

# Analysis and Index Theory on Lie Manifolds

**Lectures 2-4 at IHP**

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## Plan of lectures 2-4

I will introduce the class of **Lie manifolds**, which are possibly non-compact Riemannian manifolds that can be treated, to a large extent, the same way as we treat compact manifolds.

- 1. A first example:** Manifolds with cylindrical ends and spaces (“manifolds”) with conical points.
- 2. Differential and pseudodifferential operators on manifolds with cylindrical ends.**
- 3. Schwartz kernels and groupoids.**
- 4. Pseudodifferential operators on groupoids.**
- 5. Manifolds with a Lie structure at infinity:** Definition and examples.
- 6. Geometry:** Riemannian metric, connection.
- 7. Analysis and applications:** Fredholm operators, spectra.

The guiding problems: understand the spectrum and resolvents of  $(\lambda - \Delta)^{-1}$ .

A quick review of the main results on the Laplacean  $\Delta$  on compact manifolds and on  $\mathbb{R}^n$ .

Definition of pseudodifferential operators and of regularizing operators.

The distribution kernel of a pseudodifferential operator and their relation to regularizing operators.

## Conical singularities

$\Omega =$  an open subset of  $\mathbb{R}^n$  and

$$\partial\Omega := \bar{\Omega} \setminus \Omega.$$

The “simplest” boundary value problem, the **Poisson problem**

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases}$$

Well known: If  $f$ , and  $\partial\Omega$  are **smooth**, then  $u$  is also **smooth** (including the boundary).

**Not true if  $\partial\Omega$  is not smooth.**

**Example:** Assume  $\Omega = (0, 1)^2$ ,  $g = 0$ , and  $u$  smooth. Then

$$\partial_x^2 u(0, 0) = 0 = \partial_y^2 u(0, 0)$$

and hence  $f(0, 0) = \Delta u(0, 0) = 0$  is a necessary condition for  $u \in C^\infty(\bar{\Omega})$ , which is obviously **not** always satisfied.

## Conical points

An approach, **Kondratiev** '67:

polar coordinates  $(r, \theta)$  around the conical points.

If  $\Omega = \{\theta \in (0, \alpha)\}$ , then

$$\Delta = r^{-2} \left( (r\partial_r)^2 + \partial_\theta^2 \right)$$

$(r\partial_r)^2 + \partial_\theta^2$  is a *Fuchs-type differential operator*.

A *Fuchs-type differential operator* =

$$\sum a_{ij}(r, \theta) (r\partial_r)^i \partial_\theta^j$$

with  $a_{ij}$  smooth. Only  $r\partial_r$ , but no condition on  $\partial_\theta$ .

Fuchs-type vector fields =

$$a(r, \theta)r\partial_r + b(r, \theta)\partial_\theta.$$

They form a *Lie algebra*.

All this extends to conical points/manifolds with boundary.

## Manifolds with cylindrical ends

**Basic substitution:**  $r = e^t$ , then

$$\Delta = e^{-2t}(\partial_t^2 + \partial_\theta^2) \quad (1)$$

on a cylinder. **(PICTURES)**

**Conical points  $\leftrightarrow$  cylindrical ends.**

Let  $M_1$  be a *compact manifold with boundary*  $\partial M_1 \neq \emptyset$  and

$$M := M_1 \cup (\partial M_1 \times (-\infty, 0]), \quad \partial M_1 \equiv \partial M_1 \times \{0\}.$$

Then  $M =$  **a manifold with cylindrical ends.**

On the cylindrical end, the metric

$$g = g_\partial + (dx)^2,$$

makes  $M$  a **complete Riemannian manifold.**

**Basic property: Translation invariance at infinity.**

Let  $\Delta_M = d^*d =$  the Laplace operator on  $M$  and

$\Delta_{\partial M_1} =$  the Laplace operator on  $\partial M_1$ , defined using the metric  $g_{\partial}$ . Then

$$\Delta_M = -\partial_x^2 + \Delta_{\partial M_1}$$

on the cylindrical end  $\partial M_1 \times (-\infty, 0]$ . It is **translation invariant in a neighborhood of infinity**.

