Partial Uniqueness and Obstruction to Uniqueness in Inverse Problems for Anisotropic Elastic Media

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Abstract. We consider the inverse problem of identifying the density and elastic moduli for three-dimensional anisotropic elastic bodies, given displacement and traction measurements made at their surface. These surface measurements are modeled by the dynamic Dirichlet-to-Neumann map on a finite time interval. For linear or non-linear anisotropic hyperelastic bodies we show that the displacement-to-traction surface measurements do not change when the density and elasticity tensor in the interior are transformed tensorially by a change of coordinates fixing the surface of the body to first order. Our main tool, a new approach in inverse problems for elastic media, is the representation of the equations of motion in a covariant form (following Marsden and Hughes, 1983) that preserves the underlying physics.

In the case of classical linear elastodynamics we then investigate how the type of anisotropy changes under coordinate transformations. That is, we analyze the orbits of general linear, anisotropic elasticity tensors under the action by pull-back of diffeomorphisms that fix the surface of the elastic body to first order, and derive a pointwise characterization of parts of the orbits under this action. For example, we show that the orbit of isotropic elastic media, at any point in the body, consists of some transversely isotropic and some orthotropic elastic media. We then derive the first uniqueness result in the inverse problem for anisotropic media using surface displacement-traction data: uniqueness of three elastic moduli for tensors in the orbit of isotropic elasticity tensors.

1. Introduction

In this paper we consider the inverse problem of identifying elastic moduli for three-dimensional anisotropic elastic bodies, given displacement and traction measurements made at their surface. For the dynamic inverse

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problem, we need to consider an additional material parameter, the density. We study both linear and non-linear elastodynamics, though our main focus is on classical linear elasticity, where the elastic moduli can be defined independently of the deformation.

We are concerned with the uniqueness question, that is, whether the same collection of surface measurements can correspond to different values for the parameters. We address here a particular type of nonuniqueness, which arises when the underlying partial differential equations, describing the propagation of elastic disturbances in the body, possess a certain invariance under coordinate transformations. This natural "obstruction to uniqueness" is well known in the inverse problem of identifying material parameters from acoustic measurements in anisotropic media.

Acoustic wave propagation in anisotropic media is modeled by solutions of scalar partial differential equations of the form:

$$\frac{\partial^2}{\partial t^2}\, u \;-\; \sum_{i,j=1}^{3} \frac{1}{\sqrt{\det \mathbf{G}}}\, \frac{\partial}{\partial x^i} \left(\sqrt{\det\, \mathbf{G}}\;\, G^{ij}\, \frac{\partial u}{\partial x^j} \right) \;=\; 0, \tag{1}$$

where $\mathbf{G} = [G_{ij}]$ is a positive-definite symmetric matrix function of the spatial coordinates x_i, $i = 1, 2, 3$, and $\mathbf{G}^{-1} = [G^{ij}]$. The body is represented by an open region $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$, and we examine this equation for $x = (x_1, x_2, x_3) \in \Omega$. $\mathbf{G}$ can be interpreted as a Riemannian metric on $\mathbb{R}^3$, so that (1) becomes the wave equation

$$(\partial_t^2 \;-\; \Delta_{\mathbf{G}})u \;=\; 0, \tag{2}$$

with $\Delta_{\mathbf{G}}$ the Laplace-Beltrami operator associated with the metric $\mathbf{G}$ on Ω.

The inverse problem for acoustic media consists in determining the metric $\mathbf{G}$ from the surface measurements, in the form of the Dirichlet-to-Neumann map $\Lambda_{\mathbf{G}}$, which relate the value at the boundary of Ω of solutions to (1) to the value of their respective normal derivatives at the boundary:

$$\Lambda_{\mathbf{G}}(\, u\big|_{\partial\Omega}) \;=\; (\frac{\partial u}{\partial \nu_{\mathbf{G}}})\big|_{\partial\Omega} \;=\; \sum_{i,j=1}^{3} (\nu^i\, G^{ij}\, \frac{\partial u}{\partial x^j})\big|_{\partial\Omega}. \tag{3}$$

Above ν is the Euclidean unit outer normal vector to Ω, and $\nu_{\mathbf{G}}$ the corresponding normal with respect to $\mathbf{G}$. (For a general introduction to inverse boundary problems, we refer, for example, to [41].) Uniqueness for the inverse problem may be stated in terms of the injectivity of the map $\Lambda_{\mathbf{G}}$ with respect to $\mathbf{G}$. It is important to note that, although equation (2) is linear, $\Lambda_{\mathbf{G}}$ depends on $\mathbf{G}$ in a highly non-linear fashion, a feature common to inverse problems.

The natural obstruction to uniqueness holds for the acoustic inverse problem since for any sufficiently regular diffeomorphism $\psi : \overline{\Omega} \to \overline{\Omega}$ with

$\psi(x) = x$ for $x \in \partial\Omega$, we have that

$$\Lambda_{\psi^*\mathbf{G}} \;=\; \Lambda_{\mathbf{G}}. \tag{4}$$

In other words, the best uniqueness result possible in the inverse problem for the operator $\partial_t^2 - \Delta_{\mathbf{G}}$ is the unique determination of $\mathbf{G}$ by the Dirichlet-to-Neumann map, up to the pullback ψ^* by a diffeomorphism ψ that fixes the boundary. (Informally, the pull-back of a tensor $T(X)$ is the tensor of the same type, $\psi^* T(x)$, obtained by making the coordinate change $x = \psi^{-1}(X)$. For background material, we refer to [21].) In fact, this best possible result, uniqueness up to pull-back, has been established for the scalar wave operator, under certain conditions; see [38], [2], [36], [5], [15], [28], and [37].

It is natural to ask whether this result extends to more general models, for example, the system for elastodynamics. The uniqueness question for general anisotropic elastic media is significantly more difficult than that for the scalar wave equation, due mainly to greater complexity in the description of the rays, paths along which disturbances propagate through the medium. We show here, however, that several aspects of the uniqueness question may be addressed in the systems case without reference to rays.

Surface measurements for the elasticity problem, in the form of the correspondence between displacements of the surface of the elastic body and surface tractions that cause them, are given by the dynamic Dirichlet-to-Neumann map $\Lambda_{\mathcal{O}}$ associated with the linear or non-linear operator for elastodynamics. (See Section 2, equation (12), for classical linear elasticity and Section (3) for non-linear elasticity.) Here $\mathcal{O}$ represents the material parameters, which are the density function ρ and the elasticity tensor function $\mathbf{C}$ in the classical case.

Our main result (cf. Theorem 1), analogous to (4), states that the natural obstruction to uniqueness holds in the inverse problem for elastodynamics:

$$\Lambda_{\psi^*\mathcal{O}} \;=\; \Lambda_{\mathcal{O}}, \tag{5}$$

where ψ is any diffeomorphism of Ω that fixes the boundary to first order. Although the focus later in this paper is on results for linear elastodynamics, our approach to the proof of Theorem 1 is based on a linearization of the non-linear equations of motion, a setting especially appropriate for the study of change of coordinates. In fact, our approach is based on a covariant form for the non-linear equations of motion that respects the underlying physical laws for elastodynamics (following work of Marsden and Hughes [21]). By covariance we mean an intrinsic tensorial representation independent of any local coordinate system. This method is rather general and it applies to other problems as well, for example to unbounded elastic media. The use of this covariant form of the system of differential equations appears to be new in inverse problems for elastic media.

