = 0.1$ . We compare our formula with Black Schole's formula for different times. figure 1 and figure 2 give two different cases, they show that when  $\tau$  is small, the two solutions are undistinguishable in matlab. And even when  $\tau$  is large, the error is small. Table 1 and table 2 give a through comparison of the first order approximation with the exact Black Schole formula.

*Remark 6.1.* In practice, we can always assume that  $\tau$  is small by the change of variable (2.17). So especially near expiry our approximation is pretty good.

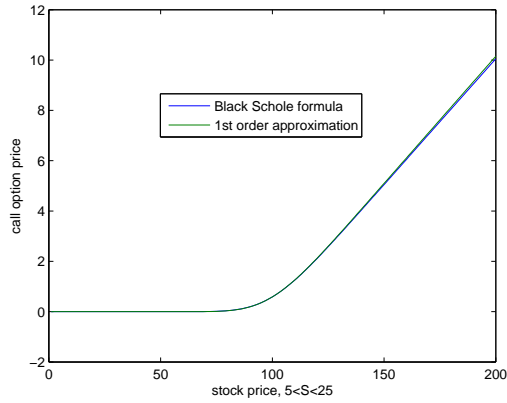


FIGURE 1.  $\tau = 0.1, K = 15, \sigma = 0.3, r = 0.1$

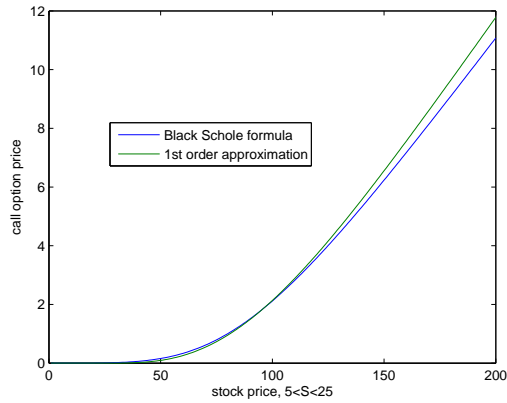


FIGURE 2.  $\tau = 0.8, K = 15, \sigma = 0.3, r = 0.1$

*Remark 6.2.* From (6.1) and (2.10) we can see that for the first order approximation, the smaller  $r$  is, the better the approximation is for the Green function that does not include the zero order term in the differential operator. This point is again elucidated in the tables 1 and 2. However, there is a way to overcome this difficulty, simply by a change of variable. Setting  $V(\tau, x) = e^{-r\tau}U(\tau, x)$  will kill the zero order term. Theoretically, after such a change of variable, our approximation should be better.

Another popular financial model is the CEV model, where the dynamics of the stock price is driven by

$$dS_t = \sigma S_t^\beta dW_t$$

It is a generalization of the Black Schole model to fit the volatility smile. The corresponding PDE is given by

$$u_\tau = \frac{\sigma^2}{2} x^{2\beta} u_{xx}, \quad \tau = T - t$$

with initial condition  $u(0, x) = (x - K)^+$ . J. Cox [13] obtained the closed form solution when  $\beta < 1$ . D.Emanuel and J. MacBeth considered the case when  $\beta > 1$  in [20], and also obtained similar results. In both cases, the closed form solutions are very complicated and involving Bessel functions. Computation for these formulas is very intensive. Schroder [45] gives a good way to compute these closed form formulas. We will show that our formula is also very accurate but computationally it is much simpler. We choose  $\beta = \frac{2}{3}, K = 15, \sigma = 0.3, r = 0.1$ . FIGURE 3 and 4 give the comparison of our method and the true solution of the CEV model for different times. Here we have chosen the basepoint  $z = x$ . The graphs are given for  $5 < x < 25$ .

*Remark 6.3.* In the CEV model, the parameter  $\beta$  can vary on the entire real line, and for any  $-\infty < \beta < +\infty$  a representation of the solution exists. Many papers show that generally  $\beta < 1$  for many stocks. For an summary of the empirical evidence, we refer to [45].

**6.2. Compare the results with different basepoints.** By equation (3.22),(3.23) and (3.8), we know that with an arbitrary dilation basepoint  $z$  the first order Green function is

$$(6.1) \quad \mathcal{G}_\tau^{[1]}(x, y; z) = \frac{1}{\sqrt{2\pi\Sigma(0, z)^2\tau}} e^{-\frac{|x-y|^2}{2\Sigma(0, z)^2\tau}} \left[ 1 + \left( \frac{3\Sigma(0, z)\Sigma'(0, z) - 2\mu(0, z)}{2\Sigma(0, z)^2} \right) (x - y) - \frac{\Sigma'(0, z)}{2\Sigma(0, z)^3\tau} (x - y)^3 + (x - z) \left( \frac{(x - y)^2}{\Sigma(0, z)^3\tau} - \frac{1}{\Sigma(0, z)} \right) \right]$$