$P$  is **translation invariant in a neighborhood of infinity** if

$$P\phi_s(f) = \phi_s(Pf)$$

where

$\phi_s =$  translation by  $s$  on the cylinder and

$f$  has support close to  $-\infty$ .

**Define:**  $\Psi_{\text{inv}}^m(M) =$  the space of properly supported pseudodifferential operators of order  $m$  on  $M$  that are **translation invariant in a neighborhood of infinity**.

**Indicial family/operator:** To characterize the Fredholmness or compactness of operators  $P \in \Psi_{\text{inv}}^m(M)$ , we need more than just its principal symbol.

$$\Phi(P) := \tilde{P}, \quad \tilde{P}(f) = \phi_{-s}P\phi_s(f),$$

$\phi_s =$  translation by  $s$  on the cylinder,  $s > 0$  very large, and  $f$  with compact support in  $\partial M_1 \times \mathbb{R}$ . ( $\tilde{P} =$  *indicial operator*)

$\tilde{P}$  is **translation invariant**  $\Rightarrow$

$$\hat{P}(\tau)g = e^{-i\tau t} \tilde{P}(e^{i\tau t}g),$$

a family of pseudodifferential operators on  $\partial M_1$ .

$\Phi$  is surjective with left inverse

$$s_0(T) := \eta T \eta,$$

$\eta = 1$  a cut-off function.

### Sobolev spaces:

$$H^s(M) := \mathcal{D}((I + \Delta_M)^{s/2}),$$

if  $s \geq 0$  ( $s < 0$  by duality).

**Resolvents.** We need inverses (fundamental solutions, recall  $\mathbb{R}^n$ )  $\Rightarrow$  enlarge  $\Psi_{\text{inv}}^\infty(M)$ . Let  $\rho \geq 1$  be a smooth function,

$$\rho(t, y) = t$$

if  $(t, y) \in (-\infty, 2] \times \partial M_1$ . Denote  $\text{ad}_\rho(T) = [\rho, T]$ .

Let  $\|T\|_{k,m}$  denote the norm of

$$\text{ad}_\rho^k(T) : H^{-m}(M) \rightarrow H^m(M).$$

**Definition.**  $\Psi_{\text{ai}}^{-\infty}(M) =$  the closure of  $\Psi_{\text{inv}}^{-\infty}(M)$  with respect to

$$T \rightarrow \|T\|_{k,m} \quad \text{and} \quad T \rightarrow \|\rho^l(T - s_0(\tilde{T}))\rho^l\|_{0,m}.$$

Then  $\Psi_{\text{ai}}^m(M) := \Psi_{\text{inv}}^m(M) + \Psi_{\text{ai}}^{-\infty}(M) =$  asymptotically translation invariant.

**Theorem.** Let  $T \in \Psi_{\text{ai}}^m(M)$ ,  $m \geq 0$ , be elliptic and invertible. Then  $T^{-1} \in \Psi_{\text{ai}}^{-m}(M)$ .

In particular, the resolvent  $(\lambda - T)^{-1} \in \Psi_{\text{ai}}^{-m}(M)$ , if  $\lambda \notin \sigma(T)$ .

**Boundedness and Fredholmness.**  $M$  = a manifold with cylindrical ends,  $P \in \Psi_{\text{ai}}^m(M)$ .

**Theorem.**[Melrose-Mendoza]

- (i)  $P : H^s(M) \rightarrow H^{s-m}(M)$  is bounded.
- (ii)  $P : H^s(M) \rightarrow H^{s-m}(M)$  is compact  $\Leftrightarrow \sigma_m(P) = 0$  and  $\tilde{P} = 0$ .
- (iii)  $P : H^s(M) \rightarrow H^{s-m}(M)$  is Fredholm  $\Leftrightarrow \sigma_m(P)$  and  $\tilde{P}$  are invertible.