A consequence of the natural obstruction (5) is that the Dirichlet-to-Neumann map does not change when elastic media are transformed by any change of coordinates that fixes the boundary to first order. It follows that uniqueness results for classes of elastic media may be extended partially to any anisotropic elastic media in the orbits of those classes. We conclude, for example, in Section 2.1 that Rachele's uniqueness results [29], [30], [31], [32] for isotropic elastodynamics (with or without residual stress) extend to certain anisotropic elastic media. Therefore, our next aim is to describe the orbits of density functions $\rho(x)$ and elasticity tensors $\mathbf{C}(x)$ at a given but arbitrary point $x \in \Omega,$ under the action by pull-back of diffeomorsphims that fix the boundary to first order. (The first-order assumption seems necessary for treating non-linear elasticity.)

We write the elasticity tensor $\mathbf{C}$ in a canonical form, obtained via the harmonic decomposition and then a Cartan decomposition of the fourth-order harmonic part, following Forte and Vianello [8]. We call this a *harmonic-Cartan decomposition*. The type of anisotropy of an elasticity tensor may be described in terms of the vanishing of certain components of a harmonic-Cartan decomposition. We then give a description of the types of anisotropy of parts of the orbits under the action by pull-back of a diffeomorphism ψ, and a complete characterization in the isotropic case. We employ the mathematical software Maple to compute the harmonic-Cartan decomposition of the transformed tensor $\psi^*\mathbf{C}$. (See Section 5.) For example, we show that at any point of Ω the orbit of isotropic elastic media consists of isotropic, some transverse isotropic, and some orthotropic elastic media. We also observe that *pointwise* some ψ can preserve the type of anisotropy for certain classes of elastic media, e.g., for transversely isotropic media.

Global constraints might place additional limits on the types of anisotropy within the orbit of any particular elastic media. In particular, it may be possible to establish that the "natural obstruction" does not occur within certain restricted classes of of anisotropic media. The approach taken here to describe these orbits does not extend directly to the global setting, as it depends on diagonalization of the symmetric part (i.e., the *stretch factor* of the polar decomposition) of the differential $D\psi$. It is interesting to note that composite elastic media, elastic media for which the type of anisotropy can vary from point to point, may lie on the orbits of non-composite elastic media.

The only uniqueness result known to the authors for the fully three-dimensional inverse parameter identification problem for nonhomogeneous anisotropic elastic media using Dirichlet-to-Neumann-type data is that of Nakamura, Tanuma, and Uhlmann [23] who prove that, in the case of transverse isotropic elasticity, the material properties of bounded elastic objects are uniquely determined by static measurements made at the surface. Nakamura, Tanuma, and Uhlmann use a layer-stripping approach (applicable to static problems).

The case of isotropic elasticity with residual stress has been studied by several authors. Robertson shows that the Dirichlet-to-Neumann map uniquely determines the residual stress tensor and one of the Lamé parameters at the boundary [33], and establishes uniqueness in the interior [34] for the linearized problem using Man's model [20] for residual stress. Hansen and Uhlmann [10] use a microlocal analytic approach to the study of reflection of singularities to prove that the scattering relation at the surface is determined by the Dirichlet-to-Neumann map, which extends Rachele's result [32] to the case that caustics are present. Isakov, Nakamura, and Wang [24], [14], consider the unique continuation property for solutions of the Cauchy problem (see also [16]), and use it to study the inverse problem of identifying inclusions or cavities in the body. Lin and Wang [17] exploit Carleman estimates and unique continuation to prove unique identification of the density by a single boundary measurement. Also, recently Ivanov, Man, and Nakamura, using the model for residual stress by Man [20], have shown that traction-displacement measurements uniquely determine the residual stress and its gradient at the surface of an unbounded, layered medium, and near the surface if the stress is in diagonal form [13], while Rachele [32] proves unique determination of the density, isotropic elasticity parameters, and the residual stress tensor at the surface in the case of a bounded elastic object, using Hoger's model [11], [12] for residual stress.

We mention other results on obstruction to uniqueness in inverse problems using Dirichlet-to-Neumann-type data. In a 2-dimensional static problem for orthotropic elastic media G. Nakamura and K. Tanuma [22] show that the Dirichlet-to-Neumann map does not uniquely determine the elastic parameters at the boundary. Also, M. Belishev [1] describes an obstruction to uniqueness in the dependence of the Dirichlet-to-Neumann map on lower-order parameters of an (isotropic) wave equation. In addition, Romanov [35] proves nonuniqueness in an inverse problem for anisotropic Maxwell's equations in the case that the permittivity ϵ and permeability μ are diagonal matrices.

We emphasize that very little is known about the orbits of elastic media, even under the action of linear transformations. Canonical forms for elasticity tensors under linear transformations in the two-dimensional case, and in the case of planar displacements of three-dimensional objects, were considered by P. Olver in [25] and [26] using Cartan's method of equivalence. Lodge [18] gives an explicit linear change of coordinates from transversely isotropic (or orthotropic) to isotropic for the equations for elastostatics.

To extend the non-uniqueness result (4) for the scalar wave equation (2) to elastic media we first describe (4) in terms of the covariance of the wave operator, i.e.,

$$\psi^*([\partial_t^2 - \Delta_{\mathbf{G}}]u) \;=\; [\partial_t^2 \;-\; \Delta_{\psi^*\mathbf{G}}](\psi^* u),$$

where ψ^* is the pullback under the diffeomorphism ψ. We then view the elastic body as a Riemannian manifold endowed with a smooth Riemannian metric $\mathbf{G}$ which we may vary as we vary the other material parameters. We need to consider a general metric on Ω since ψ will not respect the Euclidean structure in general.

We write the differential operator for hyperelastic media in a covariant form (see Marsden and Hughes [21]) so that a solution U of the elastodynamics system, with respect to a metric $\mathbf{G}$ and material parameters $\mathcal{O}$, transforms to a solution $\psi^*U = U \circ \psi$ with respect to $\psi^*\mathbf{G}$ and transformed parameters $\psi^*\mathcal{O}$. (See Theorem 1.) We conclude that the elastic bodies $(\Omega, \mathbf{G}, \mathcal{O})$ and $(\Omega, \psi^*\mathbf{G}, \psi^*\mathcal{O})$ cannot be distinguished by surface measurements given by the associated Dirichlet-to-Neumann maps if ψ preserves the boundary, that is, (5) holds. In fact, in the classical linear case this obstruction to uniqueness may be stated with respect to the Euclidean metric:

$$\Lambda_{\mathbf{e},\sqrt{\det \mathbf{G}}\ \mathcal{O}} f \;=\; \Lambda_{\mathbf{e},\det(D\psi)\sqrt{\det \mathbf{G}}\ \psi^*\mathcal{O}} f.$$

To write the elasticity equations in covariant form in a way that respects the underlying physics, we turn to the balance laws (conservation of mass, balance of momentum, balance of moment of momentum, and balance of energy) that describe the behavior of the elastic medium. In the case of non-linear elastodynamics Marsden and Hughes [21], Theorem 4.13, show that the balance laws, together with the hyperelasticity condition (that is, the existence of an internal energy E), are equivalent to the covariance of the internal energy. The balance laws (in particular, the equations of motion) may be written in either the so-called *Eulerian* (or *spatial*) *coordinates* x on the ambient space $\mathcal{S}$ (here $\mathcal{S} = \mathbb{R}^3$), or in *Lagrangian* (or *material* or *referential*) coordinates X on Ω, at least for sufficiently regular deformations. (See Section 3.) Here we use Lagrangian coordinates in order to describe changes in the material properties of the body in an intrinsic way, while avoiding the construction of global transformations of the ambient space.

