APPROXIMATE SOLUTIONS TO SECOND ORDER PARABOLIC EQUATIONS II: TIME-DEPENDENT COEFFICIENTS

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Abstract. We consider the second order parabolic equations with time dependent coefficients. Applying the dilation method, we establish an asymptotic formula for the Green's functions of the nonautonomous equations. Our procedure of approximating the Green's functions is elementary and a systematic algorithm can be obtained. Error analysis also shows that the approximation is accurate to arbitrary prescribed order in time.

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1. Introduction

Paper [7] considers the second order elliptic operator with smooth coefficients in the form

$$Lu(x) := \sum_{i,j=1}^{N} a_{ij}(x)\partial_i\partial_j u(x) + \sum_{k=1}^{N} b_k(x)\partial_k u(x) + c(x)u(x), \tag{1.1}$$

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and developed accurate approximations to the Green's function of the parabolic equation $\partial_t u - Lu = 0$. As a followed work, in this paper we consider a second order parabolic operator with non constant coefficients that also depend on time, more specifically, the operator is of the following form

$$L(t) = \sum a_{i,j}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum b_i(t,x)\frac{\partial}{\partial x_i} + c(t,x) \tag{1.2}$$

where $x = (x_1, ..., x_N) \in \mathbb{R}^N$, $\partial_k := \frac{\partial}{\partial x_k}$, and the coefficients a_{ij}, b_i, and c are smooth and all their derivatives are assumed to be uniformly bounded, in notation we write $a_{ij}(t,x), b_j(t,x), c(t,x) \in \mathcal{C}_b^\infty\left(\mathbb{R}^+ \times \mathbb{R}^N\right)$. We also impose the natural uniformly strongly elliptic condition on the operator $L(t)$, i.e. for any vector ξ, there exists $\gamma > 0$, such that

$$\sum a_{ij}(t,x)\xi_i\xi_j \geq \gamma\|\xi\|^2, \forall t \in [0,T], x \in \mathbb{R}^N \tag{1.3}$$

We shall consider the second order nonautonomous equation

$$\begin{cases} \partial_t u(t,x) - L(t)u(t,x) = g(t,x) & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = f(x), & \text{on } \{0\} \times \mathbb{R}^N \,, \end{cases} \tag{1.4}$$

where $u(t,x), f(x)$ and $g(t,x)$ are in suitable spaces. Our main goal of this paper is under minimal assumptions as we make above, to approximate the Green's function of the operator $\partial_t - L(t)$, and then approximate the solution of the initial value problem (1.4). We also want to develop error estimate for our approximation, as without error control the approximation will be of no great use in practice. With the help of the same dilation method applied in [7], we are still able to compute the mth order approximation of the Green's function even for the non-autonomous equation (1.4) formally. For error control, the main difference between the non-autonomous equation as in this paper and the usual parabolic equation discussed in [7] is that we do not have semigroups any more, instead we shall deal with evolution systems which will be denoted by $U(t,r)$ throughout this paper. The key proposition used in [7] to develop the error estimate is the mapping properties of the analytic semigroup e^{tL}, where L is a second order strongly elliptic operator with non-constant coefficient which *does not depend on time*. Under some minimal conditions of the operator (1.2), we shall show that the evolution system generated by $L(t)$ also has a similar mapping property between Sobolev spaces, which together with the same techniques as in [7] is enough to give us the error estimate. The main result of this paper is the following

Theorem 1.1. *Let m be a positive integer, $L(t)$ the operator* (1.2) *satisfying the conditions we mentioned before, and $z = z(x, y)$ an admissible function. Then $L(t)$ generates an evolution system $U(t, r)$. Define the approximated* mth *order Green's function and the error term as $\mathcal{G}_t^{m,z}(x, y)$ and $\mathcal{E}_t^{[m,z]}$ respectively(see equation* (4.3)*). Then $\mathcal{G}_t^{m,z}(x, y)$ is explicitly computable, and for any initial data $f \in W_a^{k,p}$, $a, k \in \mathbb{R}, 1 < p < \infty$, we have*

$$\|\mathcal{E}_t^{[m,z]} f\|_{W_a^{r+k,p}} \leq C t^{-r/2} \|f\|_{W_a^{k,p}} \tag{1.5}$$

for any $t \in [0, T]$, where the constant C does not depend on $t \in [0, T]$.

Potential application of our main theorem (1.1) will be numerical approximation, option pricing and model calibration, etc. See [9, 8] for applications in finance.

2. Preliminaries on the evolution system

We consider the class of second-order differential operators $L(t)$ of the form (1.2) with coefficients uniformly bounded together with all their derivatives on $\mathbb{R}^+ \times \mathbb{R}^N$. More precisely, let

$$C_b^\infty(\mathbb{R}^+ \times \mathbb{R}^N) := \{f : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}, \partial^\alpha f \text{ bounded for all } \alpha\} \tag{2.1}$$

Note that if $f \in C_b^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$, then f is actually Hölder continuous with respect to the time variable t.

Definition 2.1. *We shall denote by $\mathbb{L}$ the set of differential operators $L(t)$ of the form*

$$L(t) := \sum_{i,j=1}^N a_{ij}(t, x)\partial_i\partial_j + \sum_{k=1}^N b_k(t, x)\partial_k + c(t, x), \tag{2.2}$$

where the matrix (a_{ij}) is symmetric and $a_{ij}, b_k, c \in C_b^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ are real valued. *We shall denote by $\mathbb{L}_\gamma$ the subset of operators $L(t) \in \mathbb{L}$ satisfying the uniform strong ellipticity estimate* (1.3) *with the ellipticity constant γ*

We first introduce some notations and recall several definitions. In what follows, we denote the inner product on $\mathbb{R}^N$ by $(u, v) = \int_{\mathbb{R}^N} u(x)v(x)dx$. Let us denote $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$ and let $\hat{u}$ be the Fourier Transform of u. We also recall the definition of some basic facts about L^p-based Sobolev spaces $W^{r,p}(\mathbb{R}^N)$. For $1 < p < \infty$, $r \in \mathbb{R}$:

$$\begin{aligned} W^{r,p}(\mathbb{R}^N) &:= \{u : \mathbb{R}^N \to \mathbb{C},\ \langle\xi\rangle^r \hat{u} \in L^p(\mathbb{R}^N)\} \\ &= W^{r,p}(\mathbb{R}^N) := \{u : \mathbb{R}^N \to \mathbb{C},\ (1 - \Delta)^{r/2} u \in L^p(\mathbb{R}^N)\}, \end{aligned} \tag{2.3}$$

If $r \in \mathbb{Z}_+$,

$$W^{r,p}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{C},\ \partial^\alpha u \in L^p(\mathbb{R}^N),\ |\alpha| \leq r\}.$$

Since the dimension N is fixed throughout the paper, we will usually write $W^{r,p}$ for $W^{r,p}(\mathbb{R}^N)$. When $1 < p < \infty$, the dual of $W^{r,p}$ is the Sobolev space $W^{-r,p'}$ with $1/p + 1/p' = 1$.

We also recall that an operator A is sectorial if there are constants $\omega \in R, \theta \in (\pi/2, \pi), M > 0$ such that

$$\begin{cases} \rho(A) \supset S_{\theta,\omega} = \{\lambda \in \mathcal{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\} \\ \|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}, \forall \lambda \in S_{\theta,\omega} \end{cases}$$

where $\rho(A)$ is the resolvent set of A. Later on we will give a sufficient condition to guarantee that A is sectorial.

2.1. **Properties of the class $\mathbb{L}_\gamma$.** For elliptic operators, there are some very nice properties. The following norm equivalence property is stated in terms of pseudodifferential calculus(See [5, 11, 10] for definition and basic properties of pseudodifferential operators).

Proposition 2.2. *Let $m \geq 0$ and Q be an elliptic operator whose symbol $\sigma(Q)$ is in the symbol class $S^m_{1,0}$. Then the following two norms are equivalent*

$$\|u\|_{W^{m,p}} \sim \|u\|_{L^p} + \|Qu\|_{L^p} \tag{2.4}$$

for any $u \in W^{m,p}, 1 < p < \infty$.

Proof. On one hand, since $\sigma(Q) \in S^m_{1,0}$, Q is a bounded operator from $W^{m,p}$ to L^p. And clearly, $\|u\|_{L^p} \leq \|u\|_{W^{m,p}}$. Therefore, there exists C_1 such that

$$\|u\|_{L^p} + \|Qu\|_{L^p} \leq C_1 \|u\|_{W^{m,p}}$$

On the other hand, since T is elliptic, there exists a pseudodifferential operator R with its symbol $\sigma(R) \in S^{-m}_{1,0}$ such that $I = RQ - S$, where I is the identity operator, S is a smoothing operator with symbol in $S^{-\infty}$. Thus by mapping properties of pseudodifferential operators, we have

$$\|u\|_{W^{m,p}} \leq \|RQu\|_{W^{m,p}} + \|Su\|_{W^{m,p}} \leq C(\|Qu\|_{L^p} + \|u\|_{L^p})$$

The proof is complete. □

If $L(t)$ is in the class $\mathbb{L}_\gamma$, then from the pseudodifferential calculus point of view, it is also elliptic. And actually we have more

Corollary 2.3. *Suppose $L(t) \in \mathbb{L}_\gamma$. Then the following two norms are equivalent*

$$\|u\|_{W^{2m,p}} \sim \|u\|_{L^p} + \|L^m(t)u\|_{L^p} \tag{2.5}$$

for any $u \in W^{2m,p}, 1 < p < \infty$, where m is a nonnegative integer.

Proof. One can easily check that if $L \in \mathbb{L}_\gamma$, then $\sigma(L)$ has an elliptic symbol and is in the class $S^2_{1,0}$. By symbol calculus, $\sigma(L^m) \in S^{2m}_{1,0}$ and it is still elliptic. Then applying the above proposition completes the proof. □

It turns out that the class $\mathbb{L}_\gamma$ we defined before actually behaves very well. Next we shall show that if $L(t) \in \mathbb{L}_\gamma$, then $L(t)$ is Hölder continuous and sectorial for each $t \in [0, T]$ between suitable Sobolev spaces. These propositions will help us to replace some standard assumptions in the literature for non-autonomous equations(See [3, 6, 1]). The proof is separated in several propositions.

Proposition 2.4. *Suppose $L(t) \in \mathbb{L}_\gamma$, then $L(t) : W^{k+2,p} \mapsto W^{k,p}$ is Hölder continuous for any $k \in \mathbb{N}$.*

Proof. This is obvious, because $L(t) : W^{k+2,p} \mapsto W^{k,p}$ is a bounded operator and the coefficients are Hölder continuous. □

The following proposition gives a sufficient condition to guarantee that an operator is sectorial.

Proposition 2.5. *Let $A : D(A) \subset X \mapsto X$ be a linear operator such that $\rho(A)$ contains a half plane $\{\lambda \in \mathbb{C} : Re\lambda \geq \omega\}$, and*

$$\|\lambda R(\lambda, A)\|_{L(X)} \leq M, Re\lambda \geq \omega$$

with $\omega \in R, M > 0$. Then A is sectorial.

Proof. See [6], page 43. □

Lemma 2.6. *If $L(t) \in \mathbb{L}_\gamma$,and $L(t) : W^{2k+2,p} \mapsto W^{2k,p}$, then for each t the resolvent set of $L(t)$ contains a half plane $\{\lambda \in \mathbb{C} : Re\lambda \geq \omega\}$ for any $k \in \mathbb{N}$.*

Proof. When $k = 0$, the result is standard(See [6]). Now assume $L(t) : W^{2,p} \mapsto L^p$, and λ is in the resolvent set $\rho(L(t))$. Denote $R_\lambda = (\lambda - L(t))^{-1}$, then $R_\lambda : L^p \mapsto L^p$ is bounded by the Closed Graph Theorem. We claim that $R_\lambda\left(W^{2k,p}\right) \subset W^{2k,p}$ and R_λ is bounded. Actually, for any $f \in W^{2k,p}$, then $L^k(t)f \in L^p$ and $L^k(t)R_\lambda f = R_\lambda L^k(t)f \in L^p$, where we used the fact that $L(t)$ and R_λ commute for $f \in D(L^k(t))$(see for example [14] chapter 7). By the norm equivalence (2.5), we conclude that $R_\lambda f \in W^{2k,p}$. This fact tells us that $\rho(L(t))$ is contained in the resolvent set of $L(t) : W^{2k+2,p} \mapsto W^{2k,p}$ for $k > 0$. The proof is complete. □

Finally we prove that the operator $L(t) \in \mathbb{L}_\gamma$ is sectorial for each $t \in [0,T]$ between suitable spaces.