(See also Kondratiev, Lauter-Monthubert-N., Lockhart-Owen, Melrose-N., Schrohe, Schulze.)

**Theorem.** Let  $V \in C^\infty(M)$ ,  $V \geq 0$ , be translation invariant and  $\neq 0$  in a neighborhood of infinity. Then  $\Delta_M + V$  is invertible and  $(\Delta_M + V)^{-1} \in \Psi_{\text{ai}}^{-2}(M)$ .

## Distribution kernels

The form of the distribution kernels of invariant, properly supported pseudodifferential operators on  $N \times \mathbb{R}$ ,  $N = \partial M_1$ .

## Other examples

Edges: cylindrical coordinates  $(r, \theta, z)$  in  $\mathbb{R}^3$ .

$$\Delta = r^{-2} \left( (r\partial_r)^2 + \partial_\theta^2 + r^2\partial_z^2 \right) \quad (2)$$

Only  $r\partial_r, r\partial_z$ , but no condition on  $\partial_\theta \Rightarrow$  “Edge–type differential operators.”

Edge–type vector fields form a Lie algebra.

If we radially compactify  $\mathbb{R}^n$ , mapping to  $\|x\| < 1$ , then the constant coefficient, first order differential operators are generated over  $C^\infty(B_1)$ ,  $B_1 =$  closed unit ball, by

$$(1-r)^2\partial_r \text{ and } (1-r)\partial_y,$$

where  $y$  are local coordinates on the boundary  $\partial B_1$  and  $r$  is the distance to the boundary.

$\Rightarrow$  “Scattering differential operators” (or “SG-differential operators”) **Cordes, Melrose, Parenti.**

“Scattering vector fields” form a Lie algebra.

## Definition of Lie manifolds

A **a Lie structure at infinity** is a manifold  $M =$  the interior of a compact manifold with corners  $\overline{M}$ , together with a subspace  $\mathcal{V} \subset \Gamma(T\overline{M})$ , consisting of vector fields **tangent** to all faces of  $\overline{M}$  and satisfying:

- 1.**  $\mathcal{V}$  is closed under the Lie bracket  $[\ , \ ]$ ;
- 2.**  $\mathcal{V}$  is a  $C^\infty(\overline{M})$  module that is generated locally in the neighborhood of each point  $p \in M$  by  $n$  linearly independent vector fields  $X_1, \dots, X_n$ .
- 3.** If above  $p \in M$  (= interior of  $\overline{M}$ ), then the vector fields  $X_1, \dots, X_n$ , locally generating  $\mathcal{V}$  around  $p$ , also give a local basis of  $T_p M$ .

Implicit in **Melrose's**, *Geometric scattering theory*, 1995. Formalized by Ammann-Lauter-N. and A.-L.-N.-Vasy in order to study axiomatically the geometry of these spaces.

## Lecture 3

# Pseudodifferential operators on groupoids

Definitions and examples of:

groupoids,

Lie groupoids,

Lie algebroids,

the Lie algebroid associated to a groupoid, and

pseudodifferential operators on Lie groupoids.

Also a discussion of the integration of Lie algebroids and the results of Crainic and Debord.

## Lecture 4

### Applications to Analysis on Lie Manifolds

Lie manifold  $(\overline{M}, A) \implies$

Lie groupoid  $\mathcal{G} \rightrightarrows \overline{M}$  integrating  $A \implies$

Also, a natural class of complete Riemannian metrics of bounded geometry on  $M := \overline{M} \setminus \partial M$

Algebra of pseudodifferential operators  $\Psi^\infty(\mathcal{G})$  acting on  $\mathcal{C}_c^\infty(M)$  containing parametrices (Fredholm inverses) of the natural differential operators in  $M \implies$

Mapping properties, Fredholm conditions, invertibility for the natural geometric operators on  $M$  (additional PDE tricks are used here).