In Lagrangian coordinates, if the energy depends functionally on the Cauchy-Green deformation tensor $\mathbf{C}$ alone, then E is covariant only if the elastic medium is isotropic. To include anisotropic media in the analysis, therefore, we view E as a function of $\mathbf{C}$ and structural tensors $\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_m$, which express the specific anisotropy of the body. (See the proof of Theorem 1 in Section 4.)

For classical linear elasticity the (Lagrangian) covariant form for the equations of motion derived from the balance laws is given by [21]

$$P_{\mathbf{G},\mathsf{C},\rho}\,\mathbf{U} \;=\; \rho\frac{\partial^2\mathbf{U}}{\partial t^2} \;-\; \mathrm{DIV}_{\mathbf{G}}(\mathsf{A}\cdot\nabla\mathbf{U}) \;=\; 0, \tag{6}$$

where $\rho(X)$ is the mass density in the reference configuration, A depends on the elasticity tensor C via

$$\mathsf{A}^{aA}{}_b{}^B \;=\; 2\mathsf{C}^{EAFB}(D\iota)^a{}_E(D\iota)^c{}_F\delta_{bc}, \tag{7}$$

ι is an embedding of Ω in $\mathbb{R}^3$, $\mathbf{U}$ is the displacement vector from a material point X in the reference configuration to its new location after time t, $\mathrm{DIV}_{\mathbf{G}}(\mathsf{A}\cdot\nabla\mathbf{U})$ is the divergence on $(\Omega,\mathbf{G})$ of the two-point tensor $\mathsf{A}\cdot\nabla\mathbf{U}$, and ∇ is the covariant derivative induced by $\mathbf{G}$. Informally, in a two-point tensor some components behave like a tensor over the reference space Ω and others like a tensor over the ambient space $\mathcal{S}$.

We show in the proof of Theorem 1 that the equations of motion in local coordinates are given by

$$\rho\frac{\partial^2 U^a}{\partial t^2} \;-\; \frac{1}{\sqrt{\det\mathbf{G}}}\,\frac{\partial}{\partial X^A}\left(\sqrt{\det\,\mathbf{G}}\;\;\mathsf{A}^{aA}{}_b{}^B\frac{\partial U^b}{\partial X^B}\right) \;=\; 0, \tag{8}$$

where $\det\,\mathbf{G}$ denotes the determinant of the metric tensor G_{ij}. Here an uppercase (i.e., capitalized) index denotes the leg of a tensor in the material setting, while a lowercase index refers to the spatial setting. (See Section 3 for definitions and notation.) No lower-order-terms associated to the Christoffel symbols of $\mathbf{G}$ appear, since ∇ acts on the two-point vector $\mathbf{U}$ as if it were a scalar.

We emphasize that the equations under a change of (Lagrangian) coordinates ψ have the same form but with each of $\mathbf{G},\mathsf{C}$, and ρ replaced by its pullback.

To illustrate that not all covariant forms of a differential equation respect the underlying physics, we turn to the following example. The Navier-Stokes equations of fluid mechanics

$$\begin{cases}\partial_t u - \Delta u + u\cdot\nabla u - \nabla p \;=\; 0,\\ \operatorname{div} u \;=\; 0,\end{cases} \tag{9}$$

are in covariant form if Δ is interpreted as the Laplace-Beltrami operator and ∇ is the covariant derivative. In (9) u is a vector field describing the velocity of a fluid particle, and p is a scalar field representing the pressure.

However the correct form of the equation as derived from conservation of momentum and mass is [7]:

$$\begin{cases}\partial_t u - \operatorname{div} S + u\cdot\nabla u - \nabla p \;=\; 0,\\ \operatorname{div} u \;=\; 0.\end{cases} \tag{10}$$

Above, $S = (\nabla u + \nabla^T u)/2$ is the deformation tensor and, for divergence-free vector fields u on a Riemannian manifold, $\operatorname{div} S = \Delta\, u + 2\,\mathrm{Ric}\ u$, where Ric is the Ricci tensor.

The paper is organized as follows. In Section 2 we state the main results. In Section 3 we present notation and definitions, and write the equations of motion in covariant form for the non-linear, linearized, and classical linear cases. Obstruction to uniqueness (Theorem 1) is proved in Section 4. In Section 5 we prove Theorem 11 on the pointwise characterization of orbits in some cases.

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2. Main Results

We say that a three-dimensional linearly elastic object (represented by $\Omega \subseteq \mathbb{R}^3$) is *admissible* if the object is bounded, with smooth boundary; has smooth, positive density ρ; and has smooth elasticity tensor C which has the symmetry properties

$$\mathsf{C}^{ABCD} = \mathsf{C}^{CDAB} = \mathsf{C}^{BACD} = \mathsf{C}^{ABDC} \tag{11}$$

and is (uniformly) strongly elliptic on Ω, i.e., there is a constant $c > 0$ so that for any $X \in \Omega$ $\mathsf{C}^{ABCD}(X)V^A W^B V^C W^D \geq c|\mathbf{V}|^2|\mathbf{W}|^2$ for all vectors $\mathbf{V}, \mathbf{W} \in T^*_X\Omega$. (The symmetry condition $\mathsf{C}^{ABCD} = \mathsf{C}^{CDAB}$ is equivalent to hyperelasticity in the linear case.) Throughout the paper we follow the convention of summing over repeated indices.

In the usual Euclidean setting the strong ellipticity condition ensures well posedness of the system of equations for classical linear elastodynamics, that is, the existence, uniqueness, and regularity of solutions to the initial-boundary-value problem (13). (See [21], pages 345, 368, 369.) This result is essentially a consequence of energy estimates and Gärding's inequality in the Sobolev space $H^1(\mathbb{R}^3)$. If C is positive-definite (or *pointwise stable* [21, p. 239, 350]), that is, $\mathsf{C}(X)[\mathrm{E},\mathrm{E}] \geq c\,||\mathrm{E}||^2$, $c > 0$, for any $X \in \Omega$ and for any symmetric, rank-2 tensor $\mathrm{E} \in T_X\Omega \otimes T_X\Omega$, then, in addition, the system (13) is dynamically stable. In fact, in this case (with displacement boundary conditions) $|U| = (\mathsf{C}[\nabla U, \nabla U])^{1/2}$ defines an equivalent norm in $H^1(\mathbb{R}^3)$, and, therefore, a sharp version of Gärding's inequality holds.

For non-linear elasticity we also assume that there exist unique solutions ϕ_t of the initial-boundary-value problem (32) for regular-enough Dirichlet data f. Some partial results for well-posedness of the non-linear initial-boundary-value problem (32) are discussed, for example, in [21], p. 401, 406.

We model the behavior of any admissible nonhomogeneous elastic object $\Omega \subseteq \mathbb{R}^3$ by solutions of the equations of motion, e.g., $P_{\mathbf{G},\mathcal{O}}\mathbf{U} = 0$ for classical linear elastodynamics, where $\mathcal{O}$ (given by (C, ρ) in this case) represents the material properties of the object.