Lemma 2.7. *If $L(t) \in \mathbb{L}_\gamma$,then for each t, $L(t) : W^{2k+2,p} \mapsto W^{2k,p}$ is sectorial for any $k \in \mathbb{N}$.*

Proof. For each fixed $t = t_0$, since the sectorial property does not rely on t, we drop the time dependence and simply write $L = L(t_0)$. We shall apply proposition (2.5) to prove that L is sectorial between $W^{2k+2,p}$ and $W^{2k,p}$. We first estimate $e^{rL} : W^{2k,p} :\mapsto W^{2k,p}$. For any $f \in W^{2k+2,p}$, by the norm equivalence (2.5) we have

(2.6)
$$\begin{aligned}\|e^{rL}f\|_{W^{2k,p}} &\leq C(\|e^{rL}f\|_{L^p} + \|L^k e^{rL}f\|_{L^p}) = C(\|e^{rL}f\|_{L^p} + \|e^{rL}L^k f\|_{L^p}) \\ &\leq Ce^{\omega r}(\|f\|_{L^p} + \|L^k f\|_{L^p}) \leq Ce^{\omega r}\|f\|_{W^{2k,p}}\end{aligned}$$

where we used the standard properties of C^0-semigroups in Banach spaces that $\|e^{rL}f\|_{L^p} \leq Ce^{r\omega}\|f\|_{L^p}$. Thus $\|e^{rL}\|_{W^{2k,p}\to W^{2k,p}} \leq Ce^{\omega r}$.

Then choose any λ from the half plane contained by the resolvent set of L(by Lemma (2.6) this half plane exists). the resolvent operator $R(\lambda, L)$ is actually the Laplace transform of the semigroup generated by L(see for example [1],chapter one), i.e.

$$R(\lambda, L)f = \int_0^\infty e^{-\lambda r}e^{rL}f dr$$

Then for $Re\lambda$ large enough

(2.7)
$$\|\lambda R(\lambda, L)\|_{W^{2k,p}\to W^{2k,p}} \leq \int_0^\infty |\lambda e^{r\lambda}| \cdot \|e^{rL}\|_{W^{2k,p}\to W^{2k,p}} dr \leq \frac{C|\lambda|}{|\lambda - \omega|} \leq C$$

Then by Lemma (2.6) and Proposition (2.5), $L : W^{2k+2,p} \mapsto W^{2k,p}$ is sectorial.

□

2.2. **Existence and properties of the evolution system.** If $L(t) = L$, i.e all the coefficients are time independent, then under some hypothesis L will generate a semigroup e^{tL}, it can be considered as the solution operator. However, in the nonautonomous case, the solution operator is not a semigroup, instead an evolution system if it exists.

Definition 2.8. *A two parameter family of bounded linear operators $U(t, r), 0 \leq t \leq t \leq T$ on X is called an evolution system if the following two conditions are satisfied*

(1) $U(r,r) = I, U(t,r)U(r,l) = U(t,l)$ *for* $0 \le l \le r \le t \le T$
(2) $(t,r) \to U(t,r)$ *is strongly continuous for* $0 \le r \le t \le T$

The evolution system for (1.4) does not always exist. The standard assumption made in the literature is(see [3, 1, 6])

Assumption 2.9. $L(t)$ *is a parabolic operator, and*

(1) The domain of $L(t)$ *is dense in* X *and does not change, i.e.* $D(L(t)) = D$
(2) $L(t)$ *is uniformly bounded below and up.*
(3) $L(t)$ *is Holder continues, i.e. there exists* $\alpha \in (0,1]$*, such that*

$$\|L(t) - L(r)\|_{D \to X} \le C|t-r|^{\alpha}$$

(4) For any $t \in [0,T]$*,* $L(t)$ *is sectorial, i.e. there are constants* $\omega \in R, \theta \in (\pi/2, \pi), M > 0$ *such that*

$$\begin{cases} \rho(L(t)) \supset S_{\theta,\omega} = \{\lambda \in \mathcal{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\} \\ \|R(\lambda, L(t))\| \le \frac{M}{|\lambda - \omega|}, \forall \lambda \in S_{\theta,\omega} \end{cases}$$

Then the evolution system $U(t,r)$ exists, and there are some nice properties. Assumption (2.9) is stated in different forms by Lunardi,Amann and Pazy, but essentially they are the same. Amann and Lunardi show that the second assumption is implied by other assumptions.

To simplify our later argument, we first define the set $\mathbb{L}_\omega$

Definition 2.10. *We shall denote by* $\mathbb{L}_\omega$ *the subset of* $\mathbb{L}$ *satisfying all the four conditions in Assumption* (2.9)*.*

Therefore, any operator in $\mathbb{L}_\omega$ generates an evolution system. For later use, we record some nice properties of the evolution system in the following theorem([3, 6, 1]).

Theorem 2.11. *Suppose* $L(t) \in \mathbb{L}_\omega$*, then the evolution system* $U(t,r)$ *generated by* $L(t)$ *exists, and for* $0 \le r \le t \le T$

$$\|U(t,r)\|_X \le C, \|U(t,r)\|_{X \to D} \le \frac{C}{t-r}$$

$$\|\frac{\partial}{\partial_t} U(t,r)\| = \|L(t)U(t,r)\|_X \le \frac{C}{t-r}$$

$$\|L(t)U(t,r)\|_{D \to X} \le C$$

Proof. see, for example, Lunardi[6] corollary 6.1.8, page 219 □

In the above subsection Proposition (2.4) and Lemma (2.7) show that actually $\mathbb{L}_\gamma$ is a subset of $\mathbb{L}_\omega$. If we choose different domains and the whole spaces, we obtain some nice mapping properties of the evolution system between different Sobolev spaces.

Corollary 2.12. *Suppose $L(t) \in \mathbb{L}_\gamma$. Then $L(t)$ generates an evolution system $U(t,r), 0 \le r \le t \le T$, and for any real k*

$$\|U(t,r)\|_{W^{k,p} \mapsto W^{k,p}} \le C, \|U(t,r)\|_{W^{k,p} \mapsto W^{k+2,p}} \le \frac{C}{t-r}$$

$$\|L(t)U(t,r)\|_{W^{k+2,p} \to W^{k,p}} \le C, \|L(t)U(t,r)\|_{W^{k,p} \to W^{k,p}} \le C$$

where $1 < p < \infty$.

Proof. The proof is mainly an application of a duality argument and space interpolation. First, By Theorem (2.11), for any nonnegative even integer k and $1 < p < \infty$, the above inequalities are true.

Next we apply the duality method to pass the mapping properties to negative Sobolev spaces. Note that $U(t,s)$ satisfies the equation

$$\begin{cases} U(t,t) = 1 \\ \frac{\partial}{\partial t} U(t,r) = L(t)U(t,r) \end{cases} \tag{2.8}$$

We define the adjoint operator $V(t,r) = U(T-r, T-t)^*$, then

$$V(t,t) = U(T-t, T-t)^* = 1$$

and

(2.9)
$$\begin{aligned} &V(t,s)V(s,r) = U(T-s, T-t)^* U(T-r, T-s)^* \\ &= [U(T-r, T-s)U(T-s, T-t)]^* = [U(T-r, T-t)]^* = V(t,r) \end{aligned}$$

Therefore, $V(t,r)$ is also an evolution system. And moreover,

$$\begin{aligned} &\frac{\partial}{\partial t} V(t,r) = \frac{\partial}{\partial t} U(T-r, T-t)^* = [\frac{\partial}{\partial t} U(T-r, T-t)]^* \\ &= [U(T-r, T-t)L(T-t)]^* = L^*(T-t)U(T-r, T-t)^* \\ &= L^*(T-t)V(t,r) \end{aligned} \tag{2.10}$$

i.e. $\frac{\partial}{\partial t} V(t,r) = L^*(T-t)V(t,r)$. According to our assumptions, $L^*(T-t)$ is of the same type with $L(t)$. Thus the evolution system should satisfy the same mapping properties with $U(t,r)$. Then for any positive k, we have

$$\begin{aligned} &\|U(t,r)\|_{W^{-k,p} \to W^{-k,p}} = \|U(t,r)^*\|_{W^{k,q} \to W^{k,q}} \\ &= \|V(T-r, T-t)\|_{W^{k,q} \to W^{k,q}} \le C \end{aligned} \tag{2.11}$$

and

$$\|U(t,r)\|_{W^{-k-2,p}\to W^{-k,p}} = \|U(t,r)^*\|_{W^{k,q}\to W^{k+2,q}}$$

$$= \|V(T-r,T-t)\|_{W^{k,q}\to W^{k+2,q}} \le \frac{C}{(T-r)-(T-t)} = \frac{C}{t-r} \tag{2.12}$$

where q is the conjugate of p, i.e. $\frac{1}{p}+\frac{1}{q}=1$.

At last, we apply the spaces interpolation technique to obtain mapping properties between non-integer Sobolev spaces. For any $\ell \in \mathbb{R}$, assume $k \le \ell < k+2$ where k is an even integer. Then by the complex interpolation,

$$W^{\ell,p} = \left(W^{2k,p'}, W^{2k+2,p''}\right)_{[\theta]}, W^{\ell+2,p} = \left(W^{2k+2,p'}, W^{2k+4,p''}\right)_{[\theta]}$$

where

$$\begin{aligned} \ell &= (1-\theta)\cdot 2k + \theta\cdot(2k+2) \\ \frac{1}{p} &= \frac{1-\theta}{p'} + \frac{1}{p''} \end{aligned}$$

Therefore,

$$\begin{aligned} \|U(t,r)\|_{W^{\ell,p}\to W^{\ell,p}} &\le C_1^{1-\theta}C_2^{\theta} \le C \\ \|U(t,r)\|_{W^{\ell,p}\to W^{\ell+2,p}} &\le \left(\frac{C_1}{t-r}\right)^{1-\theta}\cdot\left(\frac{C_2}{t-r}\right)^{\theta} \le \frac{C}{t-r} \end{aligned}$$

□

Since our approximation is asymptotic near zero, without loss of generality, henceforth we assume $T=1$. We also indicate that throughout this paper C is a generic constant, it may be different at different appearance.

Back to the initial value problem (1.4), in the literature there are several types of solutions(mild, classical, strong) for (1.4). Thus we need to clarify what we mean by a solution of (1.4).

Definition 2.13. *Let $g \in \mathcal{C}([0,\infty),X)$. By a* classical solution in X *of* (1.4) *we mean a function*

$$u \in \mathcal{C}([0,\infty),X) \cap \mathcal{C}^1((0,\infty),X) \cap \mathcal{C}((0,\infty),\mathcal{D}(L(t))), \tag{2.13}$$

such that $\partial_t u(t) = L(t)u(t) + g(t)$ in X for all $t>0$ and $u(0)=f$ in X.

we are also interested in the case that f is in the possibly largest space, for in concrete applications the initial data f is not bounded, even not L^p-integrable, for example, the payoff function of the European call options. To include such cases, we therefore introduce exponentially weighted Sobolev spaces. Given a fixed point $z \in \mathbb{R}^N$, we set $\langle x \rangle_z := \langle x - z \rangle = (1 + |x - z|^2)^{1/2}$ and define $W_a^{m,p}(\mathbb{R}^N)$ for $m \in \mathbb{Z}_+$, $a \in \mathbb{R}$, $1 < p < \infty$, by

$$(2.14) \quad W_{a,z}^{r,p}(\mathbb{R}^N) := e^{-a\langle x \rangle_z} W^{r,p}(\mathbb{R}^N)$$
$$= \{u : \mathbb{R}^N \to \mathbb{C},\ \partial_x^\alpha \big(e^{a\langle x \rangle_z} u(\cdot)\big), \in L^p(\mathbb{R}^N),\ |\alpha| \le r\}, \quad \text{if } r \in \mathbb{Z}_+,$$

with norm

$$\|u\|^p_{W^{m,p}_{a,z}} := \|e^{a\langle x \rangle_z} u\|^p_{W^{m,p}} = \sum_{|\alpha| \le m} \|\partial_\xi^\alpha \big(e^{a\langle x \rangle_z} u(x)\big)\|^p_{L^p}.$$

z will be called the weight center. We consider the operator $L_a(t) = e^{a<x>_z} L(t) e^{-a<x>_z}$. Notice that proving a result for $L(t)$ acting between weighted Sobolev spaces $W_{a,z}^{s,p}$ is the same thing as proving the corresponding result for $L_a(t)$ acting between weighted Sobolev spaces $W^{s,p} = W_{0,z}^{s,p}$. But in order to pass from the conjugated operator to the ordinary operator, we require that L and L_a have the same properties.