Recall:

$(\overline{M}, A)$  is a *Lie manifold* if  $A \rightarrow M$  is a Lie algebroid such that the anchor map  $q : A \rightarrow TM$  is an isomorphism over  $M := \overline{M} \setminus \partial M$  and the Lie algebra of vector fields

$$\mathcal{V} := \Gamma(A) = q(\Gamma(A))$$

consists of vector fields tangent to all faces of the manifold with corners  $\overline{M}$ .

We can always find a Lie groupoid (possibly non Hausdorff!) integrating  $(\overline{M}, A)$  that contains the pair groupoid  $M \times M$  as a subgroupoid.

More precisely, if  $d : \mathcal{G} \rightarrow \overline{M}$  is the domain map, as before, then  $\mathcal{G}_x := d^{-1}(x) \simeq M$  for  $x \in M := \overline{M} \setminus \partial \overline{M}$ .

Let  $\Psi^\infty(\mathcal{G}) := \{(P_x), x \in \overline{M}, P_x \in \Psi^\infty(\mathcal{G}_x), \text{ smooth, right invariant, uniformly supported family } \}$ .

Then most of the **regular representations**  $(P_x) \rightarrow P_x$  of  $\Psi^\infty(\mathcal{G})$  will be **simply the action** of  $\Psi^\infty(\mathcal{G})$  on  $M$ .

**Definition.**  $\text{Diff}(\mathcal{V}) =$  the algebra of differential operators on  $M$  generated by vector fields ( $=$ derivations)  $X \in \mathcal{V}$  with coefficients in  $\mathcal{C}^\infty(\overline{M})$ .

Then  $\text{Diff}(\mathcal{V}) \subset \Psi^\infty(\mathcal{G})$  and the regular representation on  $d^{-1}(x) \simeq M$ ,  $x \in M$ , when restricted to  $\text{Diff}(\mathcal{V})$ , identifies with the natural action of  $\text{Diff}(\mathcal{V})$  on  $M$ .

Let  $\mathfrak{A}$  be the norm closure of  $\Psi^0(\mathcal{G})$ . Then  $P \in \mathfrak{A}$  is **Fredholm** if, and only if,  $\pi(P)$  is **invertible** for all irreducible  $*$ -representations of  $\mathfrak{A}$  that *do not contain* the compact operators.

**Question:** What are the irreducible  $*$ -representations of  $\mathfrak{A}$  (or of  $\Psi^0(\mathcal{G})$ )?

## Answer (partial):

Some of the irreducible  $*$ -representations (the least interesting ones) come from the principal symbol of operators in  $\Psi^0(\mathcal{G})$ .

The theory of representations of groupoid  $C^*$ -algebras (Connes, Mackey, Renault) allow us in favorable situations to identify the rest of the representations as coming from an **invariant subset** of  $\overline{M}$  and an **irreducible** representation of an associated Lie group.

Even better, these irreducible representations can be obtained (again in favorable situations), from the regular representations (= the ones on  $\mathcal{G}_x = d^{-1}(x)$ ).

## Examples of $\mathcal{V}$

**Example One.** (a)  $\overline{M}$  = a manifold with smooth boundary  $\partial\overline{M}$ ,  $M = \overline{M} \setminus \partial\overline{M}$ .

(b)  $\mathcal{V}$  = the space of vector fields on  $\overline{M}$  that are *tangent* to  $\partial\overline{M}$ .

(c) There is no condition on these vector fields in the interior.

(d) At the boundary  $\partial\overline{M} = \{x = 0\}$ , a local basis is given by  $x\partial_x, \partial_{y_2}, \dots, \partial_{y_n}$ .

( $y_2, \dots, y_n$  are local coordinates on  $\partial\overline{M}$ .)

$\mathcal{G}$  is the disjoint union of  $\overline{M} \times \overline{M}$  and  $\partial\overline{M} \times \partial\overline{M} \times \mathbb{R}$ .

The fibers  $\mathcal{G}_x := d^{-1}(x)$  are either  $M$ , the interior of  $\overline{M}$  or  $\partial\overline{M} \times \mathbb{R}$ .

The resulting algebra  $\text{Diff}(\mathcal{V})$  of differential operators is the algebra of **Fuchs-type differential operators**.