The surface measurements used to determine the material parameters $\mathcal{O}$ are given by the correspondence between forces exerted on the surface and the resulting surface displacements, taken over a finite period of time. These surface measurements are represented by the dynamic Dirichlet-to-Neumann map $\Lambda_{\mathbf{G},\mathcal{O}}$. In the case of classical linear elasticity (6) we define the Dirichlet-to-Neumann map by

(12)

$$(\Lambda_{\mathbf{G},\mathsf{C},\rho} f)^a \;=\; \langle \mathsf{A} \cdot \nabla \mathbf{U},\; \boldsymbol{\nu}_{\mathbf{G}} \rangle^a \; dS_{\mathbf{G}} \;=\; \mathsf{A}^{aA}{}_b{}^B \; (\nabla \mathbf{U})^b{}_B \; \mathbf{G}_{AC} \; \boldsymbol{\nu}_{\mathbf{G}}^C \; dS_{\mathbf{G}},$$

where $dS_{\mathbf{G}}$ is the area form on $\partial\Omega$ induced by $\mathbf{G}$, $\boldsymbol{\nu}_{\mathbf{G}}$ is the unit outer normal vector to the boundary $\partial\Omega$ with respect to $\mathbf{G}$, A is defined in (7), and $\mathbf{U}$ solves the initial-boundary-value problem

$$\begin{aligned}
&\text{(13a)} \qquad & P_{\mathbf{G},\mathsf{C},\rho}\mathbf{U} &= \rho\,\frac{\partial^2 \mathbf{U}}{\partial t^2} - \mathrm{DIV}_{\mathbf{G}}(\mathsf{A} \cdot \nabla \mathbf{U}) = 0 & &\text{on } \Omega,\\
&\text{(13b)} \qquad & \mathbf{U}(X,t) &= f(X,t) & &\text{for } X \in \partial\Omega,\\
&\text{(13c)} \qquad & \mathbf{U}(X,t) &= 0 & &\text{for } t = 0,\\
&\text{(13d)} \qquad & \frac{\partial \mathbf{U}}{\partial t}(X,t) &= 0 & &\text{for } t = 0,
\end{aligned}$$

with Dirichlet boundary data f. The Dirichlet-to-Neumann map is defined by (35) and (40) in the cases of non-linear and linearized elastodynamics, respectively.

Given any diffeomorphism ψ of $\bar{\Omega}$ that fixes the boundary to first order, i.e., $\psi \in \Psi$, where

$$\Psi \;=\; \{\; \psi \;:\; \bar{\Omega} \to \bar{\Omega} \;\;|\;\; \psi|_{\partial\Omega} \;=\; id, \quad D\psi|_{\partial\Omega} \;=\; I \;\},$$

we define $\psi^*\mathcal{O}$ to be the transformed material parameters under ψ. That is, in the classical linear case $\psi^*\mathcal{O} = (\psi^*\mathsf{C}, \psi^*\rho)$. The transformed parameters $(\psi^*\mathbf{G}, \psi^*\mathsf{C}, \psi^*\rho)$ give rise to an admissible hyperelastic object.

In the proof of Theorem 1 (in Section 4) we first show that the equations of motion are covariant in the non-linear case, and we then linearize following [21]. Consequently, if $\mathbf{U}$ is a solution of the initial-boundary-value problem (13) for linear elastodynamics with Dirichlet boundary data f, metric $\mathbf{G}$, and material parameters $\mathcal{O}$, then $\psi^*\mathbf{U} = \mathbf{U} \circ \psi$ is a solution of (13) with metric $\psi^*\mathbf{G}$ and material parameters $\psi^*\mathcal{O}$. We conclude, given the definition (12) of the Dirichlet-to-Neumann map and the fact that ψ is the identity to first order at the boundary, that the Dirichlet-to-Neumann maps for $(\mathbf{G}, \mathcal{O})$ and $(\psi^*\mathbf{G}, \psi^*\mathcal{O})$ agree. That is, the two elastic objects $(\Omega, \mathbf{G}, \mathcal{O})$, $(\Omega, \psi^*\mathbf{G}, \psi^*\mathcal{O})$ cannot be distinguished by boundary measurements.

Theorem 1 (Obstruction to uniqueness for hyperelastic media)**.** *Consider any admissible (linear, linearized, or non-linear) elastic body* $(\Omega, \mathbf{G}, \mathcal{O})$ *with material parameters given by* $\mathcal{O}$. *Then for any* $\psi \in \Psi$ *and the transformed parameters* $\psi^*\mathcal{O}$ *under* ψ

$$\Lambda_{\mathbf{G},\mathcal{O}} \;=\; \Lambda_{\psi^*\mathbf{G},\psi^*\mathcal{O}}.$$

In the classical linear case the obstruction may be stated with respect to the Euclidean metric:

Corollary 2 (Obstruction for linear elastodynamics with respect to the Euclidean metric)**.** *Consider any admissible linearly elastic body* $(\Omega, \mathbf{G}, \mathsf{C}, \rho)$. *Then*

$$(14)\qquad \begin{aligned} (\Lambda_{\mathbf{G},\mathsf{C},\rho} f\,)^a \;&=\; \mathsf{A}^{aA}{}_b{}^B\ (\nabla \mathbf{U})^b{}_B\ \delta_{AC}\ \boldsymbol{\nu}^C\ \sqrt{\det\,\mathbf{G}}\ \ dS \\ &=\; (\Lambda_{\mathbf{e},\sqrt{\det\mathbf{G}}\ (\mathsf{C},\rho)}\, f\,)^a, \end{aligned}$$

where dS *is the area form on* $\partial\Omega$ *induced by the Euclidean metric* $\mathbf{e}$, *and* $\boldsymbol{\nu}$ *is the unit outer normal vector to the boundary* $\partial\Omega$ *with respect to* $\mathbf{e}$.

In fact, for any $\psi \in \Psi$

$$(15)\qquad \Lambda_{\mathbf{e},\sqrt{\det\mathbf{G}}\,(\mathsf{C},\rho)} \;=\; \Lambda_{\mathbf{e},(\det D\psi)\sqrt{\det\mathbf{G}}\ (\psi^*\mathsf{C},\psi^*\rho)}.$$

Proof. The first equality in (14) follows from an application of the divergence theorem (e.g., [39], p. 127). In fact, on the one hand,

$$dS_{\mathbf{G}} \;=\; \sqrt{\det\mathbf{G}} < \boldsymbol{\nu}, \boldsymbol{\nu} >_{\mathbf{G}}^{1/2}\ dS,$$

where $< \boldsymbol{\nu}, \boldsymbol{\nu} >_{\mathbf{G}} = G_{AB}\nu^A\nu^B$. On the other, the unit (co)normal to $\partial\Omega$ w.r.t. $\mathbf{G}$ is related to $\boldsymbol{\nu}$ by

$$(\nu_{\mathbf{G}})_A \;= < \boldsymbol{\nu}, \boldsymbol{\nu} >_{\mathbf{G}}^{-1/2}\ \nu_A.$$

The second equality is a consequence of (8), which may be viewed as a global representation of the equations of motion (6) in Euclidean coordinates. Theorem 1 then directly implies (15). □

2.1. **Partial Uniqueness for Anisotropic Linearly Elastic Media.** We consider the orbits $\mathfrak{O}_{(\mathbf{G},\mathsf{C},\rho)} = \{\, (\psi^*\mathbf{G}, \psi^*\mathsf{C}, \psi^*\rho) \mid \psi \in \Psi \,\}$ of parameters $(\mathbf{G}, \mathsf{C}, \rho)$ of linearly elastic media under the action of Ψ. Below we prove uniqueness of some parameters of anisotropic, linearly elastic media in the orbit of isotropic elastic media and in the orbit of isotropic elastic media with residual stress.