Lemma 2.14. *If $L(t) \in \mathbb{L}_\omega$, then $L_a(t) \in \mathbb{L}_\omega$, and vice versa.*

Here ω is a generic real constant, it may be different at different context.

Proof. Suppose $u \in W^{k,p}$ and $L(t) \in \mathbb{L}_\omega$. Notice that

$$(2.15) \qquad L_a(t)u = e^{a<x>_z} L(t) e^{-a<x>_z} u = L(t)u + \gamma(a, x, z)u$$

where $\gamma(a, x, z)$ is uniformly bounded up and below, say $|\gamma(a, x, z)| < \alpha$, with respect to x as long as a and z are in bounded sets. By (2.15) it is obvious that $L_a(t)$ satisfies conditions (1)-(3) in assumption (2.9). Since

$$\lambda I - L_a(t) = (\lambda - \gamma(a, x, z))I - L(t)$$

then as long as $Re(\lambda)$ is big enough, $\lambda I - L_a(t)$ is invertible and

$$\|R(\lambda, L_a(t))\| = \|R(\lambda - \gamma(a, x, z), L(t))\| \le \frac{M}{|\lambda - \alpha - \omega|}$$

Thus condition (4) is also satisfied, and $L_a(t) \in \mathbb{L}_\omega$. Similarly, if $L_a(t) \in \mathbb{L}_\omega$, we can also show $L(t) \in \mathbb{L}_\omega$. □

Therefore, by the above lemma (2.14) and our discussion before, we may assume that $a = 0$ and z is arbitrary. In particular, $L(t) : W_a^{s+2,p} \to W_a^{s,p}$ is well defined and continuous for any a, since this is true for $a = 0$.

Lemma 2.15. *If $L(t) \in \mathbb{L}_\omega$, and $U(t,r)$ is the resulting evolution system. Then*

$$\|U(t,r) - I\|_{W_a^{k+2,p} \to W_a^{k,p}} \le C|t-r|$$

In particular, if we fix r, then

$$[r,\infty) \ni t \to U(t,r) \in \mathbf{B}(W_a^{k+2,p}, W_a^{k,p})$$

defines a continuous operator.

Proof. Notice that $\frac{\partial}{\partial_t} U(t,r) = L(t)U(t,r)$, then for any $f \in W_a^{k+2,p}$

$$\|(U(t,r)-I)f\|_{W_a^{k,p}} \le \int_r^t \|L(\tau)\cdot U(\tau,r)f\|_{W_a^{k,p}} d\tau \le C|t-r| \cdot \|f\|_{W_a^{k+2,p}}$$

by theorem (2.11). And for $t_1 \ge t_2 \ge r$

$$\|U(t_1,r)f - U(t_2,r)f\|_{W_a^{k,p}} \le \|(U(t_1,t_2)-I)f\|_{W_a^{k,p}} \cdot \|U(t_2,r)\|_{W_a^{k,p}} \le C|t_1-t_2| \cdot \|f\|_{W_a^{k+2,p}}$$

This completes the proof of the second part. □

2.3. **perturbation expansion of the operator $L(t)$.** If we split the operator $L(t)$ to the sum of two operators, namely, $L(t) = L_0 + V(t)$, where L_0 is independent on t and V depends on t. We further assume that L_0 generates an analytic semigroup $\{e^{tL_0}, t \ge 0\}$. In later applications, L_0 is actually a second order strongly elliptic operator. We can rewrite the initial value problem (1.4) as

$$\begin{cases} \partial_t u(t,x) - L_0 u(t,x) = V u(t,x) & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = f(x), & \text{on } \{0\} \times \mathbb{R}^N, \end{cases} \tag{2.16}$$

Remark 2.16. Later on we will specify L_0. Actually, to get L_0, we first choose the second order terms in $L(t)$, and send $t = 0$, and at last freeze $x = z$. The resulting operator is L_0.

The following proposition concerning mapping properties of the semigroup generated by L_0 is taken from [7].

Proposition 2.17. *Assume L_0 is actually a second order strongly elliptic operator, then e^{tL_0} is bounded on $[0,1]$ and*

$$\|e^{tL_0} f\|_{W_a^{r,p}(\mathbb{R}^N)} \le C(r,s) t^{(s-r)/2} \|f\|_{W_a^{s,p}(\mathbb{R}^N)}, r \ge s$$

An immediate consequence of the above result is

Corollary 2.18. *Let L_0 be the operator in the above remark* (2.16), *and $s, r \in \mathbb{R}$ be arbitrary. We then have that the map*

$$(0,\infty) \ni t \to e^{tL_0} \in \mathcal{B}(W_a^{s,p}, W_a^{r,p})$$

is infinitely many times differentiable.

Proof. Notice that $\partial_t^k e^{tL_0} = e^{tL_0} L_0^k$, so it suffices to show that the map $(0,\infty) \ni t \to e^{tL_0} \in \mathcal{B}(W_a^{s-k,p}, W_a^{r,p})$ is continuous. Let $t \geq \delta > 0$. Then $e^{\delta L_0}$ maps $W_a^{s-k,p}$ to $W_a^{r+2,p}$ continuously, by Proposition 2.17. Writing $e^{tL_0} = e^{(t-\delta)L_0} e^{\delta L_0}$ and using the continuity of $[\delta, \infty) \ni t \to e^{(t-\delta)L_0} \in \mathcal{B}(W_a^{r+2,p}, W_a^{r,p})$, again by proposition (2.17), we obtain the result. □

For the semigroup e^{tL}, we have seen that it behaves very well. Next for the evolution system $U(t,r)$, we proceed to develop similar mapping properties as in proposition (2.17). We start by studying the perturbation property of the evolution system. If we fix $r = 0$, it becomes a one parameter evolution system. We donote $U(t) = U(t,0)$, where $U(t,r)$ is the evolution system generated by $L(t)$. Recall L_0 generates an analytic semigroup. Now consider the following equation

$$\begin{cases} \partial_t u(t,x) - L_0 u(t,x) = g(t,x) & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = f(x), & \text{on } \{0\} \times \mathbb{R}^N, \end{cases} \tag{2.17}$$

Then we have

Lemma 2.19. *If $g \in L^1([0,1], L^p) \bigcap C((0,1], L^p)$, and $u(t,x) \in L^p$ is a classical solution to (2.17), then*

$$u(t,x) = e^{tL_0} f + \int_0^t e^{(t-\tau)L_0} g(\tau) d\tau, 0 \leq t \leq 1$$

for any initial data $f \in L^p$. Moreover, assume $L(t)$ generates an evolution system $U(t,r)$ and $U(t) = U(t,0)$ is the one parameter system, then the classical L^p-solution to the equation (2.16) is given by

$$u(t) = U(t)f = e^{tL_0} f + \int_0^t e^{(t-\tau)L_0} V(\tau) u(\tau) d\tau \tag{2.18}$$

for any $f \in L^p$.

Proof. Define

$$h(\tau) = e^{(t-\tau)L_0} u(\tau), 0 \leq \tau \leq t$$

since $u(t,x)$ is a classical solution, $h(\tau)$ is continuously differentiable when $\tau > 0$

$$h(0) = e^{tL_0} f, h(t) = u(t)$$

and

$$h'(\tau) = -L_0 e^{(t-\tau)L_0} u(\tau) + e^{(t-\tau)L_0} u'(\tau) = e^{(t-\tau)L_0} g(\tau), 0 < \tau < t$$

integrating it from ϵ to $t-\epsilon$, we have

$$u(t-\epsilon) = e^{(t-\epsilon)L_0} f + \int_\epsilon^{t-\epsilon} e^{(t-\epsilon-\tau)L_0} g(\tau) d\tau$$

Sending ϵ to zero completes the proof of the first part. For the second part, we first assume that $f \in W^{2,p}$, since $L(t)$ generates an evolution system, suppose $u(t,x)$ is the classical solution of (2.16). Define

$$g(t) = Vu(t,x) = (L(t) - L_0)u(t,x) = u_t(t,x) - L_0 U(t) f$$

Then $g(t)$ is continuous, as $U(t) : W^{2,p} \to W^{2,p}$ is continuous and bounded, and $L_0 : W^{2,p} \to L^p$ is bounded. By the result of the first part, $u(t,x)$ has the form (2.18). For the general case when $f \in L^p$, since $W^{2,p}$ is dense in L^p, thus (2.18) still holds. The proof is complete. □

Lemma 2.20. *(Mapping properties of $U(t,r)$)Suppose $U(t,r)$ is the two parameter evolution system introduced before, and $k \le r, t > 0, a \in \mathbb{R}$. Then*

$$\|U(t_1,t_2)\|_{W_a^{k,p} \to W_a^{r,p}} \le C(t_1-t_2)^{(k-r)/2}$$

Proof. As discussed before, we only need to prove the case $a = 0$. We first assume that $k \le r < k+2$, then by Corollary (2.12) and Proposition (2.17), starting from equation (2.18) we have

(2.19)

$$\|U(t_1,t_2)\|_{W^{k,p}\to W^{r,p}} \le \|e^{(t_1-t_2)L_0}\|_{W^{k,p}\to W^{r,p}}$$
$$+ \int_0^{(t_1-t_2)/2} \|e^{(t_1-t_2-\tau)L_0}\|_{W^{k-2,p}\to W^{r,p}} \|V\|_{W^{k,p}\to W^{k-2,p}} \|U(\tau)\|_{W^{k,p}\to W^{k,p}} d\tau$$
$$+ \int_{(t_1-t_2)/2}^{t_1-t_2} \|e^{(t_1-t_2-\tau)L_0}\|_{W^{k,p}\to W^{r,p}} \|V\|_{W^{k+2,p}\to W^{k,p}} \|U(\tau)\|_{W^{k,p}\to W^{k+2,p}} d\tau$$
$$\le C\left((t_1-t_2)^{\frac{(k-r)}{2}} + \int_0^{\frac{(t_1-t_2)}{2}} (t_1-t_2-\tau)^{\frac{(k-2-r)}{2}} d\tau + \int_{\frac{(t_1-t_2)}{2}}^{t_1-t_2} (t_1-t_2-\tau)^{\frac{(k-r)}{2}} \cdot \frac{1}{\tau} d\tau \right)$$
$$\le C(t_1-t_2)^{(k-r)/2}$$

that is

$$\|U(t_1,t_2)\|_{W^{k,p}\to W^{r,p}} \le C(t_1-t_2)^{(k-r)/2}, t_1 \ge t_2 \ge 0$$

For the general case, let $\delta = \frac{r-k}{m}$, where m is an integer and $m > \frac{r-k}{2}$. Then by our above argument, for $j = 1, 2, \cdots, m$

$$\|U(t_1-(j-1)\frac{t_1-t_2}{m}, t_1-j\frac{t_1-t_2}{m})\|_{W^{k+(j-1)\delta,p}\to W^{k+j\delta,p}} \le C\left(\frac{t_1-t_2}{m}\right)^{\frac{k-r}{2m}}$$

Therefore,

$$\|U(t_1,t_2)\|_{W^{k,p}\to W^{r,p}} \le C\left(\frac{t_1-t_2}{m}\right)^{m\cdot\frac{k-r}{2m}} = C(t_1-t_2)^{(k-r)/2}$$

where C depends on k, r, p and a. $\square$

In particular, consider the one parameter evolution system $U(t)$, then the meaning of Proposition (2.17) and Lemma (2.20) is that both e^{tL_0} and $U(t)$ are smoothing operators as long as $t \ge \delta > 0$, i.e. they map a function from any Sobolev space to another Sobolev space continuously. And they do not decrease the regularity of this function for any $t \ge 0$. The worst case is that $t = 0$, and e^{tL_0} and $U(t)$ become the identity operator. This fact will be useful in later applications. Another consequence of Lemma (2.20) is that the Green function for any $L(t) \in \mathbb{L}_\gamma$ exists by Schwartz Kernel Theorem, i.e. there exists $\mathcal{G}_t^L(x,y) \in \mathcal{C}^\infty((0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N)$ such that

$$U(t)f(x) = \int_{\mathbb{R}^N} \mathcal{G}_t^L(x,y)f(y)dy \tag{2.20}$$

and explicitly, we have

$$\mathcal{G}_t^L(x,y) =< \delta_x, U(t)\delta_y >$$

where $< \cdot, \cdot >$ is the duality pairing between $C^\infty(\mathbb{R}^N)$ and compactly supported distributions $\mathcal{E}'(\mathbb{R}^N)$, δ_z is the Dirac delta distribution. As we mentioned before, one purpose of this paper is to approximate $\mathcal{G}_t^L(x,y)$.

