# Analysis and Index Theory on Singular spaces: A Groupoid Approach

**Lectures at IHP**

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# Plan of First Lecture

- Introduction (motivation, examples).

- The Poisson problem ($-\Delta u = f$, $u = 0$ on the boundary) and its weak formulation.

- Discretization of the Poisson problem (linear system).

- A basic Finite Element Estimate.

- Comparison of methods (complexity).

# Introduction

The course will be devoted to some results in Analysis and Index Theory on Singular Spaces

A common point of view on the analysis on singular spaces is to do analysis on the smooth part of the singular space, information on the singular space being nevertheless present in the form of the differential (or pseudodifferential operators used). Since the smooth part of a singular space is a non-compact manifold, we can treat the problem of studying the analysis on a singular space as a particular case of studying the analysis on a non-compact manifold:

*Singular space* ⟶ *Non-compact manifold*

The subject of non-compact manifolds is too large to be treated in any reasonable amount of time (and no single person could cover all the interesting results known in this area). Because of this, in this course we will concentrate on Lie Manifolds.

For Lie manifolds, we can essentially reverse the arrow

$$\textit{Lie manifold} \longrightarrow \textit{Singular space}$$

because a Lie manifold will have, by definition, a compactification to a manifold with corners. (We postpone the definition for a subsequent lecture, once we will have introduced some example and discussed the motivation for studying Lie manifolds.)

The motivation for studying Lie manifolds comes from both (pure) mathematics and from applications. We will next discuss some of these motivations.

**1.** *Smooth, compact* manifolds play a central role in a large part of mathematical results and have applications both within mathematics and outside mathematics (differential geometry, topology, physics, ... ). It is then a natural question to try to generalize the main results on smooth, compact manifolds to non-compact manifolds.

Unfortunately, the class of *all* non-compact manifolds is too large for many interesting results to extend to this case. The class of *complete* manifolds is smaller, but still to large. So is the subclass of *bounded geometry* manifolds. On the other hand, the class of *Lie manifolds*, a subclass of manifolds with bounded geometry, is small enough so that many interesting properties of smooth compact manifolds extend to this class. Most notable are the characterization of Fredholm operators and the construction of relevant algebras of pseudodifferential operators.

The inclusion mentioned above

$$\text{“Lie manif.”} \subset \text{“Bounded geom.”} \subset \text{“Complete manif.”}$$

**2.** Many areas that benefit from the analysis on smooth, compact manifolds, also lead to some questions about certain particular, explicit non-compact manifolds. Examples include:

- gauge theory;

- number theory (the relation of the eta invariant with special values of $L$-functions);

- representation theory (the analysis on the double quotient $\Gamma \setminus G / K$, where $G$ is a connected reductive Lie group, $K$ is its maximal compact subgroup, and $\Gamma$ is a suitable discrete subgroup, in which case we often obtain a space that is both non-compact and singular);

- algebraic geometry ($L^2$-cohomology);

- topology (Ricci flow).

Most of these particular examples (as far as I know) can be studied using Lie manifolds (sometimes by conformally changing the metric).

**3.** Orbifolds and conformally compact manifolds have recently become interesting in Physics. Groupoids bring an important contribution to the study of these spaces. For example, the Chern-Ruan cohomology of an orbifold is the same as the cyclic homology of the convolution algebra of the holonomy groupoid associated to that orbifold. Conformally compact manifolds (to be discussed in detail in the part on Lie manifolds and Fredholm conditions) appear in the AdS-CFT conjecture.

**4.** Polyhedral domains can be used to illustrate the simplest and most typical examples of singular spaces: manifolds with conical points and manifolds with edges. For example, a polygonal domain in the plane is the analog with boundary of a two-dimensional manifold with conical points. Polyhedral domains will be discussed in more detail due to their relavance in Numerical Analysis.

# The Poisson problem

Consider the Poisson problem

$$\begin{cases} -\Delta u := -\partial_1^2 u - \ldots - \partial_d^2 u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

on a bounded open subset $\Omega \subset \mathbb{R}^d$.

Results typical for elliptic problems. (Ex. Elasticity.)

One is interested in estimating the solution $u$ of this Poisson problem.

One of the most successful method is the Finite Element Method.

We need Sobolev spaces and the weak formulation of our Poisson problem (1).