The elasticity tensor for isotropic elastic media with respect to the Euclidean metric $\mathbf{e}$ (which we denote by C_{iso}) has the following components:

$$\mathsf{C}_{iso}^{ABCD} \;=\; \lambda(X)\delta^{AB}\delta^{CD} \;+\; \mu(X)\big[\delta^{AC}\delta^{BD} \;+\; \delta^{AD}\delta^{BC}\big] \tag{16}$$

with λ, μ the *Lamé parameters*. In this case we denote the Dirichlet-to-Neumann map by $\Lambda_{\mathbf{e},\lambda,\mu,\rho}$. Similarly, $\widehat{\mathsf{C}}_{iso}$ denotes the isotropic elasticity tensor with Lamé parameters $\widehat{\lambda}$ and $\widehat{\mu}$. (In general, C depends on the metric $\mathbf{G}$.)

We first recall Rachele's uniqueness result for linear, isotropic elastic media in Euclidean space. In particular, we consider an isotropic, linearly elastic medium $(\Omega,\, \lambda,\, \mu,\, \rho)$ (respectively, $(\Omega,\, \widehat{\lambda},\, \widehat{\mu},\, \widehat{\rho})$) that satisfies the admissibility conditions given at the start of Section 2, e.g., $\mu > 0$ and $3\lambda + 2\mu > 0$ (positive definiteness of C) on $\bar{\Omega}$. In addition, we assume that the metrics $g_p = ([\lambda + 2\mu]/\rho)^{-1}\,\mathbf{e}$ and $g_s = (\mu/\rho)^{-1}\,\mathbf{e}$ (respectively, $\widehat{g}_p = ([\widehat{\lambda} + 2\widehat{\mu}]/\widehat{\rho})^{-1}\,\mathbf{e}$ and $\widehat{g}_s = (\widehat{\mu}/\widehat{\rho})^{-1}\,\mathbf{e}$) have the property that $g_{p/s}$ does not give rise to caustics; the geodesics of $g_{p/s}$ all exit Ω in finite time; and the geodesics do not graze $\partial\Omega$. Finally, we assume that $\lambda = 2\mu$ (i.e., $c_p = 2c_s$) on a small enough set, e.g., at only isolated points in $\bar{\Omega}$, and that the conditions for one of the boundary rigidity results [38], [2], [36], [5], [15], and [37] holds. (See also Pestov [27] for weaker conditions needed to invert the ray transform for tensor fields.)

Theorem 3 (Rachele [31], [29], [30], Uniqueness for isotropic elastodynamics in Euclidean space)**.** *Suppose that* $(\Omega,\, \lambda,\, \mu,\, \rho)$ *(respectively,* $(\Omega,\, \widehat{\lambda},\, \widehat{\mu},\, \widehat{\rho})$*) are isotropic, linearly elastic media satisfying the conditions given in the previous paragraph. In addition, suppose that*

$$\Lambda_{\mathbf{e},\lambda,\mu,\rho} \;=\; \Lambda_{\mathbf{e},\widehat{\lambda},\widehat{\mu},\widehat{\rho}} \quad \text{on} \;\; (0,T),$$

where $T < \infty$ *is greater than the geodesic distance (with respect to each of* $g_p,\, g_s,\, \widehat{g}_p,\, \widehat{g}_s$*) between any two boundary points of* Ω*. Then for any* $X \in \bar{\Omega}$

$$\lambda(X) = \widehat{\lambda}(X), \qquad \mu(X) = \widehat{\mu}(X), \quad \text{and} \qquad \rho(X) = \widehat{\rho}(X).$$

We denote by

$$\mathfrak{O}_{iso} \;=\; \{(\psi^*\mathbf{G}, \psi^*\mathsf{C}_{iso}, \psi^*\rho) \mid \psi \in \Psi\}$$

the orbit of isotropic, linearly elastic media. We now prove uniqueness for the anisotropic, linearly elastic media in the orbit $\mathfrak{O}_{iso}$ by applying Theorem 1, equation (14), and Theorem 3. (In Theorem 11 we give a pointwise characterization of $\mathfrak{O}_{iso}$ and a pointwise description of parts of the orbits of other types of anisotropic elastic media.)

Corollary 4 (Partial uniqueness within the orbit generated by isotropic elastic media)**.** *Consider admissible, linearly elastic media* $(\Omega, \psi^*\mathbf{G}, \psi^*\mathsf{C}_{iso},$ $\psi^*\rho)$*,* $(\Omega,\, \widehat{\psi}^*\widehat{\mathbf{G}},\, \widehat{\psi}^*\widehat{\mathsf{C}}_{iso},\, \widehat{\psi}^*\widehat{\rho}) \in \mathfrak{O}_{iso}$*. Suppose that the transformed media*

$\big(\Omega, \mathbf{e}, \sqrt{\det\,\mathbf{G}}(\lambda,\mu,\rho)\big)$ *and* $\big(\Omega, \mathbf{e}, \sqrt{\det\,\widehat{\mathbf{G}}}(\widehat{\lambda},\widehat{\mu},\widehat{\rho})\big)$ *satisfy the conditions of Theorem 3. In addition, suppose that*

$$\Lambda_{\psi^*\mathbf{G},\psi^*\mathbf{C}_{iso},\psi^*\rho} \;=\; \Lambda_{\widehat{\psi}^*\widehat{\mathbf{G}},\,\widehat{\psi}^*\widehat{\mathbf{C}}_{iso},\,\widehat{\psi}^*\widehat{\rho}}\,. \tag{17}$$

Then for $X \in \overline{\Omega}$ *and* $\delta = \sqrt{\det\,\widehat{\mathbf{G}}}/\sqrt{\det\,\mathbf{G}}$

$$\lambda(X) \;=\; \delta\,\widehat{\lambda}(X), \qquad \mu(X) \;=\; \delta\,\widehat{\mu}(X), \quad \textit{and} \qquad \rho(X) \;=\; \delta\,\widehat{\rho}(X).$$

Proof. By Theorem 1 the left side $\Lambda_{\psi^*\mathbf{G},\psi^*\mathbf{C}_{iso},\psi^*\rho}$ of (17) may be written as $\Lambda_{\mathbf{G},\mathbf{C}_{iso},\rho}$, which by (14) agrees with $\Lambda_{\mathbf{e},\sqrt{\det\mathbf{G}}(\mathbf{C}_{iso},\rho)}$. Similarly, the right side of (17) may be written as $\Lambda_{\mathbf{e},\sqrt{\det\widehat{\mathbf{G}}}(\widehat{\mathbf{C}}_{iso},\widehat{\rho})}$. The result now follows by (17) and Theorem 3. □

Remark 5. In particular, we may apply Corollary 4 to prove uniqueness for isotropic elastodynamics in the setting of any arbitrary, flat metric $\bar{\mathbf{G}}$ of the form $\psi^*\mathbf{e}$, $\psi \in \Psi$. In this case the elasticity tensor $\mathbf{C}$ (isotropic with respect to $\bar{\mathbf{G}}$) has the form

$$\mathsf{C}^{ABCD} \;=\; \lambda(X)\bar{G}^{AB}\bar{G}^{CD} \;+\; \mu(X)\big[\bar{G}^{AC}\bar{G}^{BD} \;+\; \bar{G}^{AD}\bar{G}^{BC}\big].$$

In fact, $\mathbf{C} = \psi^*\mathbf{C}_{iso}$.