As a consequence of Lemma (2.20), we also have the following corollary similar to Corollary (2.18)

Corollary 2.21. *If $L(t) \in \mathbb{L}_\omega$, and $U(t,r), t \ge r \ge 0$ is the resulting two parameter evolution system, then*

$$(r,+\infty) \ni t \to U(t,r) \in \mathcal{B}(W_a^{s,p}, W_a^{m,p})$$

is infinitely many times differentiable.

Proof. For any positive integer k, it is easy to show that formally $\partial_t^k U(t,r) = h(L(t),\partial_t L(t))U(t,r)$ where $h(L(t),\partial_t L(t))$ is a $2k$ order differential operators with smooth and bounded coefficients. For any fixed δ with $t \ge \delta > r$,$U(\delta,r)$ is a continuous map from $W_a^{s,p}$ to $W_a^{m+2k+2,p}$ by lemma (2.20). And $U(t,\delta)$ is continuous from $W_a^{m+2k+2,p}$ to $W_a^{m+2k,p}$ by lemma (2.15). Last, clearly $h(L(t),\partial_t L(t)) \in \mathcal{B}(W_a^{r+2k,p}, W_a^{r,p})$

is also continuous. Therefore, combining the three operators and using the definition of evolution system we conclude that $\partial_t^k U(t,r)$ is continuous from $W_a^{s,p}$ to $W_a^{r,p}$ $\square$

Next we proceed to expand the operator $U(t)$ at $t = 1$. For each $k \in \mathbb{Z}_+$, we denote

$$\begin{aligned}\Sigma_k &:= \{\tau = (\tau_0, \tau_1, \ldots, \tau_k) \in \mathbb{R}^{k+1},\ \tau_j \geq 0, \sum \tau_j = 1\} \\ &\simeq \{\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{R}^k,\ 1 \geq \sigma_1 \geq \sigma_2 \geq \ldots \sigma_{k-1} \geq \sigma_k \geq 0\}\end{aligned}$$

the *standard unit simplex* of dimension k. The bijection above is given by $\sigma_j = \tau_j + \tau_{j+1} + \ldots + \tau_k$. Using this bijection, for any operator-valued function f of $\mathbb{R}^N$ we can write

$$\int_{\Sigma_k} f(\tau) d\tau = \int_0^1 \int_0^{\sigma_1} \cdots \int_0^{\sigma_{k-1}} f(1-\sigma_1, \sigma_1-\sigma_2, \ldots, \sigma_{k-1}-\sigma_k, \sigma_k) d\sigma_k \ldots d\sigma_1$$

Throughout, operator-valued integrals are taken in the sense of Bochner.

Lemma 2.22. *Let $L_j \in \mathbb{L}_\gamma$ and let V_j be such that $e^{-b_j\langle x\rangle} V_j \in \mathbb{L}$, $j = 1, \ldots, k$, for some $b = (b_1, \ldots, b_k) \in \mathbb{R}_+^k$, $k \in \mathbb{Z}_+$. Then*

$$\Phi(\tau) = e^{\tau_0 L_0} V_1 e^{\tau_1 L_1} \ldots e^{\tau_{k-1} L_{k-1}} V_k E(\tau_k), \qquad \tau \in \Sigma_k$$

defines a continuous function $\Phi : \Sigma_k \to \mathcal{B}(W_a^{s,p}(\mathbb{R}^N), W_{a-|b|}^{r,p}(\mathbb{R}^N))$ for any $a \in \mathbb{R}$ $r \geq s$, and $1 < p < \infty$, where either $E(\tau_k) = e^{\tau_k L_k}$ or $E(\tau_k) = U(\tau_k)$. Recall $U(t)$ is the one parameter evolution system defined before.

In this lemma we used the standard multi index notation $|b| = \sum_{j=1}^k b_j$.

Proof. Our proof is based on the fact that both $e^{\tau L}$ and $U(\tau)$ are smoothing operators when $\tau > 0$, and they have the same type mapping properties(see proposition (2.17) and lemma (2.20)). It suffices to prove that Φ is continuous on each of the sets $\mathcal{V}_j := \{\tau_j > 1/(k+2)\}$, $j = 0, \ldots, k$, since they cover Σ_k. Let us assume that $j = 0$, for the simplicity of notation.(If $j = k$, it is slightly different. We will indicate later.)

By assumption and by proposition (2.17) and Lemma 2.15, each of the functions

$$[0, \infty) \ni \tau_j \to V_j e^{\tau_j L_j} \in \mathcal{B}(W_{c_j}^{r_j+4,p}, W_{c_j-b_j}^{r_j,p}), \quad 1 \leq j < k,$$

$$[0, \infty) \ni \tau_k \to V_k E(\tau_k) \in \mathcal{B}(W_{c_k}^{r_k+4,p}, W_{c_k-b_k}^{r_k,p})$$

is continuous. For a suitable choice of c_j and r_j (more precisely, $c_j = c_{j+1} - b_{j+1}$, $c_k = a$, $r_j = r_{j+1} - 4$, $r_k = s$), we obtain that the map

$$[0,\infty)^k \ni (\tau_j) =: \tau' \to \Psi(\tau') := V_1 e^{\tau_1 L_1} ... V_k e^{\tau_k L_k} \in \mathcal{B}(W_a^{s,p}, W_{a-|b|}^{s-4k,p})$$

is continuous.

Corollary 2.18 gives that the map $\tau_0 \to e^{\tau_0 L_0} \in \mathcal{B}(W_{a-|b|}^{s-4k,p}, W_{a-|b|}^{r,p})$ is continuous for $\tau_0 \geq 1/(k+2)$(If $j = k$, we shall use Corollary (2.21)). This proves the continuity of Φ on $\mathcal{V}_0$ and completes the proof of the lemma. □

In particular, if $L_j = L_0, j = 1, 2, \cdots, k$ as in remark (2.16) and the coefficients of V_j are of polynomial growth, an immediate result of lemma (2.22) is

Corollary 2.23. *If $L(t)$ is defined by* (1.2)*, and let L_0 be the operator discussed in remark* (2.16) *and the coefficients of L_j are polynomials in x. Then for some $b = (b_1, \ldots, b_k) \in \mathbb{R}_+^k$, $k \in \mathbb{Z}_+$,*

$$\Phi(\tau) = e^{\tau_0 L_0} L_1 e^{\tau_1 L_0} \ldots e^{\tau_{k-1} L_0} L_k E(\tau_k), \qquad \tau \in \Sigma_k$$

defines a continuous function $\Phi : \Sigma_k \to \mathcal{B}(W_a^{s,p}(\mathbb{R}^N), W_{a-|b|}^{r,p}(\mathbb{R}^N))$ for any $a \in \mathbb{R}$ $r \geq s$, and $1 < p < \infty$, where either $E(\tau_k) = e^{\tau_k L_k}$ or $E(\tau_k) = U(\tau_k)$.

Later on, in the expansion of the Green's function of $L(t)$, the operators will fit in the conditions of the above corollary.

Proposition 2.24. *Let $d \in \mathbb{Z}_+$, and $L(t)$ is split as in equation* (2.16)*, then*

$$\begin{aligned} U(t) =\ & e^{tL_0} + t\int_{\Sigma_1} e^{t\tau_0 L_0} V(t\tau_1) e^{t\tau_1 L_0} d\tau \\ & + t^2 \int_{\Sigma_2} e^{t\tau_0 L_0} V(t\tau_1) e^{t\tau_1 L_0} V(t\tau_2) e^{t\tau_2 L_0} d\tau + \cdots + \\ & + t^d \int_{\Sigma_d} e^{t\tau_0 L_0} V(t\tau_1) e^{t\tau_1 L_0} \ldots e^{t\tau_{d-1} L_0} V(t\tau_d) e^{t\tau_d L_0} d\tau \\ & + t^{d+1} \int_{\Sigma_{d+1}} e^{t\tau_0 L_0} V(t\tau_1) e^{t\tau_1 L_0} \ldots e^{t\tau_d L_0} V(t\tau_{d+1}) U(t\tau_{d+1}) d\tau, \end{aligned} \tag{2.21}$$

and each integral is a well-defined Bochner integral.

The positive integer d will be called the *iteration level* of the approximation. Later on, V will be replaced by a Taylor approximation of L, so that V will have polynomial coefficients in x and t.

Proof. Recall that Lemma 2.19 gives

$$U(t) - e^{tL_0} = \int_0^t e^{(t-\zeta)L_0} V(\zeta)U(\zeta)d\zeta = \int_0^1 e^{t(1-\tau)(L_0)} V(t\tau)U(t\tau)td\tau.$$

with the substitution $\zeta = t\tau$. This is in fact our result for $p = 1$.

The result for any p then follows by induction using the above formula for $\xi = t\sigma_p$. Explicitly, for $t = 1$:

(2.22)

$$U(1) = e^{L_0} + \int_{\Sigma_1} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{\sigma_1 L_0} d\sigma + \int_{\Sigma_2} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} V(\sigma_2) e^{\sigma_2 L_0} d\sigma$$
$$+ \cdots + \int_{\Sigma_{d-1}} e^{(1-\sigma_1)L_0} V(\sigma_1) \dots e^{(\sigma_{d-2}-\sigma_{n-1})L_0} V(\sigma_{d-1}) U(\sigma_{d-1}) d\sigma$$
$$= e^{L_0} + \int_{\Sigma_1} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{\sigma_1 L_0} d\sigma + \int_{\Sigma_2} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} V(\sigma_2) e^{\sigma_2 L_0} d\sigma$$
$$+ \cdots + \int_{\Sigma_{d-1}} e^{(1-\sigma_1)L_0} V(\sigma_1) \dots e^{(\sigma_{d-2}-\sigma_{n-1})L_0} V(\sigma_{d-1}) e^{\sigma_{d-1} L_0} d\sigma + \dots$$
$$+ \int_{\Sigma_{d-1}} \int_0^{\sigma_{d-1}} e^{(1-\sigma_1)L_0} V(\sigma_1) \dots e^{(\sigma_{d-2}-\sigma_{d-1})L_0} V(\sigma_{d-1}) e^{(\sigma_{d-1}-\sigma_d)L_0} V(\sigma_d) U(\sigma_d) d\sigma d\sigma_n$$
$$= e^{L_0} + \int_{\Sigma_1} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{\sigma_1 L_0} d\sigma + \int_{\Sigma_2} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} V(\sigma_2) e^{\sigma_2 L_0} d\sigma$$
$$+ \cdots + \int_{\Sigma_d} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_{d-1}-\sigma_d)L_0} V(\sigma_d) U(\sigma_d) d\sigma,$$

where each integral is well defined as a Bochner integral by the Lemma. □

3. Dilation of the operator

In this section, we shall drop the time dependence to simplify our notation and write $L(t) = L$. For any function $v(t,x)$ and $f(x)$, we choose an arbitrary but fixed basepoint z and dilate them in the following sense

$$v^s(t,x) = v(s^2 t, z + s(x-z))$$
$$f^s(x) = f(z + s(x-z))$$

For the operator L, we set

(3.1)

$$L^s = \sum a^s_{i,j}(s^2 t, z+s(x-z)) \frac{\partial^2}{\partial x_i \partial x_j} + s \sum b^s_i(s^2 t, z+s(x-z)) \frac{\partial}{\partial x_i} + s^2 c^s(s^2 t, z+s(x-z))$$

It is not difficult to show that if $u(t,x)$ is a solution of the equation (1.4), then $u^s(t,x)$ is a solution of the following equation

$$
(3.2) \qquad \begin{cases} \partial_t u^s(t,x) - L^s u^s(t,x) = 0 & \text{in } (0,\infty)\times\mathbb{R}^N \\ u^s(0,x) = f^s(x), \quad f\in\mathcal{C}^\infty(\mathbb{R}^n) & \text{on } \{0\}\times\mathbb{R}^N\,, \end{cases}
$$

Clearly, L^s satisfies all the conditions we assumed above. Denote the evolution system generated by L^s by $U^{L^s}(t)$. And let $\mathcal{G}_t^L(x,y)$ and $\mathcal{G}_t^{L^s}(x,y)$ be the Green functions or fundamental solutions for the operator $\partial_t - L$ and $\partial_t - L^s$ respectively. We want to relate $\mathcal{G}_t^L(x,y)$ and $\mathcal{G}_t^{L^s}(x,y)$.