# Sobolev spaces

Let $\Omega \subset \mathbb{R}^d$ be an open subset, $\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$ be a basic differential monomial, $|\alpha| = \alpha_1 + \ldots + \alpha_d$.

$$L^2(\Omega) = \{u : \Omega \to \mathbb{C}, \int_\Omega |u(x)|^2 dx < \infty\}.$$

$m$th Sobolev spaces $H^m(\Omega)$:

$$H^m(\Omega) := \{u, \ \partial^\alpha u \in L^2(\Omega), \ \forall |\alpha| \leq m\}.$$

An important variant (for $\Omega$ nice)

$$H_0^1(\Omega) := H^1(\Omega) \cap \{u = 0, \text{ on } \partial\Omega\}.$$

Let

$$\nabla u = (\partial_1 u, \partial_2 u, \ldots, \partial_d u).$$

Basic bilinear form:

$$\begin{aligned} B(u,v) &:= \int_\Omega \nabla u \cdot \nabla v dx \\ &= \int_\Omega \Big(\partial_1 u \partial_1 v + \partial_2 u \partial_2 v + \ldots + \partial_d u \partial_d v\Big) dx. \end{aligned}$$

# Weak formulation

We shall take $u \in H_0^1(\Omega)$, which includes the (Dirichlet) boundary conditions.

$$
\begin{aligned}
-\Delta u = f,\ u \in H_0^1(\Omega) \\
\Leftrightarrow -\int_\Omega (\Delta u) v dx = \int_\Omega f v dx,\ \forall v \in H_0^1(\Omega), \\
\Leftrightarrow \int_\Omega \nabla u \cdot \nabla v dx =: B(u,v) = \int_\Omega f v dx.
\end{aligned}
$$

The weak formulation of our Poisson problem:

*Find* $u \in H_0^1(\Omega)$ *such that*

$$B(u,v) = \int_\Omega f v dx,$$

for all $v \in H_0^1(\Omega)$.

(Poincaré inequality and Lax-Milgram Lemma imply that $u$ exists, is unique, and depends continuously on $f \in H^{-1}(\Omega) := H_0^1(\Omega)^*$, well posedness for the Poisson problem with Dirichlet boundary conditions.)

# Discretization

Assume we are given a finite dimensional subspace $S \subset H_0^1(\Omega)$. Then we define the discrete solution of our Poisson problem (1) as the unique $u_S \in S$ satisfying

$$B(u_S, v_S) = \int_\Omega f v_S dx, \ \forall v_S \in S.$$

Consider a basis $\phi_j$ of $S$, so that $u_S = \sum x_j \phi_j$. Take then $v_S = \phi_k$ for each $k$ to obtain a system

$$K_S u_S = f_S$$

of size $\dim(S) \times \dim(S)$.

Basic question: What is the relation between the discrete solution $u_S \in S$ and the actual solution $u \in H_0^1(\Omega)$ of our Poisson equation:

$u_S$ is the projection of the solution $u$ onto $S$ (in the inner product defined by the bilinear form $B$).

Consider a sequence of subspaces $S_k \in H_0^1(\Omega)$ and denote $u_k = u_{S_k}$.

If $S_k \subset S_{k+1} \subset \ldots$ and $\cup S_k$ is dense in $H_0^1(\Omega)$, then

$$\|u - u_k\|_{H^1(\Omega)} \to 0, \quad \text{as } k \to \infty.$$

Solving the system $K_S u_S = f_S$ is expensive and the amount of work required growth with $\mathrm{dim}(S)$.

The amount of work needed to solve the system is $\sim \mathrm{dim}(S)^{7/3}$ for direct elimination, $d = 3$ (L. Greengard) and $\sim \mathrm{dim}(S)$ for Multi-grid (optimal).

We want $\mathrm{dim}(S)$ as small as possible.

More on solving this system below, when we will compare several methods.

# Basic FEM Estimate

We partition $\Omega$ in triangles or tetrahedra (mesh) $\mathcal{T}$.

Take $S =$the space of continuous, piecewise polynomials of degree $m$ on some mesh $\mathcal{T}$ on $\Omega$.

We assume:

• the angles appearing in mesh are bounded from below by $\theta > 0$ and

• the sizes of the triangles or tetrahedra $T$ appearing in the mesh are comparable:

$$\operatorname{diam}(T)/\operatorname{diam}(T') \le R.$$

A quasi-uniform sequence of meshes $\mathcal{T}_k$ if we can chose $\theta$ and $R$ *independent* of $n$.

We let $S_k$ be the associated finite element spaces of continuous piecewise polynomials of degree $m$ and denote by $u_k = u_{S_k}$ the associated discrete solution.

Assume that $u \in H^{m+1}(\Omega)$ (i.e. $(m+1)$-square integrable derivatives) and let $h_k$ be the maximum diameter of $T \in \mathcal{T}_k$. Then we have the following quasi-optimal rate of convergence

$$\|u - u_k\|_{H^1} \leq C h_k^m \|u\|_{H^{m+1}},$$

with $C$ independent of $k$ and $f$ ($\Omega \subset \mathbb{R}^d$, $d = 2, 3$ in our case).

Fortunately, a basic result in Partial Differential Equations provides us with conditions on $f$ that will insure $u \in H^{m+1}(\Omega)$. Let us recall this basic result next.