We next consider the associated inverse boundary problem in Euclidean space for linear, isotropic elastic media *with residual stress.* In particular, we consider an isotropic linearly elastic medium $(\Omega,\,\lambda,\,\mu,\,\rho,\,\mathcal{T})$ (and another $(\Omega,\,\widehat{\lambda},\,\widehat{\mu},\,\widehat{\rho},\,\widehat{\mathcal{T}})$) with initial stress $\mathbf{S}$ (respectively, $\widehat{\mathbf{S}}$) given by the symmetric, C^∞-smooth 2-tensor $\mathcal{T}$ ($\widehat{\mathcal{T}}$), called the *residual stress tensor*, which satisfies $\nabla_x \cdot \mathcal{T} \,=\, 0$ in Ω (divergence-free in the interior) and $\mathcal{T}\cdot\vec{\nu} \,=\, 0$ on $\partial\Omega$ (zero surface traction), where ν is the unit outer normal to $\partial\Omega$. The same conditions hold for $\widehat{\mathcal{T}}$. We assume that the admissibility conditions given at the start of Section 2 hold. (The strong ellipticity condition here for $\mathring{\mathsf{A}}^{aA}{}_b{}^B = \mathring{\mathbf{C}}_{iso}^{A'AB'B}\delta^a{}_{A'}\delta_{bB'} + \delta^a{}_b\mathcal{T}^{AB}$ reduces to the conditions that $\lambda+\mu > 0$ on $\bar{\Omega}$ and that $\mu(x)I + \mathcal{T}$ be uniformly positive-definite on $\bar{\Omega}$. We use the model for residual stress derived by Hoger [11], [12] here. (See [32] for details.) In addition, we assume that the normal derivatives of all orders of each of $\lambda,\ \mu,\ \rho,\ \mathcal{T},$ and its counterpart from $\widehat{\lambda},\ \widehat{\mu},\ \widehat{\rho},\ \widehat{\mathcal{T}},$ are the same at $\partial\Omega$; and that the conditions for one of the boundary rigidity results [38], [2], [36], [5], [15], and [37] holds.

Theorem 6 (Rachele [32], Partial uniqueness for $(\lambda,\mu,\rho,\mathcal{T})$ in Euclidean space)**.** *Suppose that* $(\Omega,\,\lambda,\,\mu,\,\rho,\,\mathcal{T})$ *(respectively,* $(\Omega,\,\widehat{\lambda},\,\widehat{\mu},\,\widehat{\rho},\,\widehat{\mathcal{T}})$*) are isotropic, linearly elastic media with residual stress satisfying the conditions of the preceeding paragraph. Let*

$$g_p \;=\; \Big(\frac{\lambda+2\mu}{\rho}I + \frac{1}{\rho}\mathcal{T}\Big)^{-1} \quad \textit{and} \quad g_s \;=\; \Big(\frac{\mu}{\rho}I + \frac{1}{\rho}\mathcal{T}\Big)^{-1}$$

denote the metrics whose geodesics are the paths of wave propagation in these media (respectively, $\widehat{g}_p$ *and* $\widehat{g}_s$*). In addition, suppose that*

$$\Lambda_{\mathbf{e},\lambda,\mu,\rho,\mathcal{T}} \;=\; \Lambda_{\mathbf{e},\widehat{\lambda},\widehat{\mu},\widehat{\rho},\widehat{\mathcal{T}}} \quad on \;\; (0,T),$$

where $T < \infty$ *is greater than the geodesic distance (with respect to each of* $g_p,\, g_s,\, \widehat{g}_p,\, \widehat{g}_s$*) between any two boundary points of* Ω*. Then*

$$\psi_p^* g_p \;=\; \widehat{g}_p \quad and \quad \psi_s^* g_s \;=\; \widehat{g}_s \quad in \;\; \Omega$$

for some diffeomorphisms ψ_p, ψ_s *of* $\bar{\Omega}$ *which are the identity on* $\partial\Omega$*.*

Under slightly different conditions Hansen and Uhlmann [10] show that Theorem 6 holds even in the presence of caustics.

We conclude from Theorem 1, equation (14), and Theorem 6, that uniqueness holds for the anisotropic (with respect to $\mathbf{e}$) elastic media in the orbit

$$\mathfrak{O}_{resid} \;=\; \{\; (\psi^*\mathbf{G}, \psi^*\mathbf{C}_{iso}, \psi^*\rho, \psi^*\mathcal{T}) \mid \psi \in \Psi \;\}$$

of $\mathbf{e}$-isotropic elastic media with residual stress. The equations of motion for such media are given by (38) with second Piola-Kirchhoff stress tensor $\mathring{\mathbf{S}}$ given by the residual stress tensor $\mathcal{T}$.

Corollary 7 (Partial uniqueness within the orbit generated by isotropic elastic media with residual stress)**.** *Consider admissible linearized elastic media* $(\Omega, \psi^*\mathbf{G}, \psi^*\mathbf{C}_{iso}, \psi^*\rho, \psi^*\mathcal{T})$, $(\Omega,\; \widehat{\psi}^*\widehat{\mathbf{G}},\; \widehat{\psi}^*\widehat{\mathbf{C}}_{iso},\; \widehat{\psi}^*\widehat{\rho},\; \widehat{\psi}^*\widehat{\mathcal{T}}) \in \mathfrak{O}_{resid}$ *with the property that the transformed media* $\big(\Omega, \mathbf{e}, \sqrt{\det\, \mathbf{G}}(\lambda,\mu,\rho,\mathcal{T})\big)$ *and* $\big(\Omega, \mathbf{e}, \sqrt{\det\, \widehat{\mathbf{G}}}(\widehat{\lambda},\widehat{\mu},\widehat{\rho},\widehat{\mathcal{T}})\big)$ *satisfy the conditions of Theorem 6. In addition, suppose that*

$$\Lambda_{\,\psi^*\mathbf{G},\psi^*\mathbf{C}_{iso},\psi^*\rho,\psi^*\mathcal{T}} \;=\; \Lambda_{\,\widehat{\psi}^*\widehat{\mathbf{G}},\; \widehat{\psi}^*\widehat{\mathbf{C}}_{iso},\; \widehat{\psi}^*\widehat{\rho},\; \widehat{\psi}^*\widehat{\mathcal{T}}}\;.$$

Then for $g_p,\, g_s,\, \widehat{g}_p,\, \widehat{g}_s$ *as defined in Theorem 6*

$$\psi_p^* g_p \;=\; \widehat{g}_p \quad and \quad \psi_s^* g_s \;=\; \widehat{g}_s \quad in \;\; \Omega$$

for some diffeomorphisms ψ_p, ψ_s *of* $\bar{\Omega}$ *which are the identity on* $\partial\Omega$*.*

In the same spirit any interior uniqueness result that uses displacement-traction data for a class of isotropic media with residual stress gives rise to partial uniqueness for any density and elastic moduli on the orbit of that class. For example, we expect these results to apply to [13], [17], [34], which we describe in the introduction.

Corollaries 4 and 7 illustrate an approach that may be taken to prove partial uniqueness for more general anisotropic (linear or linearized) elastic media. The strategy is to prove uniqueness for the simplest class of anisotropic media in an orbit $\mathfrak{O}$, and then to derive partial uniqueness, via Theorem 1, for other anisotropic media in $\mathfrak{O}$.

2.2. **The Symmetry Group of $\psi^*\mathbf{C}$.** In the previous section we extend uniqueness results for elastic media to their orbits under the action of Ψ. Consequently, we are interested in characterizing the orbits of each type of (an)isotropic elastic media, especially the orbit of isotropic elastic media (for which uniqueness results are already known). In this section we characterize the orbit of isotropic elastic media. (At any $X \in \Omega$ the orbit of the isotropic elasticity tensors consists of the isotropic, some transverse isotropic, and some orthotropic elasticity tensors.) We also describe $\psi^*\mathbf{C}(X)$ for most other types of anisotropic media, for certain $\psi \in \Psi$. In particular, we describe the type of anisotropy of $\psi^*\mathbf{C}$ via its material symmetry group with respect to the Euclidean metric.