Lemma 3.1. *Let z be a fixed but arbitrary point in $\mathbb{R}^N$. Then for any $s>0$, we have*

$$
\mathcal{G}_t^L(x,y) = s^{-N}\mathcal{G}_{s^{-2}t}^{L^s}(z+\frac{x-z}{s}, z+\frac{y-z}{s})
$$

In particular, when $s=\sqrt{t}$,

$$
(3.3) \qquad \mathcal{G}_t^L(x,y) = t^{-N/2}\mathcal{G}_1^{L^{\sqrt{t}}}(z+\frac{x-z}{\sqrt{t}}, z+\frac{y-z}{\sqrt{t}})
$$

Proof. Without loss of generality, we assume $z=0$. On one hand, by definition of Green's function,

$$
\begin{aligned}
u^s(t,x) &= \int_{\mathbb{R}^N} \mathcal{G}_t^{L^s}(x,y) f^s(y)dy = \int_{\mathbb{R}^N} \mathcal{G}_t^{L^s}(x,y) f(sy)dy \\
&= s^{-N}\int_{\mathbb{R}^N} \mathcal{G}_t^{L^s}(x,\frac{y}{s}) f(y)dy
\end{aligned}
$$

On the other hand,

$$
u^s(t,x) = u(s^2t, sx) = \int_{\mathbb{R}^N} \mathcal{G}_{s^2t}^L(sx,y)f(y)dy
$$

Therefore,

$$
s^{-N}\int_{\mathbb{R}^N} \mathcal{G}_t^{L^s}(x,\frac{y}{s}) f(y)dy = \int_{\mathbb{R}^N} \mathcal{G}_{s^2t}^L(sx,y)f(y)dy
$$

which implies,

$$
s^{-N}\mathcal{G}_t^{L^s}(x,\frac{y}{s}) = \mathcal{G}_{s^2t}^L(sx,y)
$$

After a change of variable, we get

$$
\mathcal{G}_t^L(x,y) = s^{-N}\mathcal{G}_{s^{-2}t}^{L^s}(\frac{x}{s},\frac{y}{s})
$$

□

With this lemma in hand, in order to approximate $\mathcal{G}_t^L(x,y)$, it suffices to approximate $\mathcal{G}_1^{L^{\sqrt{t}}}(x,y)$. We shall apply the perturbation technique illustrated as follows.

3.1. **formal expansion of the operator L^s.** Suppose L^s is given by (3.1), we taylor-expand it with respect to the parameter s,

$$L^s = L_0 + \sum_{m=1}^{n} s^m L_m + V_{n+1}^s$$

where

$$L_m = \frac{1}{m!}\left(\frac{d^m}{ds^m}L^s\right)\Big|_{s=0} \tag{3.4}$$

and they are independent of s. But V_{n+1}^s does depend on s, and all of the terms depend on z even it does not appear in the notation. We also use another way to denote V_{n+1}^s, i.e.

$$V_{n+1}^s = s^{n+1} L_{n+1}^{s,z}$$

We shall look for a general formula for L_m. For a function $f(t,x)$ smooth enough, we can Taylor expand it around $(0,z)$ as

$$f(t,x) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^l (x-z)^k}{l!k!}\frac{\partial^l}{\partial t^l}\frac{\partial^k}{\partial x^k} f(0,z)$$

Therefore,

$$f(s^2 t, z + s(x-z)) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(s^2 t)^l s^k (x-z)^k}{l!k!}\frac{\partial^l}{\partial t^l}\frac{\partial^k}{\partial x^k} f(0,z)$$

(3.5)

$$\frac{1}{m!}\frac{d^m}{ds^m} f(s^2 t, z + s(x-z))\Big|_{s=0} = \sum_{l=0}^{[\frac{m}{2}]} \frac{t^l (x-z)^{m-2l}}{l!(m-2l)!}\frac{\partial^l}{\partial t^l}\frac{\partial^{m-2l}}{\partial x^{m-2l}} f(0,z)$$

Combine this with (3.4), we can explicitly write L_m as

$$\begin{aligned} L_m \;=\; & \sum_{i,j=1}^{d}\sum_{l=0}^{[\frac{m}{2}]} \frac{t^l (x-z)^{m-2l}}{l!(m-2l)!}\frac{\partial^l}{\partial t^l}\frac{\partial^{m-2l}}{\partial x^{m-2l}} a_{i,j}(0,z)\frac{\partial^2}{\partial x_i \partial x_j} \\ & + \sum_{i}^{d}\sum_{l=0}^{[\frac{m-1}{2}]} \frac{t^l (x-z)^{m-1-2l}}{l!(m-1-2l)!}\frac{\partial^l}{\partial t^l}\frac{\partial^{m-1-2l}}{\partial x^{m-1-2l}} b_i(0,z)\frac{\partial}{\partial x_i} \\ & + \sum_{l=0}^{[\frac{m-2}{2}]} \frac{t^l (x-z)^{m-2-2l}}{l!(m-2-2l)!}\frac{\partial^l}{\partial t^l}\frac{\partial^{m-2-2l}}{\partial x^{m-2-2l}} c(0,z) \end{aligned}$$

where $m \geq 2$. So L_m is a second order differential operator with polynomial coefficients with degree at most m with respect to $x - z$ and $[\frac{m}{2}]$ with respect to t. When $m = 0, 1, 2$,We can give the first few terms in the taylor expansion explicitly,

$$\begin{aligned}
L_0 &= \sum_{i,j=1}^{d} a_{i,j}(0,z)\frac{\partial^2}{\partial x_i \partial x_j} \\
L_1 &= \sum_{i,j=1}^{d} (x-z)\cdot \nabla a_{i,j}(0,z)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(0,z)\frac{\partial}{\partial x_i} \\
L_2 &= \sum_{i,j=1}^{d} \left(\frac{1}{2}(x-z)^T \nabla^2 a_{ij}(0,z)(x-z) + t\cdot \partial_t a_{ij}(0,z)\right)\frac{\partial^2}{\partial x_i \partial x_j} \\
&\quad + \sum_{i=1}^{d}((x-z)\cdot \nabla b_i(0,z))\frac{\partial}{\partial x_i} + c(0,z)
\end{aligned}$$

Remark 3.2. Clearly, L_0 is an operator with constant coefficients. By our assumption (1.3), L_0 generates an analytic semigroup, explicitly, we have

$$e^{tL_0} = \frac{1}{\sqrt{(4\pi t)^N \det A^0}} e^{\frac{(x-y)^t (A^0)^{-1}(x-y)}{4t}}, \tag{3.6}$$

where $A^0 := A(0, z)$, and N is the dimension.

3.2. **Asymptotic expansion of the evolution system.** Recall proposition 2.24, we want to explicitly express $U(t)$ in a nice way, and now $L^s - L_0 = \sum_{m=1}^{n} s^m L_m + V^s_{n+1}$ will serve the role as V does in proposition 2.24, and L_0 is what we introduced above.

To compute the integrals in the above lemma, we need

Lemma 3.3. *(Baker-Campbell-Hausdorf formula) A and B are two operators, then*

$$[e^A, B] = \left([A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots\right) e^A. \tag{3.7}$$

In general this formula is an infinite series. But in our later application, it will be a finite series. This formula tells us how to commute e^A and B:

$$e^A B = \left(B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots\right) e^A.$$

If we apply this over and over again in proposition 2.24, we can reduce the integrals.

Note that from the relation (3.3), eventually we will set t equal to one, so we do not really care about the parameter t appearing in the operators $L_m, m = 0, 1, 2, \cdots$. Now let's give some similar definitions as those in [7].

Definition 3.4 (Spaces of Differentiations)**.** *For any nonnegative integers a, b we denote by $\mathcal{D}(a, b)$ the vector space of all differentiations of degree at most a and order at most b. We extend this definition to negative indices by defining $\mathcal{D}(a, b) = \{0\}$ if either a or b is negative. By degree of A we mean the highest power of the polynomials appearing as coefficients in A.*

Definition 3.5 (Adjoint Representation)**.** *For any two differentiations $A_1 \in \mathcal{D}(a_1, b_1)$ and $A_2 \in \mathcal{D}(a_2, b_2)$ we define $\mathrm{ad}_{A_1}(A_2)$ by*

$$\mathrm{ad}_{A_1}(A_2) := [A_1, A_2] = A_1A_2 - A_2A_1 \tag{3.8}$$

and for any integer $j \geq 1$ we define $\mathrm{ad}^j_{A_1}(A_2)$ recursively by

$$\mathrm{ad}^j_{A_1}(A_2) := \mathrm{ad}_{A_1}(\mathrm{ad}^{j-1}_{A_1}(A_2)) \tag{3.9}$$

Proposition 3.6. *Suppose $A_1 \in \mathcal{D}(a_1, b_1)$ and $A_2 \in \mathcal{D}(a_2, b_2)$. Then for any integer $k \geq 1$,*

$$\mathrm{ad}^k_{A_1}(A_2) \in \mathcal{D}(k(a_1 - 1) + a_2, k(b_1 - 1) + b_2). \tag{3.10}$$

Proof. We first notice that

$$\mathrm{ad}_{A_1}(A_2) \in \mathcal{D}(a_1 - 1 + a_2, b_1 - 1 + b_2). \tag{3.11}$$

Next, from (3.9) we have

$$\mathrm{ad}^k_{A_1}(A_2) = \mathrm{ad}_{A_1}(\mathrm{ad}_{A_1}(\mathrm{ad}_{A_1}(\mathrm{ad}_{A_1}(\dots)))) \tag{3.12}$$

so that an application of (3.11) k times yields the result.

□

Lemma 3.7. *Let m, k be fixed integers ≥ 1. Let $L^z_0 \in \mathcal{D}(0, 2)$ be the constant coefficient operator and $L^z_m \in \mathcal{D}(m, 2)$ be the operator given above, Then,*

$$\mathrm{ad}^k_{L_0}(L_m) \in \mathcal{D}(m - k, m + 2). \tag{3.13}$$

In particular,

$$\mathrm{ad}^k_{L_0}(L_m) = 0 \quad \forall k > m. \tag{3.14}$$

Proof. Applying Lemma 3.7 we see that $\mathrm{ad}^k_{L^z_0}(L^z_m) \in \mathcal{D}(m - k, m + 2)$. If $k > m$, then by definition $\mathcal{D}(m - k, m + 2) = \{0\}$ and we obtain (3.14). □

Lemma 3.8. *Let L_0 and L_m be defined above, then for any $\theta \in (0,1)$,*

$$e^{(1-\theta)L_0} L_m(\theta) = P_m(\theta, x-z, \partial) e^{(1-\theta)L_0}$$

where

$$P_m(\theta, x-z, \partial) := L_m(\theta) + \sum_{i=1}^{m} \frac{(1-\theta)^i}{i!} \mathrm{ad}^i_{L_0}(L_m(\theta)) \in \mathcal{D}(m, m+2)$$

is a finite sum of terms with the form $a(z)(1-\theta)^i \theta^j (x-z)^k \partial_x^\alpha$, in which $a(z)$ and all its derivatives are bounded, α is a multi-index.