(Anticipating) The geometry is that of a **manifold with cylindrical ends**. This is obtained by setting  $x\partial_x$  to have length **1** in a tubular neighborhood of the boundary endowed with a product metric.

The pseudodifferential calculus is (essentially) Melrose's  **$b$ -calculus**.

We can identify  $\Psi_{\text{inv}}^{\infty}(M)$  with a suitable subalgebra of  $\Psi^{\infty}(\mathcal{G})$ .

For  $Q = (P_x) \in \Psi_{\text{inv}}^{\infty}(M) \subset \Psi^{\infty}(\mathcal{G})$ , we have  $P_x \simeq P$ , all isomorphic for  $x \in M$ , and  $P_x \simeq \tilde{P}$  for  $x \in \partial\bar{M}$ .

The interesting irreducible representations are associated to the boundary (invariant subset) and to the space  $\hat{\mathbb{R}} = \mathbb{R}$  of representations of  $\mathbb{R}$ . Then  $P \rightarrow \hat{P}(t)$  the **indicial family**.

**See page 8.**

## Metric and geometry

Fix a Lie manifold  $(\overline{M}, \mathcal{V})$ , where  $\mathcal{V}$  is a Lie algebra of vector fields that is also a finitely generated, projective  $C^\infty(\overline{M})$ -module, and hence

$$\mathcal{V} \simeq \Gamma(A),$$

for some vector bundle  $A \rightarrow \overline{M}$  extending  $TM$  to  $\overline{M}$ :

$$A|_M \simeq TM.$$

In particular,  $A \rightarrow \overline{M}$  is a Lie algebroid.

In particular, a metric on  $A$  will induce a Riemannian metric on  $TM$  (i.e. a metric on  $M$ ).  $\Rightarrow$  cylindrical ends for our first example.

Lie algebroids have a theory similar to that of Lie algebras. The “global objects” are the Lie groupoids.

To any differential groupoid there is associated a Lie algebroid, however, Lie’s third theorem fails in this setting: not every Lie algebroid corresponds to a differential groupoid.

# Connections

The **Levi-Civita connection**

$$\nabla : \Gamma(TM) \rightarrow \Gamma(TM \otimes T^*M),$$

extends to an  **$A^*$ -valued connection**

$$\nabla : \Gamma(A) \rightarrow \Gamma(A \otimes A^*),$$

satisfying the usual **Leibnitz rule**:

$$\nabla_X(fY) = X(f)Y + f\nabla_X(Y) \quad \text{and}$$

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle,$$

for all  $X, Y, Z \in \mathcal{V} = \Gamma(A)$ .

This is seen as follows.

Recall that the formula for the **Levi-Civita connection**,  $\nabla_X Y$ , is given by

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle &= \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \\ &\quad + X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle. \end{aligned}$$

Suppose  $X, Y, Z \in \Gamma(A)$  in the above formula.

The function  $2\langle \nabla_X Y, Z \rangle$ , defined initially only on  $M$ , extends to a smooth function on  $\overline{M}$ . Since the inner product  $\langle \cdot, \cdot \rangle$  is the same on  $A$  and on  $TM$ , the above equation determines  $\nabla_X Y$  as a smooth section of  $A$ .

## Differential operators

Recall that  $\text{Diff}(\mathcal{V}) =$  the algebra of differential operators on  $M$  generated by  $C^\infty(\overline{M})$  and vector fields  $X \in \mathcal{V}$ .

We can extend the definition of  $\text{Diff}(\mathcal{V})$  to include operators  $\text{Diff}(\mathcal{V}; E, F)$  acting between vector bundles  $E, F \rightarrow \overline{M}$ .

$$d \in \text{Diff}(\mathcal{V}; \Lambda^q A^*, \Lambda^{q+1} A^*)$$

and

$$\nabla \in \text{Diff}(\mathcal{V}; A, A \otimes A^*).$$

Then (Ammann–Lauter–N.)

$$\Delta \in \text{Diff}(\mathcal{V}).$$

Similarly, all **geometric** differential operators on  $\overline{M}$  are generated by  $\mathcal{V}$ .