The *material symmetry group* $\mathcal{G}$ associated with an elastic medium is defined to be the subgroup of $\mathbf{G}$-orthogonal transformations $Q : T_X\Omega \to T_X\Omega$ (i.e., with $Q^T = Q^{-1}$, where, in coordinates, $Q^T(X)^A{}_B = G_{BB'}Q^{B'}{}_{A'}G^{A'A}$) whose action on the reference configuration does not change the material response. In particular, the symmetry group $\mathrm{Sym}_{\mathbf{e}}\mathbf{T}$ at $X \in \Omega$ of an m-tensor field $\mathbf{T}$ on Ω (e.g., an elasticity tensor $\mathbf{C}$) is defined to be the collection of $\mathbf{e}$-orthogonal transformations Q of $T_X\Omega$ (i.e., with $Q^T(X)^A{}_B = \delta_{BB'}Q^{B'}{}_{A'}\delta^{A'A}$) such that

$$Q * \mathbf{T}(X) \;=\; \mathbf{T}(X),$$

where for any 3×3 matrix M (which we identify with a $(1,1)$-tensor) we define $M * \mathbf{T}(X)$ to be the m-tensor with components

$$(M * \mathbf{T}(X))^{ABC\ldots D} \;=\; M^A{}_{\underline{A}}\, M^B{}_{\underline{B}}\, M^C{}_{\underline{C}} \ldots M^D{}_{\underline{D}}\, \mathsf{T}(X)^{\underline{A}\,\underline{B}\,\underline{C}\ldots\underline{D}}. \tag{18}$$

We remark that the $*$ operation respects the symmetries of $\mathbf{T}$ for any invertible matrix M; in particular, it defines an action of the orthogonal group $O(3)$ on the space of elasticity tensors.

In addition, the $*$ operation defines an action of $O(3)$ on the symmetry groups of elasticity tensors. In fact, the action of $Q \in O(3)$ on the symmetry group of a tensor $\mathbf{T}$ is given by the symmetry group of $Q * \mathbf{T}$:

$$\mathrm{Sym}_e(Q * \mathbf{T}) \;=\; Q(\,\mathrm{Sym}_e\mathbf{T})Q^T \;=\; Q * (\,\mathrm{Sym}_e\mathbf{T}) \tag{19}$$

since $\overline{Q} \in \mathrm{Sym}_e\mathbf{T}$ if and only if $Q\overline{Q}Q^T * (Q * \mathbf{T}) = Q * \mathbf{T}$. Here we write $M * G = MGM^T$ in terms of conjugation by M for any rank-2 tensor G, and we write

$$M * \mathcal{G} \;=\; M\,\mathcal{G}\,M^T \;=\; \{MGM^T \mid G \in \mathcal{G}\,\} \tag{20}$$

for any collection $\mathcal{G}$ of linear transformations.

Given an elasticity tensor $\mathbf{C}$ with a certain type of anisotropy (one of eight types, given [8]), there exists a right-handed orthonormal basis $\{e_1, e_2, e_3\}$ so that the symmetry group of $\mathbf{C}$ is generated by $-I$ and the orthogonal transformations listed in Table 1. (See [4], for example.) The generators in Table 1 have the form $Q^{\phi}_{e_j}$, $j = 1, 2, 3$, where $Q^{\phi}_{e_j}$ is the rotation of angle ϕ

about e_j; e.g., $Q^{\phi}_{e_3}(xe_1{+}ye_2{+}ze_3) = [x\cos\phi{-}y\sin\phi]e_1{+}[y\cos\phi{+}x\sin\phi]e_2{+}ze_3$. Throughout, $\mathrm{sg}\{Q^{\pi}_{e_i}, Q^{\pi}_{e_j}\}$ denotes the subgroup of $O(3)$ generated by $Q^{\pi}_{e_i}, Q^{\pi}_{e_j}$. These generators are not independent in the sense that, for example, $\mathrm{sg}\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}\} = \mathrm{sg}\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\}$ since $Q^{\pi}_{e_1}\circ Q^{\pi}_{e_2}(xe_1{+}ye_2{+}ze_3) = Q^{\pi}_{e_1}(-xe_1 + ye_2 - ze_3) = (-xe_1 - ye_2 + ze_3) = Q^{\pi}_{e_3}(xe_1 + ye_2 + ze_3)$.

Two elasticity tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ have the same anisotropy type at $X \in \Omega$ if and only if there exists an orthogonal transformation Q such that

$$\mathbf{C}_1(X) \; = \; Q * \mathbf{C}_2(X).$$

To describe the symmetry group of $\psi^*\mathbf{C}$ for $\psi \in \Psi$ at a fixed point $X \in \Omega$, we write $D\psi^{-1}(X)$ in terms of its polar decomposition with the positive, symmetric S diagonalized as $S = UDU^T$. Here U, V are orthogonal, and $D = \operatorname{diag}(a, b, c)$ is a diagonal 3×3 matrix with a, b, c the (positive) eigenvalues of S. Without loss of generality, we assume either that $a = b = c$, that $a = b \neq c$, or that the a, b, c are distinct.

We observe that at $X \in \Omega$

$$\psi^*\mathbf{C} = (D\psi^{-1}) * (\mathbf{C}\circ\psi) = UDU^TV * (\mathbf{C}\circ\psi) = UD * \bar{\mathbf{C}} = U * \hat{\bar{\mathbf{C}}}, \tag{21}$$

where

$$\bar{\mathbf{C}} \; = \; U^TV * (\mathbf{C}\circ\psi) \qquad \text{and} \qquad \hat{\bar{\mathbf{C}}} = D * \bar{\mathbf{C}}. \tag{22}$$

It follows, given (19), that we may reduce the task of describing $\mathrm{Sym}_e\psi^*\mathbf{C}$ to that of describing $\mathrm{Sym}_e(D * \bar{\mathbf{C}})$ for any elasticity tensor $\bar{\mathbf{C}}$.

Proposition 8. *Let $\mathbf{C}$ be an admissible elasticity tensor on Ω. For $X \in \Omega$ and $\psi \in \Psi$ with polar decomposition at X in the form described above*

$$\mathrm{Sym}_e\psi^*\mathbf{C} \; = \; U * \mathrm{Sym}_e(D * \bar{\mathbf{C}}), \tag{23}$$

$$\mathrm{Sym}_e\bar{\mathbf{C}} \; = \; U^TV * \mathrm{Sym}_e(\mathbf{C}\circ\psi). \tag{24}$$

We note that the action on $\mathrm{Sym}_e\bar{\mathbf{C}}$ of the pullback by D is accomplished at each point $X \in \Omega$ by varying only two parameters a/c and b/c.

A partial description of the relationship at $X \in \Omega$ between $\mathrm{Sym}_{\mathbf{e}}\bar{\mathbf{C}}$ and $\mathrm{Sym}_{\mathbf{e}}(D * \bar{\mathbf{C}})$ is given in the following proposition.