Proof. setting $A = (1-\theta)L_0$ and $B = L_m(\theta)$ in Baker-Campbell-Hausdorf formula yields the results. □

Next, we shall rewrite equation (2.22) in a more computable and explicit form. In abuse of notations, it is convenient to write $L^{s,z}_{n+1} = L_{n+1}$. Recall that $L_m = L_m(t)$ is a function of t, thus so is V. Plug $V = \sum_{m=1}^{n+1} s^m L_m(t)$ into (2.22) and expand it, we obtain

(3.15)

$$\begin{aligned}
U(1) =\ & e^{L_0} + \sum_{k=1}^{d} \sum_{1 \le \alpha_i \le n+1} \int_{\Sigma_k} e^{(1-\sigma_1)L_0} s^{\alpha_1} L_{\alpha_1}(\sigma_1) e^{(\sigma_1 - \sigma_2)L_0} \dots e^{(\sigma_{k-1}-\sigma_k)L_0} s^{\alpha_k} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma \\
&+ \sum_{1 \le \alpha_i \le n+1} \int_{\Sigma_{d+1}} e^{(1-\sigma_1)L_0} s^{\alpha_1} L_{\alpha_1}(\sigma_1) e^{(\sigma_1 - \sigma_2)L_0} \dots e^{(\sigma_d - \sigma_{d+1})L_0} s^{\alpha_{d+1}} L_{\alpha_{d+1}}(\sigma_{d+1}) U(\tau_{d+1}) d\sigma, \\
&= e^{L_0} + \sum_{k=1}^{d} \sum_{1 \le \alpha_i \le n+1} s^{\alpha_1 + \dots + \alpha_k} \int_{\Sigma_k} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_{k-1}-\sigma_k)L_0} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma \\
&+ \sum_{1 \le \alpha_i \le n+1} s^{\alpha_1 + \dots + \alpha_{d+1}} \int_{\Sigma_{d+1}} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_d - \sigma_{d+1})L_0} L_{\alpha_{d+1}}(\sigma_{d+1}) U(\sigma_{d+1}) d\sigma,
\end{aligned}$$

To simplify the above formula, we first introduce the notations as follows

Definition 3.9. *For any integers $1 \le k \le d+1$ and ℓ, we shall denote by $\mathfrak{A}_{k,\ell}$ the set of multi-indexes $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{N}^k$, such that $|\alpha| := \sum \alpha_j = \ell$. Furthermore, we denote $\mathfrak{A}_\ell := \bigcup_{k=1}^{\ell} \mathfrak{A}_{k,\ell}$. For symmetry, it will be convenient to set $\mathfrak{A}_0 = \{\emptyset\}$.*

Clearly, since $\alpha_i \ge 1$, the set $\mathfrak{A}_{k,\ell}$ is empty if $\ell < k$. The meaning of ℓ is that of the corresponding power of s and the meaning of k is that of the expansion order. For each $\alpha \in \mathfrak{A}_{k,\ell}$, we denote if $k < d+1$

$$\Lambda_{\alpha,z} = \int_{\Sigma_k} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_{k-1}-\sigma_k)L_0} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma$$

if $k = d+1$,

$$\begin{aligned}
\Lambda_{\alpha,z} &= \int_{\Sigma_{d+1}} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_d-\sigma_{d+1})L_0} L_{\alpha_{d+1}}(\sigma_{d+1}) U(\sigma_{d+1}) d\sigma \\
&= \int_{\Sigma_{d+1}} P_{\alpha_1}(\sigma_1, x-z, \partial) e^{(1-\sigma_2)L_0} \cdots e^{(\sigma_d-\sigma_{d+1})L_0} L_{\alpha_{d+1}}(\sigma_{d+1}) U(\sigma_{d+1}) d\sigma \\
&= \cdots\cdots \\
&= \int_{\Sigma_{d+1}} P_{\alpha_1}(\sigma_1, x-z, \partial) \cdots P_{\alpha_{d+1}}(\sigma_d, x-z, \partial) e^{(1-\sigma_{d+1})L_0} U(\sigma_{d+1}) d\sigma
\end{aligned}$$

A simple but useful lemma about $\Lambda_{\alpha,z}$ is the following, which we record for later use

Lemma 3.10. *For any given multi-index $\alpha \in \mathfrak{A}_{k,\ell}$ with $k \le d$, then*

$$\Lambda_{\alpha,z} = \mathcal{P}_\alpha(x, z, \partial) e^{L_0}$$

where the product is the composition of operators and $\mathcal{P}_\alpha(x, z, \partial)$ is a differential operator of order $2k + \ell$ and polynomial degree $\le \ell = |\alpha|$. More precisely, we have

(3.16)

$$\begin{aligned}
\mathcal{P}_\alpha(x, z, \partial) &= \int_{\Sigma_k} P_{\alpha_1}(\sigma_1, x-z, \partial) P_{\alpha_2}(\sigma_2, x-z, \partial) \cdots P_{\alpha_k}(\sigma_k, x-z, \partial) d\sigma \\
&= \sum_{|\beta \le \ell|} \sum_{|\gamma| \le \ell + 2k} a_{\beta,\gamma}(z) (x-z)^\beta \partial_x^\gamma
\end{aligned}$$

where $a_{\beta,\gamma}(z) \in \mathcal{C}_b^\infty(\mathbb{R})$.

Proof. Applying Lemma (3.8) over and over again, we have

$$\begin{aligned}
\Lambda_{\alpha,z} &= \int_{\Sigma_k} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_{k-1}-\sigma_k)L_0} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma \\
&= \int_{\Sigma_k} P_{\alpha_1}(\sigma_1, x-z, \partial) e^{(1-\sigma_2)L_0} \cdots e^{(\sigma_{k-1}-\sigma_k)L_0} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma \\
&= \cdots\cdots \\
&= \int_{\Sigma_k} P_{\alpha_1}(\sigma_1, x-z, \partial) P_{\alpha_2}(\sigma_2, x-z, \partial) \cdots P_{\alpha_k}(\sigma_k, x-z, \partial) e^{L_0} d\sigma \\
&= \left(\int_{\Sigma_k} P_{\alpha_1}(\sigma_1, x-z, \partial) P_{\alpha_2}(\sigma_2, x-z, \partial) \cdots P_{\alpha_k}(\sigma_k, x-z, \partial) d\sigma \right) e^{L_0}
\end{aligned}$$

where $\sigma_j = \tau_j + \tau_{j+1} + \cdots + \tau_k$. By Lemma (3.8) and Lemma (3.7), we know that each operator $P_{\alpha_i}(\sigma_i, x-z, \partial) \in \mathcal{D}(\alpha_i, \alpha_i+2), i = 1, 2, \cdots, k$. Thus $\mathcal{P}_\alpha(x, z, \partial) \in \mathcal{D}(|\alpha|, |\alpha| + 2k) = \mathcal{D}(\ell, \ell + 2k)$. Also notice that each $P_{\alpha_i}(\sigma_i, x-z, \partial)$ has polynomial coefficients in σ_i, so the integration with respect to σ will be exact, and $\mathcal{P}_\alpha(x, z, \partial)$ is of the desired form. The proof is complete. □

We also set

$$\Lambda_z^{k,\ell} = \sum_{\alpha \in \mathfrak{A}_{k,\ell}} \Lambda_{\alpha,z} \tag{3.17}$$

and

$$\Lambda_z^{\ell} = \sum_{k=1}^{\min(\ell,d+1)} \Lambda_z^{k,\ell} \tag{3.18}$$

For convenience, let $\Lambda_z^0 = e^{L_0}$.
Now we record the above calculation as the following main theorem of this section

Theorem 3.11. *Let $M = (d+1)(n+1)$. Suppose $U(t)$ is the one parameter evolution system, and $d \geq n$, then it has the expansion*

$$U(1) = e^{L_0} + \sum_{\ell=1}^{m} s^\ell \Lambda_z^\ell + s^{m+1} E_{d,n}^{s,z}$$

where $E_{d,n}^{s,z} = \sum_{\ell=m+1}^{M} s^{\ell-m-1} \Lambda_z^\ell$ is the error term. Recall that d is the iteration level and n is the expansion order of $L(t)$.

Proof. The proof is straightforward. Rewrite (3.15) with the above notations, it becomes

$$U(1) = \sum_{\ell=0}^{M} \sum_{k=1}^{\min(d+1,\ell)} \sum_{\alpha \in \mathfrak{A}_{k,\ell}} s^\ell \Lambda_{\alpha,z} \tag{3.19}$$

Picking up the terms with powers of s less than $m+1$, and putting all the other higher order terms in the last term $E_{d,n}^{s,z}$ completes the proof. □

Remark 3.12. If $\ell \leq n$, we indicate that $\Lambda_{\alpha,z}$ and Λ_z^ℓ are both independent of d, the iteration level, as long as $d \geq n$. And Λ_z^ℓ is independent of n also as long as $n \geq \max(\alpha_i)$. These facts will be useful in our error analysis.

4. Error Analysis

In this section we shall mainly apply the pseudodifferential operator techniques to justify that our approximation yields accurate solution to arbitrary prescribed order in time. For all relevant properties of pseudodifferential operators, we refer to [10]. We start from the operator L_m in the expansion (3.4). As we mentioned before, $L_m (0 \leq m \leq n+1)$ is a second order differential operator with polynomial coefficients with degree at most m with respect to $x - z$. An immediate consequence of this fact is that

Lemma 4.1. *The family*

$$\{\langle x\rangle_z^{-j} L_j^z,\ \langle x\rangle_z^{-n-1} L_{n+1}^{s,z};\ s \in (0,1],\ z \in \mathbb{R}^N,\ j = 0, \dots, n+1\}$$

defines a bounded subset of $\mathbb{L}$.

Recall that for convenience, we also denote $L_{n+1}^{s,z}$ by L_{n+1}^z, which actually depends on s and the dilation center z as well.

Lemma 4.2. *For each given* $\epsilon > 0$*, the family*

$$\{e^{-\epsilon < z-w >} e^{-\epsilon \langle x\rangle_w} L_j^z,\ s \in (0,1],\ z \in \mathbb{R}^N,\ j = 0, \dots, n+1\}$$

is a bounded subset of $\mathbb{L}$.

Proof. If $w = z$, then the desired result follows directly from Lemma 4.1 and the simple observation that $\langle x\rangle_z^j e^{-\epsilon \langle x\rangle_z} \leq C$, with C independent of z and j.

If $w \neq z$, then

$$\begin{aligned} & < x - z > - < x - w > = \sqrt{1 + |x-z|^2} - \sqrt{1 + |x-w|^2} \\ & = \frac{(|x-z| - |x-w|)(|x-z| + |x-w|)}{\sqrt{1+|x-z|^2} + \sqrt{1+|x-w|^2}} \\ & \leq |w - z| \leq < w - z > \quad \text{(triangle inequality)} \end{aligned} \tag{4.1}$$

Therefore $e^{\epsilon(<x-z> - <x-w> - <w-z>)} \leq 1$, and the family

$$e^{\epsilon(<x-z> - <x-w> - <w-z>)} e^{-\epsilon <x>_z} L_j^z = e^{-\epsilon <z-w>} e^{-\epsilon \langle x\rangle_w} L_j^z$$

is bounded for $s \in (0,1]$ and $j = 0, 1, 2, \cdots, n+1$ as claimed. □

Lemma 2.22 and lemma (4.2) then give

Corollary 4.3. *For any* $\alpha_1, \alpha_2, \cdots, \alpha_k$ *with* $\sum_{i=1}^k \alpha_i = \ell$*, the operators*

$$\Lambda_{\alpha,\ell} = \int_{\Sigma_k} e^{\tau_0 L_0} L_{\alpha_1}(\tau_1) e^{\tau_1 L_0} \cdots e^{\tau_{k-1} L_0} L_{\alpha_k}(\tau_k) e^{\tau_k L_0} d\tau, \quad k \leq d$$

and

$$\Lambda_{\alpha,\ell} = \int_{\Sigma_{d+1}} e^{\tau_0 L_0} L_{\alpha_1}(\tau_1) e^{\tau_1 L_0} \cdots e^{\tau_d L_0} L_{\alpha_{d+1}}(\tau_{d+1}) U(\tau_{d+1}) d\tau$$

are bounded linear operators from $W_a^{s,p}$ *to* $W_{a-\epsilon}^{r,p}$ *for any* $z \in \mathbb{R}^N$, $r, s \in \mathbb{R}$, $1 < p < \infty$, *and* $\epsilon > 0$. *Moreover, we have that*

$$\|\Lambda_{\alpha,\ell}\|_{W_a^{s,p} \to W_{a-\epsilon}^{r,p}} \leq C_{s,r,p,a,\epsilon} e^{k\epsilon < z-w >},$$

for a constant $C_{s,r,p,a,\epsilon}$ *that does not depend on* z. *In particular, each* $\Lambda_{\alpha,\ell}$ *is an operator with smooth kernel* $\Lambda_{\alpha,\ell}(x, y)$.