**Theorem.** If $f \in H^{m-1}(\Omega)$ and $\Omega$ has smooth boundary, then the solution $u$ of the Poisson equation (1) satisfies $u \in H^{m+1}(\Omega)$ and, moreover

$$\|u\|_{H^{m+1}} \leq C\|f\|_{H^{m-1}},$$

This result is the well posedness of the Poisson problem with Dirichlet boundary conditions in the $H^{m+1}$ spaces. The inequality of the above theorem is called an *a priori estimate*.

Combining the basic Finite Element Estimate with the apriory estimate of the above theorem yields that for quasi-uniform meshes we have

$$\|u - u_k\|_{H^1} \leq Ch_k^m\|u\|_{H^{m+1}} \leq Ch_k^m\|f\|_{H^{m-1}}.$$

As we will see, the framework of quasi-uniform meshes is not sufficient, so we want to replace the above estimate with something that is independent of the mesh size. Indeed, taking into account that, for a quasi-uniform family of meshes, the number of vertices, edges, triangles (and tetrahedra in 3D, i.e. $d = 3$) has the order of $h_k^{-d}$, we obtain that $\dim(S_k) \sim h_k^{-d}$, or $h_k \sim \dim(S_k)^{-1/d}$. Since the relevant quantity is $\dim(S_k)$, whether the sequence of meshes is quasi-uniform or not, we see that the above equation can be replaced with

$$\|u - u_k\|_{H^1} \leq C \dim(S_k)^{-m/d} \|f\|_{H^{m-1}}, \qquad (2)$$

which is the equation that gives quasi-optimal rates of convergence for an arbitrary sequence of meshes, whether quasi-uniform or not.

We have seen that this happens for $u \in H^{m+1}(\Omega)$, but this is where our problem begins.

The problem is that, in general, $u \notin H^{m+1}(\Omega)$; we are guaranteed only $u \in H^{s+1-\delta}(\Omega)$, where $s = \pi/\alpha_{MAX}$ for polygonal domains. (For more general on Lipschitz domains see: **Costabel, Dauge, Grisvard, Jerison, Kenig, Mitrea, Verchota, Vogel, Taylor.**)

The loss of regularity can very easily be seen by looking again at the Poisson problem $\Delta u = f$, $u \in H^1(\Omega)$. The above well-posedness Theorem for the Poisson equation gives, in particular, that if $f$, $g$, and $\partial\Omega$ are smooth, then $u$ is also smooth (including the boundary).

This is not true if $\partial\Omega$ is not smooth, as seen from the following simple example. Let $\Omega = (0,1)^2$ and assume that$u$ is smooth. Then

$$\partial_x^2 u(0,0) = 0 = \partial_y^2 u(0,0)$$

and hence $f(0,0) = \Delta u(0,0) = 0$ is a necessary condition for $u \in \mathcal{C}^\infty(\overline{\Omega}$, which is however not always satisfied.

In fact, not only regularity is lost, we also obtain much lower convergence rate

$$\mathrm{dim}(S_k)^{-s/d}$$

when quasi-uniform (QU) meshes are used.

# Comparison of methods

Assume our problem is to approximate $u$ within $\epsilon$.

I will show how to replace the quasi-uniform meshes with adaptive (AD) type of meshes, so that the quasi-optimal rate of convergence are restored.

Take $\alpha_{MAX} = 3\pi/2$, so $s = 2/3$ in $\dim(S_k)^{-s/d}$.

| $n$ | mesh | $m$ | DE work | MG work |
|---|---|---|---|---|
| 2 | QU | – | $(1/\epsilon)^6$ | $(1/\epsilon)^3$ |
| 2 | AD | 1 | $(1/\epsilon)^4$ | $(1/\epsilon)^2$ |
| 2 | AD | 3 | $(1/\epsilon)^{1.3}$ | $(1/\epsilon)^{0.6}$ |
| 3 | QU | – | $(1/\epsilon)^{10.5}$ | $(1/\epsilon)^{4.5}$ |
| 3 | AD | 1 | $(1/\epsilon)^7$ | $(1/\epsilon)^3$ |
| 3 | AD | 3 | $(1/\epsilon)^{2.3}$ | $(1/\epsilon)$ |

QU="Quasi-uniform meshes", AD="Adaptive meshes", DE="Direct elimination solver", MG="Multigrid solver".

most work $\sim$ (least work)$^{10}$.

If $(1/\epsilon) = 1000$, $m = 3$, then AD may work 1 billion times faster than QU.

In practice, $(1/\epsilon)$ can be $>>$ 1000.

This completes our introduction to the Finite Element Method and motivates the problem of restoring the quasi-optimal rate of convergence:

$$\|u - u_k\|_{H^1(\Omega)} \leq C\,\mathrm{dim}(S_k)^{-m/3}\|f\|_{H^{m-1}(\Omega)},$$

This will be discussed in the last lecture, the main result(s) being

- Apriori estimates for boundary value problems on polyhedral domains

- Mesh refinement