CHAPTER 12 - FORMULA SHEET 2

POWER SERIES

Recall the notion of an infinite series. Power series are basically like infinitely long polynomials.

1. A power series in the variable $x$ and centered at $a$ is the infinite series

$$
\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots + c_n(x-a)^n + \ldots
$$

This means that for every value of $x$, we get an ordinary infinite series $\sum_{n=0}^{\infty} a_n$.

2. Let $f$ be the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. The domain of $f$ consists of the set of $x$-values for which the series converges. This means that for each $x$ in the domain, the value of $f(x)$ is a finite real number. The domain of $f$ is called the interval of convergence of the power series.

3. The Radius of Convergence of a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is the number $R \geq 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Please make note of the following important facts:

(a) $R = \infty$ when the series converges for all values of $x$.

(b) $R = 0$ means the series converges only at the single point $x = a$.

(c) The behavior of the series at the endpoints $a - R$ and $a + R$ of a finite interval of convergence must be checked individually since the series could either converge or diverge at these points.

(d) $R$ is usually determined by using the ratio test.

4. Power series may differentiated and integrated one term at a time, just like polynomials, and the radius of convergence is unaltered.
REPRESENTING FUNCTIONS USING POWER SERIES

1. The Taylor series of a function \( f(x) \) that is centered at \( x = a \) is the infinite series

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + ... \]

2. The MacLaurin Series of a function is its Taylor series centered at \( a = 0 \).

3. The Maclaurin series for some basic functions. These are very important examples, so you must memorize them.

   (a) \[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ... \]
   (b) \[ \sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \]
   (c) \[ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \]
   (d) \[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \]

   Note: If you remember the series for \( \sin(x) \), then the series for \( \cos(x) \) can be obtained by simply taking its derivative. Furthermore, the power series expansion of \( \frac{a}{b + cx}, \sin(kx), \cos(kx) \) and \( e^{kx} \) can be obtained from expressions a,b,c, and d, by using the appropriate substitution.

APPROXIMATING FUNCTIONS USING TAYLOR POLYNOMIALS

Many modern calculators use Taylor polynomials, which are just Taylor series with only finitely many non-zero terms, to compute the values of \( e^x, \sin(x), \) etc.

1. the \( n \)-th degree Taylor polynomial of a function \( f \) is

\[
T_n(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + ... + \frac{f^{(n)}(a)(x-a)^n}{n!}
\]

2. The accuracy of the Taylor polynomial \( T_n(x) \) over the interval \( (a-d, a+d) \) can be computed via the ”Taylor Estimate” of the remainder function \( R_n(x) \):

\[
R_n(x) = |f(x) - T_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}
\]

   Here \( M \) denotes the maximum value of \( f^{(n+1)}(x) \) for \( |x-a| < d \).

3. The power series of the function \( f \) converges to the value of the function, over the interval \( (a-d, a+d) \), if \( \lim_{n \to \infty} R_n = 0 \). This is usually checked using the Taylor estimate(2) and the fact that \[ \lim_{n \to \infty} \frac{x^n}{n!} = 0. \]