(Done “by hand” for earlier particular examples of manifolds with a Lie structure at infinity.)

## Applications

Assume  $(\overline{M}, A)$  “nice”.

**Theorem.** We can associate to  $(\overline{M}, A)$  a family  $M_\alpha$  of manifolds such that, if  $P \in \text{Diff}(\mathcal{V})$ , then there exist operators  $P_\alpha$  on  $M_\alpha$  satisfying:

$P$  is Fredholm  $\Leftrightarrow P$  is elliptic and all  $P_\alpha$  are invertible.

(Lauter-Monthubert-N., earlier: **Kondratiev, Mazya, Plamenevski, Mazzeo, Melrose, Mendoza, Schrohe, Schulze ...**)

Each manifold  $M_\alpha = Z_\alpha \times G_\alpha$ , where  $Z_\alpha$  is a *lower dimensional manifold* and  $G_\alpha$  is a Lie group.

Each operator  $P_\alpha$  is  $G_\alpha$ -invariant and “of the same kind” as the operator  $P$  (Laplace, Dirac, ... ).

Questions about  $M$  are reduced to questions about its  $P_\alpha$  and  $M_\alpha \Rightarrow$  **Harmonic analysis on various Lie groups.**

$\Rightarrow$  an **inductive** procedure to study geometric operators on  $M$ .

## Singular kernels

Fix  $(\overline{M}, A, M)$  = a manifold with a Lie structure at infinity.

Let  $P$  be a properly supported, classical pseudodifferential operator on  $M$ . Then its distribution kernel is in  $I^m(M^2, M)$ .

Fix  $r > 0$  small so that the exponential map defines a bijection

$$(TM)_r := \{v \in TM, \|v\| < r\} \rightarrow M_r^2.$$

We have that

$$(TM)_r \subset (A)_r := \{w \in A, \|w\| < r\}.$$

So  $(A)_r$  extends  $TM$  and  $\overline{M}$  extends  $M$ .

**Definition.** We say that  $P$  extends to  $A$  if its distribution kernel extends to a compactly supported distribution in  $I^m((A)_r, \overline{M})$ .

# Pseudodifferential operators

Define:

$$\Psi_{\mathcal{V}}^{-\infty}(\overline{M}) = \left\{ \sum P \exp(X_1) \dots \exp(X_k), \right. \\ \left. P \in \Psi^{-\infty}(M) \text{ extends to } A \text{ and } X_j \in \mathcal{V} \right\}.$$

and

$$\Psi_{\mathcal{V}}^m(\overline{M}) = \{P \in \Psi^m(M) \text{ extends to } A\} + \Psi_{\mathcal{V}}^{-\infty}(\overline{M}).$$

**Theorem.**  $\Psi_{\mathcal{V}}^{\infty}(\overline{M}) := \cup_{m \in \mathbb{Z}} \Psi_{\mathcal{V}}^m(\overline{M})$  is an algebra closed under adjoints that “quantizes” the Lie algebra  $\mathcal{V}$ .

(Ammann-Lauter-Vasy-N, answers a question of Melrose; based on important earlier work of **Melrose**, **Mazzeo**, **Weinstein**, and **Ping Xu**. Also: **Schulze**, **Schrohe**, **Grisvard**, **Mazya**, **Plamenevsk** ...)

A major ingredient in the proof is to establish first **Lie's third theorem** for  $\mathcal{V}$  (or  $A$ ), N. in the cases of interest for  $\Psi$ do's, and then completely by Crainic-Fernandez.

“Quantizes” means:

1. The differential operators in  $\Psi_{\mathcal{V}}^{\infty}(\overline{M})$  are exactly the ones “generated” by  $\mathcal{V}$ .
2. It has the usual symbolic and analytic properties of  $\Psi$ do’s.

Fix an arbitrary metric on  $TM$  coming from a metric on  $A \rightarrow \overline{M}$ .