Therefore, the above corollary gives

$$\Lambda_{\alpha,\ell} f(x) = \int_{\mathbb{R}^N} \Lambda_{\alpha,\ell}(x, y) f(y) dy. \tag{4.2}$$

From now on, we shall denote by $T(x, y)$ the kernel of an operator T with smooth kernel. Then in terms of kernels, theorem (3.11) becomes

$$U(1)(x, y) = e^{L_0}(x, y) + \sum_{\ell=1}^{m} s^\ell \Lambda_z^\ell(x, y) + s^{m+1} E_{d,n}^{s,z}(x, y)$$

By lemma (3.1), if we do the substitution $x = z + s^{-1}(x - z)$ and $y = z + s^{-1}(y - z)$ in the above equation, we have

(4.3)

$$\mathcal{G}_t^L(x, y) = s^{-N} \Big(e^{L_0}(z + s^{-1}(x - z), z + s^{-1}(y - z)) +$$
$$\sum_{\ell=1}^{m} s^\ell \Lambda_z^\ell(z + s^{-1}(x - z), z + s^{-1}(y - z)) + s^{m+1} E_{d,n}^{s,z}(z + s^{-1}(x - z), z + s^{-1}(y - z)) \Big)$$
$$:= \mathcal{G}_t^{[m,z]}(x, y) + s^{m+1} E_{d,n}^{t,z}(x, y)$$

where $s = \sqrt{t}$, and recall $\mathcal{G}_t^L(x, y)$ is the Green function of the operator $\partial_t - L(t)$. We can compute $\mathcal{G}_t^{[m,z]}(x, y)$ explicitly, then

$$s^{m+1} E_{d,n}^{t,z}(x, y) = \mathcal{G}_t^L(x, y) - \mathcal{G}_t^{[m,z]}(x, y)$$

is the error term which we need to bound. We define the error operator as

$$\mathcal{E}_{d,n}^{[m,z]} f = \int E_{d,n}^{t,z}(x, y) f(y) dy \tag{4.4}$$

In abuse of notations, if $\mathcal{G}_t^{[m,z]}$ denotes the operator with kernel $\mathcal{G}_t^{[m,z]}(x, y)$, then

$$U(t, 0) = \mathcal{G}_t^{[m,z]} + s^{m+1} \mathcal{E}_{d,n}^{[m,z]}$$

By the definition of the error operator (4.4) and equation (3.19), we have

$$\mathcal{E}_t^{[m,z]} = \sum_{\ell=m+1}^{M} \sum_{k=1}^{\min(d+1,\ell)} \sum_{\alpha\in\mathfrak{A}_{k,\ell}} s^{\ell-m-1}\Lambda_{\alpha,z} \tag{4.5}$$

Later on we shall estimate the error by splitting $\mathcal{E}_t^{[m,z]}$ into two parts, namely

$$\mathcal{E}_t^{[m,z]} = \sum_{\ell=m+1}^{h} \sum_{k=1}^{\ell} \sum_{\alpha\in\mathfrak{A}_{k,\ell}} s^{\ell-m-1}\Lambda_{\alpha,z} + s^{h-m}\mathcal{E}_t^{[h,z]} \tag{4.6}$$

In all the above formulas, we do not specify the dilation center z, it is arbitrary. And generally, it is a function of x and y, i.e. $z = z(x,y)$. For our error analysis, we need to specify the dilation center

Definition 4.4. *A function $z : \mathbb{R}^{2N} \to \mathbb{R}^N$ will be called* admissible *if*

(i) $z(x,x) = x$, for all $x \in \mathbb{R}^N$.
(ii) All derivatives of z are bounded.

A typical example is $z(x,y) = \lambda x + (1-\lambda)y$, for some fixed parameter λ. A simple application of the mean value theorem gives that $\langle z - x\rangle \le C\langle y - x\rangle$ for some constant $C > 0$. From the application point of view, $z(x,y) = x$ will give us the simplest formula to approximate the Green function([9, 8]). However, it may not be the optimal choice([8]).

4.1. **Bound the desired term.** In this subsection, we consider the desired term

$$\mathcal{G}_t^{[m,z]}(x,y) = \sum_{\ell=0}^{m} s^\ell \Lambda_z^\ell(z + s^{-1}(x-z), z + s^{-1}(y-z))$$

and we shall fix the function $z = z(x,y)$ which is admissible. Recall that

$$\Lambda_z^\ell = \sum_{k=1}^{\min(d+1,\ell)} \sum_{\alpha\in\mathfrak{A}_{k,\ell}} s^\ell \Lambda_{\alpha,z}$$

We treat each operator $\Lambda_{\alpha,z}$ in one time. Define the approximation operator

$$\mathcal{L}_{s,\alpha} f(x) = s^{-N}\int_{\mathbb{R}^N} \Lambda_{\alpha,z}(z + s^{-1}(x-z), z + s^{-1}(y-z)) f(y)dy, \tag{4.7}$$

We will show below that for an admissible function z, and $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathfrak{A}_{k,\ell}$, $k \le n$, $\alpha_i \le n$, the operator $\mathcal{L}_{s,\alpha}$ is a pseudodifferential operator whose symbol is well behaved. We shall then use symbol calculus to derive the desired error estimates. Let's first recall some standard definitions and results from pseudodifferential calculus. Let $m \in \mathbb{R}$. We define $S^m_{1,0}$ to be the set of all functions $\sigma(x,\xi)$ in $C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ such that for any two multi-indices α and β, there is positive constant $C_{\alpha,\beta}$, depending on α and β only, such that

$$\left|\left(D_x^\alpha D_\xi^\beta \sigma(x,\xi)\right)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}$$

then we call any function σ in $\bigcup_{m\in\mathbb{R}} S^m_{1,0}$ a symbol, and we denote $S^{-\infty} = \bigcap_{m\in\mathbb{R}} S^m_{1,0}$. Any operator whose symbol in $S^{-\infty}$ is a smoothing operator. Now if $\sigma(x,\xi)$ is a symbol. Then the pseudodifferential operator $\sigma(x,D)$ associated to $\sigma(x,\xi)$ is defined by

$$(\sigma(x,D)\psi)(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix\cdot\xi} \sigma(x,\xi)\hat{\psi}(\xi) d\xi$$

where $D = \frac{1}{i}\partial$ and

$$\mathcal{F}\psi(x) = \hat{\psi}(\xi) := \int_{\mathbb{R}^N} e^{-\imath\xi\cdot x}\psi(x)dx \tag{4.8}$$

the usual Fourier transform of ψ. Next let's relate the operator $\sigma(x,D)$ with its distributional kernel, actually we can recover one from the other under some conditions. Denote by $\mathcal{F}_2$ the Fourier transform in the second variable of a function of two variables. For $\sigma(x,\xi) \in S^{-\infty}$, the operator $\sigma(x,D)$ is smoothing with distribution kernel

$$\sigma(x,D)(x,y) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{\imath(x-y)\cdot\xi}\sigma(x,\xi)d\xi = (\mathcal{F}_2^{-1}\sigma)(x, x-y).$$

Let us denote by K a smooth function on $\mathbb{R}^N \times \mathbb{R}^N$, if the integral (smoothing) operator defined by K is in fact a pseudodifferential operator $\sigma(x,D)$, then we can recover σ from K by the formula $\mathcal{F}_2^{-1}\sigma(x,y) = K(x,x-y)$, so

$$\sigma(x,\xi) = \int_{\mathbb{R}^N} e^{-\imath\xi\cdot y} K(x, x-y)dy. \tag{4.9}$$

Concerning the class $S^{-\infty}$, the following result is also standard and we are going to use it later on.

Lemma 4.5. *(i) The Fourier transform in the second variable establishes an isomorphism $\mathcal{F}_2 : S^{-\infty} := S^{-\infty}(\mathbb{R}^N \times \mathbb{R}^N) \to S^{-\infty}$.*
(ii) Multiplication defines a continuous map $S^m_{(1,0)} \times S^{-\infty} \to S^{-\infty}$.

For more about pseudodifferential calculus, we refer to the works of Taylor [5, 10] and Wong [11].

With this tool in hand, we move on to do the analysis. Recall that the function $G(z;x) = (4\pi)^{-N/2}\det(A(z))^{-1/2}e^{-x^TA(z)^{-1}x/4}$ introduced in Equation (3.6): Then the distribution kernel of $e^{L_0^z}$ is given by

$$e^{L_0^z}(x,y) = G(z;x-y), \tag{4.10}$$

A direct computation gives the following lemma, which coincides with the fact that $e^{L_0^z}$ is a convolution operator.

Lemma 4.6. *Let $z \in \mathbb{R}^N$ be a parameter and let us consider the operator $T = a(z)(x-z)^\beta \partial_x^\gamma e^{L_0^z}$, where β and γ are multi-indices and $a \in \mathcal{C}_b^\infty(\mathbb{R}^N)$. Then the distribution kernel of T is given by*

$$T(x,y) = a(z)(x-z)^\beta(\partial_x^\gamma G)(z;x-y).$$

The next theorem characterizes the symbol of $\mathcal{L}_{s,\alpha}$.

Theorem 4.7. *Let $\alpha \in \mathfrak{A}_{k,\ell}$, $k \le n$, $\alpha_i \le n$. Assume that $z : \mathbb{R}^N \times \mathbb{R}^N$ satisfies $z(x,x) = x$ and $\partial^\alpha z$ is bounded for all $\alpha \neq 0$. Then there exists a uniformly bounded family $\{\sigma_s\}_{s\in(0,1]}$ in $S^{-\infty}$ such that*

$$\mathcal{L}_{s,\alpha} = \sigma_s(x,sD) := \varrho_s(x,D), \quad \varrho_s(x,\xi) = \sigma_s(x,s\xi).$$

Proof. By Lemma 3.10, we know that $\Lambda_{\alpha,z}$ is a finite sum of terms of the form $a(z)(x-z)^\beta\partial_x^\gamma e^{L_0^z}$. We recall that $a(z)$ is a function that itself and all its derivatives are bounded. Suppose $k_z(x,y)$ is the distribution kernel of $a(z)(x-z)^\beta\partial_x^\gamma e^{L_0^z}$ and let

$$K_s(x,y) := s^{-N}k_z(z+s^{-1}(x-z), z+s^{-1}(y-z)), \quad z = z(x,y).$$

By abuse of notation, we shall denote also by K_s the integral operator defined by K_s. It suffices to prove our theorem for K_s. Namely, it is enough to show that there exists a uniformly bounded family $\{\sigma_s\}_{s\in(0,1]}$ in $S^{-\infty}$ such that

$$K_s = \sigma_s(x,sD).$$

By lemma 4.6, we have that the distribution kernel of $\partial_x^\gamma e^{L_0^z}$ is of the form $\varsigma(z,x-y)$ for some $\varsigma \in S^{-\infty}$. More precisely $\varsigma(z,x)$ is the Fourier tranform of the function $(\imath\xi)^\gamma e^{\xi\cdot A(z)\cdot\xi}$. This gives

$$K_s(x,y) = a(z(x,y))s^{-|\beta|-N}(x-z(x,y))^\beta\varsigma(z(x,y),s^{-1}(x-y)) =:$$
$$a(z)s^{|\beta|-N}(x-z)^\beta\varsigma(z,s^{-1}(x-y)), \quad z = z(x,y).$$

Then by (4.9), $\sigma_s(x,\xi)$ is given by

$$\sigma_s(x,\xi) = \int_{\mathbb{R}^N} e^{-\imath y\cdot\xi}a(z)s^{-|\beta|-N}(x-z)^\beta\varsigma(z,s^{-1}y)dy, \quad z = z(x,x-y).$$

Let us substitute y with sy and let us denote

$$\varrho_s(x,\xi)=\int_{\mathbb{R}^N} e^{-\imath y\cdot\xi}a(z)s^{-|\beta|}(x-z)^\beta\varsigma(z,y)dy,\quad z=z(x,x-sy).$$

Then $\sigma_s(x,\xi)=\varrho_s(x,s\xi)$, so we need to show that ϱ_s is a bounded family in $S^{-\infty}$.

Notice that $a(z)\in S^0_{(1,0)}$ and $s^{-1}(x_j-z_j(x,x-sy))\in S^0_{(1,0)}$ and they form bounded families for $s\in[0,1]$, then by Lemma (4.5) the proof is complete. ☐

The next lemma is obvious

Lemma 4.8. *Let $\sigma(x,\xi)$ be a symbol in $S^{-\infty}$, then $s^k\sigma(x,s\xi)$ is a symbol in $S^{-k}_{1,0}$ uniformly bounded in $(0,1]$ with respect to s.*

Proof. Denote ∂_1 and ∂_2 the derivatives of $\sigma(x,\xi)$ with respect to the first and second variable respectively. Since $\sigma(x,\xi)\in S^{-\infty}$, of course $\sigma(x,\xi)\in S^{-k}_{1,0}$, thus for any α and β we have

$$|\partial_x^\alpha\partial_\xi^\beta\sigma(x,\xi)|\le C(1+|\xi|)^{-k-|\beta|}$$

Therefore,

$$\begin{aligned}&|\partial_x^\alpha\partial_\xi^\beta(s^k\sigma(x,s\xi))|=|s^{k+\beta}\partial_1^\alpha\partial_2^\beta\sigma(x,s\xi)|\\&\le Cs^{k+\beta}(1+|s\xi|)^{-k-|\beta|}\le\tilde C(1+|\xi|)^{-k-|\beta|}\end{aligned}$$

where $\tilde C$ does not depend on s. Thus $s^k\sigma(x,s\xi)$ is uniformly bounded in $S^{-k}_{1,0}$ for $s\in(0,1]$. ☐

We now obtain the main result of this subsection, the desired refined mapping property estimate by standard results from pseudodifferential operators theory.