By (5.2), we only need to integrate this Green function against the terminal data, and then we obtain the call option price for any one dimension model. In this section we will choose different basepoints, i.e.  $z = x, \frac{x+y}{2}$  and  $z = \text{geodesic midpoint}$ , and compare the results with the true solution. Since for the Black-Scholes-Merton equation, an exact and also simple representation to the true solution exists (1.3) we will compare formulas for the case  $\Sigma(\tau, x) = \sigma x$  and  $\mu(\tau, x) = rx$ . Denote  $\mathcal{G}_{BSM}^{[1]}(\tau, x, y; z = x), \mathcal{G}_{BSM}^{[1]}(\tau, x, y; z = \frac{x+y}{2})$  and  $\mathcal{G}_{BSM}^{[1]}(\tau, x, y; z = \sqrt{xy})$  by the three different approximated Green's functions, their meanings are clear. In all the following comparison from figure 5 to figure 10, we fix  $K = 15, r = 0.1, \sigma = 0.3$ . We plot the figures at different times  $\tau = 0.1, 0.3, 0.5$ . For the next six figures, the left ones are plotted for  $5 < x < 25$  again. And We zoom near the strike price  $K$ , which is most relevant point for practitioners. From these graphs we can draw the conclusion that  $\mathcal{G}_{BS}^{[1]}(\tau, x, y; z = x)$  is the best approximation among the first order Green's function approximations.

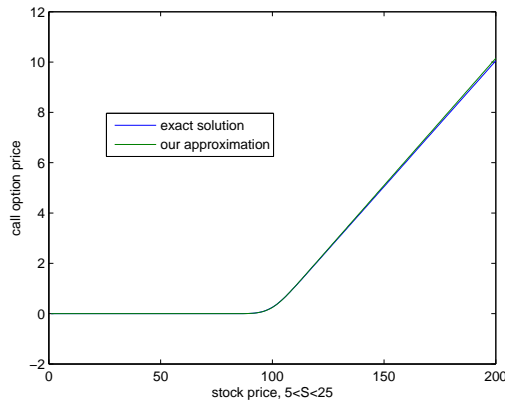


FIGURE 3.  $\tau = 0.1$

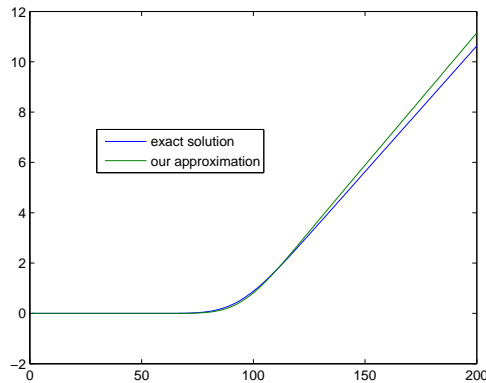


FIGURE 4.  $\tau = 0.5$

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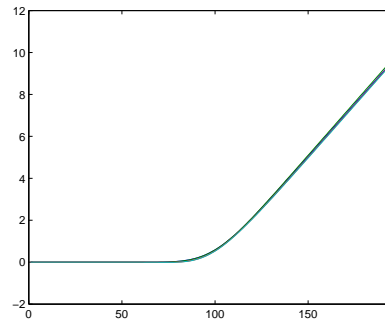


FIGURE  
5.  $\tau = 0.1$

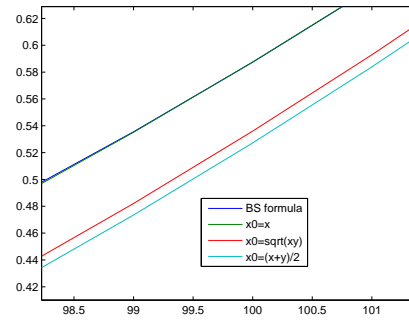


FIGURE  
6.  $\tau = 0.1$

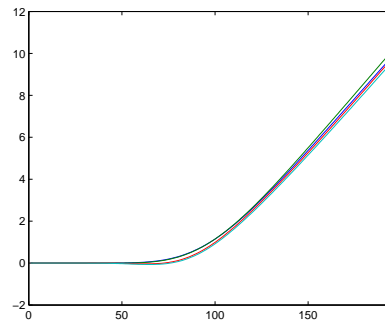


FIGURE  
7.  $\tau = 0.3$

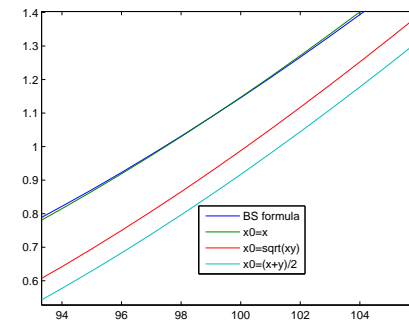


FIGURE  
8.  $\tau = 0.3$

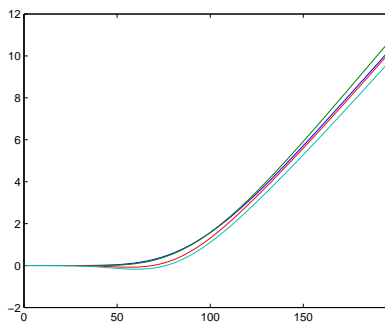


FIGURE  
9.  $\tau = 0.5$

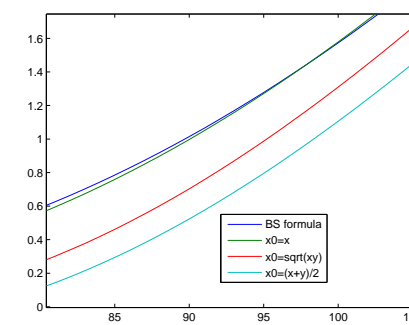


FIGURE  
10.  $\tau = 0.5$

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