Then  $L^2(M)$  is (as a vector space) independent of the metric on  $A$  and  $\Psi^0(\overline{M})$  acts by bounded operators on  $L^2(M)$ .

The norm closure of  $\Psi^0(\overline{M})$  generalizes the **comparison algebras** considered by **Cordes**. Relevant in proving the Fredholm property.

Similarly, let

$$H^s(M) := \{u \in L^2(M), (1 + \Delta)^{s/2}u \in L^2(M)\}, \quad s \geq 0.$$

Then these spaces are independent of the metric on  $A$  and  $P : H^s(M) \rightarrow H^{s-m}(M)$  if  $P \in \Psi^m(\overline{M})$ .

**Example 2.** (a)  $\overline{M}$  = a manifold with smooth boundary  $\partial\overline{M}$ ,  
 $M = \overline{M} \setminus \partial\overline{M}$ .

(b)  $\mathcal{V}$  = the space of vector fields on  $\overline{M}$  that **vanish on**  $\partial\overline{M}$ .

(c) There is no condition on these vector fields in the interior.

(d) At the boundary  $\partial\overline{M} = \{x = 0\}$  a local basis is given by  
 $x\partial_x, x\partial_{y_2}, \dots, x\partial_{y_n}$ .

The resulting geometry is that of an **asymptotically hyperbolic** manifold.

$\{\alpha\} = \partial\overline{M}$ , each  $M_\alpha = G_\alpha = T_\alpha(\partial\overline{M}) \rtimes \mathbb{R}$  is a **solvable** Lie group, and each  $P_\alpha$  is  $G_\alpha$  invariant.

Recently these manifolds have been used in Math. physics in connection to the AdS–CFT correspondence (Witten, Anderson, Lee, Mazzeo, ... ).

Earlier pseudodifferential calculus: Mazzeo's, Schulze's **edge-calculus**.

**Example 3.** (a)  $\overline{M}$  = a manifold with smooth boundary  $\partial\overline{M}$ ,  
 $M = \overline{M} \setminus \partial\overline{M}$ .

(b)  $\mathcal{V}$  = the space of vector fields on  $\overline{M}$  that vanish on  $\partial\overline{M}$   
and the normal covariant to the boundary also vanishes at the  
boundary.

(c) There is no condition on these vector fields in the interior.

(d) At the boundary  $\partial\overline{M} = \{x = 0\}$  a local basis is given by  
 $x^2\partial_x, x\partial_{y_2}, \dots, x\partial_{y_n}$ .

The resulting geometry is that of an asymptotically Euclidean  
manifold.

$\{\alpha\} = \partial\overline{M}$ , each  $M_\alpha = G_\alpha = T_\alpha(\partial\overline{M}) \times \mathbb{R}$  is an abelian Lie  
group, and each  $P_\alpha$  is  $G_\alpha$  invariant.

Earlier pseudodifferential calculus: Parenti's SG-calculus= Mel-  
rose's scattering-calculus.

**Example 4. (a)**  $\overline{M}$  = a manifold with smooth boundary  $\partial\overline{M}$ ,  $M = \overline{M} \setminus \partial\overline{M}$ , together with a fibration  $\pi : \partial M \rightarrow B$ .

**(b)**  $\mathcal{V}$  = the space of vector fields on  $\overline{M}$  that are tangent to the fibers of  $\partial\overline{M} \rightarrow B$ .

**(c)** There is no condition on these vector fields in the interior.

**(d)** At the boundary  $\partial\overline{M} = \{x = 0\}$  a local basis is given by  $x\partial_x, x\partial_{y_2}, \dots, x\partial_{y_k}, \partial_{y_{k+1}}, \dots, \partial_{y_n}$ .

The resulting geometry is related to that of an locally symmetric spaces. It is also relevant for boundary value problems.

$\{\alpha\} = B$ , each  $Z_\alpha = \pi^{-1}(\alpha), \alpha \in B$ ,  $M_\alpha = Z_\alpha \times G_\alpha$ ,  $G_\alpha = T_\alpha B \rtimes \mathbb{R}$  is a solvable Lie group, and each  $P_\alpha$  is  $G_\alpha$  invariant.