Theorem 4.9. *For any $1<p<\infty$, any $r\in\mathbb{R}$,*

$$s^k\|\mathcal{L}_{s,\alpha}\|_{W^{r,p}\to W^{r+k,p}}\le C_{k,r,p},\tag{4.11}$$

for a constant $C_{k,r,p}$ independent of $t\in(0,1]$.

From (4.3), we immediately obtain the desired estimate on the principal part of the asymptotic expansion.

Corollary 4.10. *For each $1<p<\infty$, $r\in\mathbb{R}$, and any $f\in W^{r,p}$*

$$\int_{\mathbb{R}^N}\mathcal{G}_t^{[m,z]}(x,y)f(y)\,dy,$$

is uniformly bounded in $W^{r,p}$ for $t\in(0,1]$.

4.2. **Bound the error term.** In this subsection, we shall bound the error term $E^{t,z}_{d,n}$, which is the sum of two kinds of operators. The first is the one we discussed in the last subsection, which is actually a pseudodifferential operator and behaves well(Theorem (4.9)). The second is the operator $\Lambda_{\alpha,l}$ with either $\alpha \in \mathfrak{A}_{n+1,\ell}$ or for some $\alpha_i = n+1$. In order to bounded $E^{t,z}_{d,n}$, it suffices to obtain mapping properties of the latter operator. In this case, the operator will depend on t also. Generally, we do not know whether $\Lambda_{\alpha,l}$ is a pseudodifferential operator or not. However, we are going to show that $\Lambda_{\alpha,l}$ also behaves well, and has a similar mapping property with Theorem (4.9) but a little bit rougher. It turns out that this rough estimate is enough to give us the desired error control. In stead of pseudodifferential calculus applied in the last subsection, the main technique we shall use is the so called Riesz's Lemma(See for example [4])

Lemma 4.11. *(Riesz) Assume K is an integral operator with kernel $k(x, y)$, i.e.*

$$Ku(x) = \int_X k(x, y)u(y)d\mu(y)$$

where (X, μ) is a mearsure space. If $k(x, y)$ is measurable on $X \times X$ and

$$\int_X |k(x, y)|d\mu(x) \leq C_1, \int_X |k(x, y)|d\mu(y) \leq C_2 \tag{4.12}$$

for all y and for all x respectively. Then K is a bounded operator on $L^p(X, \mu)$ for each $p \in [1, \infty]$. Moreover,

$$\|K\| \leq C_1^{1/p} C_2^{1/q}$$

where q is the conjugate of p.

The main result of this subsection is as follows

Theorem 4.12. *Let z be admissible, $r \geq 0$. Then we have*

$$s^{k+r}\|\mathcal{L}_{s,\alpha}\|_{W^{r,p}\to W^{r+k,p}} \leq C_{k,r,p}, \tag{4.13}$$

Note that the main difference between Theorem (4.9) and Theorem (4.12) is the additional r. This result is rougher, but as we mentioned before, it is enough to give us the main theorem of this paper.

Before we prove this theorem, we first recall some notations and prove a lemma. We shall denote by $W^{r,p}_a = W^{r,p}_{a,w}$ as before, where w is the center of the weight $\langle x\rangle_w = \langle x - w\rangle$ used to define our exponentially weighted Sobolev spaces (2.14). We shall write $L^p_a = W^{0,p}_a$. The following lemma is a special case of Theorem (4.12).

Lemma 4.13. *Assume that $z : \mathbb{R}^N \times \mathbb{R}^N$ is admissible. For any α, any $1 < p < \infty$, $k \in \mathbb{Z}_+$, $r \geq 0$, and $a \in \mathbb{R}$*

$$s^k \|\mathcal{L}_{s,\alpha}\|_{L^p_a \to W^{k,p}_a} \leq C_{k,p}, \tag{4.14}$$

for a constant $C_{k,p}$ independent of $t \in (0,1]$, independent of a in a bounded set, and independent of the center of the weight that defines the weighted Sobolev spaces.

Proof. The proof will be mainly an application of the Riesz's Lemma. Because of the reason we mentioned before, we may assume that $a = 0$. Recall that

$$\mathcal{L}_{s,\alpha}(x,y) = s^{-N}\Lambda_{\alpha,z}(z + s^{-1}(x-z), z + s^{-1}(y-z))$$

is the kernel of the operator $\mathcal{L}_{s,\alpha}$, where $z = z(x,y)$. Then by Riesz's Lemma it suffices to show that for any multi-index γ with $|\gamma| \leq k$,

$$\int_{\mathbb{R}^N} s^{|\gamma|}|\partial_x^\gamma \mathcal{L}_{s,\alpha}(x,y)|dy \leq C_1, \int_{\mathbb{R}^N} s^{|\gamma|}|\partial_x^\gamma \mathcal{L}_{s,\alpha}(x,y)|dx \leq C_2 \tag{4.15}$$

where the constants C_1 and C_2 should be independent of x and y respectively. Generally, we need to estimate the growth rate of $s^{-N}\partial_x^\gamma \Lambda_{\alpha,z}(z+ s^{-1}(x-z), z + s^{-1}(y-z))$. with respect to x and y. We need to use weighted Sobolev spaces introduced in (2.14). Recall that the mapping properties between the weighted Sobolev spaces are uniform in terms of the weight center, thus we can choose z as the weight center. Notice that $\partial_x^\gamma \mathcal{L}_{s,\alpha}(x,y)$ is the sum of terms of the form

$$s^{-N-j}\partial_x^\beta \partial_z^{\beta'} \partial_y^{\beta''} \Lambda_{\alpha,z}(z + s^{-1}(x-z), z + s^{-1}(y-z)) \cdot \xi(z) \tag{4.16}$$

where $j \leq |\gamma|$ and $\xi(z)$ is the product of derivatives of z with respect to x, it is bounded as z is admissible. While

$$\begin{aligned} |\partial_x^\beta \partial_z^{\beta'} \partial_y^{\beta''} \Lambda_{\alpha,z}(x,y)| &= |< \partial^\beta \delta_x, \partial_z^{\beta'} \Lambda_{\alpha,z} \partial^{\beta''} \delta_y > | \\ &\leq C\|\partial^\beta \delta_x\|_{H^{-q}_{-a-\epsilon}} \|\partial_z^{\beta'} \Lambda_{\alpha,z}\|_{H^{-q}_{-a} \to H^q_{-a-\epsilon}} \|\partial^{\beta''} \delta_y\|_{H^{-q}_{-a}} \end{aligned} \tag{4.17}$$

Next we shall estimate the three norms at the right hand side of the above estimate. For each multi-index β, $\partial_x^\beta \in H^{-q}(\mathbb{R}^N)$ as long as $q > N + |\beta|$. Therefore, if we choose z as the base point and $q > N + |\beta|$. Then for all $a \in \mathbb{R}$ and $\epsilon > 0$

$$\|\partial^\beta \delta_x\|_{H^{-q}_{-a-\epsilon}} := \|e^{-(a+\epsilon)<x-z>}\partial^\beta \delta_x\|_{H^{-q}} \leq Ce^{-(a+\epsilon)<x-z>}$$

and

$$\|\partial^{\beta''} \delta_y\|_{H^{-q}_{-a}} := \|e^{-a<y-z>}\partial^{\beta''} \delta_y\|_{H^{-q}} \leq Ce^{-a<y-z>}$$

For the second term $\|\partial_z^{\beta'}\Lambda_{\alpha,z}\|_{H^{-q}_{-a}\to H^{q}_{-q-\epsilon}}$, since all the coefficients and their derivatives of $L(t)$ are bounded, $\partial_z^{\beta'}\Lambda_{\alpha,z}$ will satisfy the same mapping properties as $\Lambda_{\alpha,z}$. Thus by Corollary (4.3),

$$\|\partial_z^{\beta'}\Lambda_{\alpha,z}\|_{H^{-q}_{-a}\to H^{q}_{-a-\epsilon}} \le Ce^{\epsilon<z-x>}$$

Now get back to (4.17), we have

$$|\partial_x^{\beta}\partial_z^{\beta'}\partial_y^{\beta''}\Lambda_{\alpha,z}(x,y)| \le Ce^{\epsilon<z-x>-a<y-z>-(a+\epsilon)<x-z>} = Ce^{-a<y-z>-a<x-z>}$$

Therefore, we obtain

$$\begin{aligned} &|s^{-N-j}\partial_x^{\beta}\partial_z^{\beta'}\partial_y^{\beta''}\Lambda_{\alpha,z}(z+s^{-1}(x-z),z+s^{-1}(y-z))\cdot\xi(z)| \\ &\le Cs^{-N-|\gamma|}e^{-a<s^{-1}(y-z)>-a<s^{-1}(x-z)>} \\ &\le Cs^{-N-|\gamma|}e^{-a<s^{-1}(y-x)>} \end{aligned} \tag{4.18}$$

In the last inequality, we used the triangle inequality $< y-z > + < x-z > \ge < y-x >$. Then after the change of variable $\lambda = \frac{y-x}{s}$, we find that (4.15) holds. The proof is complete. □

proof of Theorem (4.12)*:* Notice that $W^{r,p}$ is compactly supported in L^p for any $r \ge 0$(for non-integer r, it is a consequence of the interpolation argument). Then if we consider $\mathcal{L}_{s,\alpha}$ as an operator from L^p to $W^{r+k,p}$ in stead of from $W^{r,p}$ to $W^{r+k,p}$, the result follows directly from Lemma (4.13). □

Recall that $\mathcal{E}_t^{[m,z]}$ is the sum of two kinds of operators we mentioned before, then a direct corollary of Theorem (4.12) and Theorem (4.9) is the following

Corollary 4.14. *Assume z is admissible and $r \ge 0$, then $\mathcal{E}_t^{[m,z]}$ satisfies the following mapping property*

$$\|\mathcal{E}_t^{[m,z]}\|_{W^{r,p}\to W^{r+k,p}} \le Cs^{-r-k} \tag{4.19}$$

Surprisingly, it turns out that the r at the right hand side of equation (4.19) is redundant, we can get rid of it to obtain a more refined estimate.

Theorem 4.15. *Assume z is admissible and $r \ge 0$, then $\mathcal{E}_t^{[m,z]}$ satisfies the following mapping property*

$$\|\mathcal{E}_t^{[m,z]}\|_{W^{r,p}\to W^{r+k,p}} \le Cs^{-k} \tag{4.20}$$

Proof. Recall that as long as $d \geq n > m$, $\mathcal{G}_t^{[m]}(x,y)$ does not depend on d and n. Thus as the difference

$$\mathcal{E}_t^{[m,z]}(x,y) = U(1)(z + \frac{x-z}{\sqrt{t}}, z + \frac{y-z}{\sqrt{t}}) - \mathcal{G}_t^{[m,z]}(x,y)$$

also does not depend on n and d. In the expansion (3.19), we expand it to much more terms, specifically, such that $M \geq m + r - 1$. Then by Theorem (4.9) and Corollary (4.14)

$$\|\mathcal{E}_t^{[m,z]}\|_{W^{r,p}\to W^{r+k,p}} \leq \sum_{\ell=\mu+1}^{M} s^{\ell-m-1} \sum_{k=m+1}^{\ell} \sum_{\alpha\in\mathfrak{A}_{k,\ell}} \|\mathcal{L}_{\alpha,z}\|_{W^{r,p}\to W^{r+k,p}}$$
$$+ s^{M+1-m}\|\mathcal{E}_t^{[M,z]}\|_{W^{r,p}\to W^{r+k,p}} \leq Cs^{-k}(1 + s^{M+1-m}s^{-r-k}) \leq Cs^{-k}.$$

□

This completes the main Theorem (1.1) of this paper.

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