CHAPTER 12 - FORMULA SHEET 1

INFINITE SEQUENCES

Let f be a function defined through $f(n) = a_n$, for $n = 1, 2, 3, ...$ The range of f is called an *infinite sequence* and is written as $\{a_n\}$ or as $\{a_1, a_2, a_3, ...\}$.

1. A sequence is *convergent* if it has a finite limit as n tends to ∞.
2. A sequence is *divergent* if it either has an infinite limit or if the limit fails to exist, as n tends to ∞.
3. A sequence is *increasing* if the following condition holds: $a_m \geq a_n$ whenever $m \geq n$. Similarly, it is *decreasing* if it is true that $a_m \leq a_n$ whenever $m \geq n$.
4. A sequence is *bounded above* if the following condition holds: $a_n \leq M$ for every value of n. Similarly, it is *bounded below* if it is true that $a_n \geq m$ for all n.
5. A sequence is *monotonic* if it is either increasing or decreasing. It is *bounded* if it is both bounded above and bounded below. The following fact is a key theorem: *EVERY BOUNDED MONOTONIC SEQUENCE IS CONVERGENT.*

INFINITE SERIES

An *infinite series* is an infinite sum of the form $a_1 + a_2 + a_3 + ...$ and is denoted as $\sum_{n=1}^{n=\infty} a_n$.

1. Define $s_n = a_1 + a_2 + a_3 + ... + a_n$ to be the n-th partial sum.
2. The series is *convergent* if the sequence of partial sums is convergent.

   This means $lim_{n\to\infty} s_n = L$ with L being a *finite* number called the *limit of the series.* The series is *divergent* if L does not exist or is $\pm\infty$.
3. If the series $\sum_{n=1}^{\infty} |a_n|$ converges then so does the series $\sum a_n$.

THE GEOMETRIC SERIES

The geometric series is the series

$$\sum_{n=1}^{\infty} ar^{n-1}$$

with common ratio r and first-term a. It converges if $|r| < 1$. It diverges if $|r| \geq 1$.

THE p-SERIES

The p-series is the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

where p is the power of n. It converges if $p > 1$ and diverges if $p \leq 1$.

ALTERNATING SERIES

An alternating series is defined as $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ where $\sum_{n=1}^{\infty} b_n$ is a series with *positive* terms.

1. *Alternating Series Test* The alternating series converges if both of the following conditions are true:

   (i) $b_{n+1} \leq b_n$ for all values of n and

   (ii) $\lim_{n\to\infty} b_n = 0$.

   If either one (or both) of these conditions fails to hold, then the series diverges.

2. The alternating series is called *absolutely convergent* if $\sum |a_n|$ is convergent.

3. The alternating series is called *conditionally convergent* if it is convergent, but the series $\sum |a_n|$ is divergent. For example, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^{n-1}(1/n)$ is conditionally convergent.

AN IMPORTANT FACT:
-IF A SERIES IS ABSOLUTELY CONVERGENT, THEN IT IS CONVERGENT.
-BUT, IF A SERIES IS CONVERGENT, IT NEED *NOT* BE ABSOLUTELY CONVERGENT.

TESTS FOR CONVERGENCE OF INFINITE SERIES

1. *Divergence Test:* If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges.

2. *Integral Test:* Let f be a continuous, positive- valued, decreasing function with $f(n) = a_n$.

   (i) If $\int_1^{\infty} f(x)dx$ converges, then the series $\sum a_n$ also converges.

   (ii) If $\int_1^{\infty} f(x)dx$ diverges, then the series $\sum a_n$ also diverges.

3. *Comparison Test:* Consider 2 series $\sum a_n$ and $\sum b_n$, with $a_n \geq 0$, $b_n \geq 0$ for all values of n.

   (i) If $a_n \leq b_n$ and the series $\sum b_n$ converges, then the series $\sum a_n$ also converges.

   (ii) If $a_n \geq b_n$ and the series $\sum b_n$ diverges, then the series $\sum a_n$ also diverges.

   (iii) If $\lim_{n\to\infty} a_n/b_n$ exists and is non-zero, then

   - if $\sum b_n$ converges then $\sum a_n$ also converges,
   - if $\sum b_n$ diverges then $\sum a_n$ also diverges.

4. *Ratio Test:* Let $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$.

   If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent.* If $L > 1$, or is ∞, then it *diverges.* If $L = 1$ then the test fails.

5. *Root Test:* Let $L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$.

   If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent.* If $L > 1$, or is ∞, then it diverges. It $L = 1$ then the test fails.

ESTIMATING ERRORS FOR INFINITE SERIES

Let $R_n = a_{n+1}+a_{n+2}+a_{n+3}+...$ be the remainder, or error, in approximating the sum of a convergent series $\sum_{n=1}^{n=\infty} a_n$ by its n-th partial sum. Let f be a continuous, positive- valued, decreasing function with $f(n) = a_n$. Then, it is true that

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx.$$