Earlier pseudodifferential calculus: Mazzeo's, Schulze's edge-calculus.

It is important to generalize this example to higher rank spaces (=corners of higher codimension).

**Example 5.** (a)  $\overline{M}$  = a manifold with smooth boundary  $\partial\overline{M}$ ,  $M = \overline{M} \setminus \partial\overline{M}$ , together with a foliation  $F \subset T\partial\overline{M}$ .

(b)  $\mathcal{V}$  = the space of vector fields on  $\overline{M}$  that are tangent to the leaves of the foliation  $F$ .

(c) There is no condition on these vector fields in the interior.

(d) At the boundary  $\partial\overline{M} = \{x = 0\}$  a local basis is given by  $x\partial_x, x\partial_{y_2}, \dots, x\partial_{y_k}, \partial_{y_{k+1}}, \dots, \partial_{y_n}$ .

This manifold with a Lie structure at infinity is, however, not “nice.” There is no description of Fredholm operators in terms of lower dimensional spaces and operators invariant with respect to groups. (There exists however a different characterization of Fredholm operators.)

Earlier pseudodifferential calculus?

Questions on the analysis on foliations spaces arise also from Riemannian geometry.

## Spectra

Let  $P = \Delta - \lambda = \Delta_M - \lambda$ . Then

$$\hat{P}(\tau) = \Delta_{\partial M} + \tau^2 - \lambda.$$

Since the spectrum of  $\Delta_{\partial M}$  is

$$\Delta_{\partial M} = \{0, \lambda_1, \lambda_2, \dots\} \subset [0, \infty),$$

we obtain that  $\hat{P}(\tau) = \Delta_{\partial M} + \tau^2 - \lambda$  is invertible for any  $\tau \in \mathbb{R}$  if, and only if,  $\lambda < 0$ . Hence  $\Delta_M - \lambda$  is Fredholm, if, and only if,  $\lambda < 0$ .

This shows that  $\sigma_e(\Delta_M) = [0, \infty)$ . But then

$$[0, \infty) \subset \sigma_e(\Delta_M) \subset \sigma(\Delta_M) \subset [0, \infty)$$

and hence

$$\sigma(\Delta_M) = [0, \infty).$$

This argument generalizes to higher “rank spaces.”

**Example 6.** (a)  $\overline{M}$  = a compact manifold with corners,  $M$  = the interior of  $\overline{M}$ .

(b)  $\mathcal{V}$  = the space of vector fields on  $\overline{M}$  that are **tangent** to all hyperfaces of  $\overline{M}$ .

(generalizes our first example)

(c) There is no condition on these vector fields in the interior.

(d) At the codimension  $k$  face

$$\{x_1 = \dots = x_k = 0\},$$

a local basis is

$$x_1 \partial_{x_1}, \dots, x_k \partial_{x_k}, \partial_{y_{k+1}}, \dots, \partial_{y_n}.$$

Earlier pseudodifferential calculus: Melrose and Piazza.

The manifolds  $Z_\alpha$  are the open faces of  $\overline{M}$ . The group  $G_\alpha$  is  $\mathbb{R}^k$ , with  $k$  the *codimension* of  $Z_\alpha$ .

**Theorem.**[Lauter-N]

$$\sigma(\Delta_M) = [0, \infty)$$

The **complete characterization of the spectrum** (multiplicity of the spectral measure, discreteness of the point spectrum, absence of continuous singular spectrum) is **still open**, although it is of great importance in Number Theory and Representation Theory (automorphic forms, Langlands program).

Similarly, let  $D$  be the Dirac operator associated to a **Clifford(A)**-bundle over  $\overline{M}$ . Then

**Theorem.**[N] The Dirac operator  $D$  is invertible if, and only if, the Dirac operator  $D_F$  associated to any open face  $F$  of  $\overline{M}$  (including  $M$ ), has no harmonic spinors (=zero kernel).