Proposition 9. *Let $\bar{\mathbf{C}}$ be an admissible elasticity tensor on Ω. Let D be a 3×3 matrix. Then*

$$\mathrm{Sym}_{\mathbf{e}}\bar{\mathbf{C}} \;\; \cap \;\; \mathrm{Sym}_e D \qquad \subseteq \qquad \mathrm{Sym}_{\mathbf{e}}(D * \bar{\mathbf{C}}). \tag{25}$$

Proof. For $Q \in \mathrm{Sym}_{\mathbf{e}}\bar{\mathbf{C}} \cap \mathrm{Sym}_{\mathbf{e}}D$ we have $Q * (D * \bar{\mathbf{C}}) = QD * \bar{\mathbf{C}} = DQ * \bar{\mathbf{C}} = D * \bar{\mathbf{C}}$. That is, $Q \in \mathrm{Sym}_{\mathbf{e}}(D * \bar{\mathbf{C}})$. □

The simplest situation in (25), and the one we consider here, occurs when S is diagonal with respect to the basis $\{e_1, e_2, e_3\}$ used to describe $\mathrm{Sym}_e V * (\mathbf{C}\circ\psi)$. The computational complexity of the other cases is significantly greater. This motivates the following definition. We say that ψ

is *aligned with* $\mathbf{C}$ (at $X' \in \Omega$) if there is a polar decomposition of $D\psi^{-1}$ and diagonalization of the stretch factor S as described above so that, with respect to the standard basis for Euclidean space, the generators of $\mathrm{Sym}_e \bar{\mathbf{C}}(X')$ are listed in Table 1. We observe below that for certain aligned ψ and $\mathbf{C}$ equality holds in (25).

Remark 10. For $\mathbf{C}$ isotropic, every $\psi \in \Psi$ is aligned with $\mathbf{C}$.

Without loss of generality, we set $c = 1$ below.

Theorem 11 (Computing $D{*}\bar{\mathbf{C}}$ in the aligned case)**.** *Let $\bar{\mathbf{C}}$ be an admissible isotropic, transverse isotropic, cubic, or tetragonal elasticity tensor on Ω represented by parameters $(\bar{\alpha}, \bar{\beta}, \bar{A}^{11}, \bar{B}^{11}, \bar{\alpha}_0, \bar{\alpha}_4)$ via the harmonic-Cartan decomposition (53) and (57). Suppose that, with respect to the standard basis for Euclidean space, the generators of $\mathrm{Sym}_e\bar{\mathbf{C}}$ are listed in Table 1. Let $D = \operatorname{diag}(a, b, 1)$ be a diagonal 3×3 matrix with a, b positive and $a = b \neq 1$ or $a, b, 1$ distinct. Then via the harmonic-Cartan decomposition we may represent $D * \bar{\mathbf{C}}$ by $(\widehat{\bar{\alpha}}, \widehat{\bar{\beta}}, \widehat{\bar{A}}^{11}, \widehat{\bar{B}}^{11}, \widehat{\bar{\alpha}}_0, \widehat{\bar{\alpha}}_2, \widehat{\bar{\alpha}}_4)$, and the (scaled) parameters $(15\widehat{\bar{\alpha}}, 15\widehat{\bar{\beta}}, 21\widehat{\bar{A}}^{11}, 21\widehat{\bar{A}}^{22}, 21\widehat{\bar{B}}^{11}, 21\widehat{\bar{B}}^{22}, 280\widehat{\bar{\alpha}}_0, 14\widehat{\bar{\alpha}}_2, 8\widehat{\bar{\alpha}}_4)$ for $D * \bar{\mathbf{C}}$ are given by*

$$\begin{aligned} &\bar{\alpha} \cdot P_{\bar{\alpha}}(a,b) \;+\; \bar{\beta} \cdot P_{\bar{\beta}}(a,b) \;+\; \bar{A}^{11} \cdot P_{\bar{A}^{11}}(a,b) \\ &\quad + \bar{B}^{11} \cdot P_{\bar{B}^{11}}(a,b) \;+\; \bar{\alpha}_0 \cdot P_{\bar{\alpha}_0}(a,b) \;+\; \bar{\alpha}_4 \cdot P_{\bar{\alpha}_4}(a,b), \end{aligned} \tag{26}$$

where the $P(a,b)$, defined by (77), are linearly independent for fixed a, b.

In fact, $\mathrm{Sym}_e\bar{\mathbf{C}} \not\subseteq \mathrm{Sym}_e D$ implies

$$\mathrm{Sym}_e\bar{\mathbf{C}} \;\cap\; \mathrm{Sym}_e D \;=\; \mathrm{Sym}_e(D * \bar{\mathbf{C}}). \tag{27}$$

Remark 12. Note that in the case that $a = b = c$ we have that $D{*}\bar{\mathbf{C}} = a^4\bar{\mathbf{C}}$, and so the pullback by ψ preserves the type of anisotropy of the elasticity tensor.

The proof of Theorem 11, given in Section 5, is based on the harmonic decomposition (53) of the elasticity tensor $\bar{\mathbf{C}}$, followed by the Cartan decomposition (57) of the fourth-order, harmonic part $\bar{\mathbf{H}}$ of $\bar{\mathbf{C}}$, as described in [8]. Due to this decomposition we may write exact formulas for the (an)isotropic elasticity tensors $\bar{\mathbf{C}}$ and $D * \bar{\mathbf{C}}$. We then use the mathematical software Maple to compute the harmonic-Cartan decomposition of $D{*}\bar{\mathbf{C}}$, and analyze the (possible) vanishing of certain terms in order to relate $\mathrm{Sym}_e\bar{\mathbf{C}}$ with $\mathrm{Sym}_e D * \bar{\mathbf{C}}$.

Below we relate the type of (an)isotropy in the aligned case of isotropic, transversely isotropic, cubic, and tetragonal $\bar{\mathbf{C}}$ and the corresponding $D * \bar{\mathbf{C}}$.

Corollary 13 (Characterizing the range of the $D*$ operation on some aligned $\bar{\mathbf{C}}$)**.** *Suppose that $\bar{\mathbf{C}}$ satisfies the conditions of Theorem 11. Then:*

i). *For $\bar{\mathbf{C}}$ isotropic and $a = b \neq 1$, the range of $D * \bar{\mathbf{C}}$ is a 3-parameter family of the form $\bar{\alpha} \cdot P_{\bar{\alpha}}(a,b) + \bar{\beta} \cdot P_{\bar{\beta}}(a,b)$ in the 5-parameter family of transverse isotropic elasticity tensors.*

ii). *For* $\bar{\mathsf{C}}$ *cubic and* $a = b \neq 1$, *the range of* $D * \bar{\mathsf{C}}$ *is a 4-parameter family of the form* $\bar{\alpha} \cdot P_{\bar{\alpha}}(a,b) + \bar{\beta} \cdot P_{\bar{\beta}}(a,b) + \bar{\alpha}_0 \cdot [P_{\bar{\alpha}_0}(a,b) + 5P_{\bar{\alpha}_4}(a,b)]$ *in the 5-parameter family of tetragonal elasticity tensors.*

iii). *If* $a, b, 1$ *are distinct, then for isotropic, transverse isotropic, cubic, or tetragonal* $\bar{\mathsf{C}}$, *the range of* $D*\bar{\mathsf{C}}$ *is a family of orthotropic elasticity tensors of the form (26).*

Corollary 13 is a consequence of Theorem 11 and Section 5.2, and is derived in Section 5.4. The following proposition is also proved in Section 5.4.

Proposition 14. *If* C *is isotropic, transverse isotropic, or orthotropic, and* ψ *is aligned with* C *at* $X' \in \Omega$, *then* ψ^{-1} *is aligned with* $\psi^*\mathsf{C}$ *at* $X = \psi(X')$.

Corollary 15 ($D*$ preserves transverse isotropy for some aligned $\bar{\mathsf{C}}(X)$)**.** *For* $X \in \Omega$ *there is a(n at least two-parameter) non-trivial family of transverse isotropic* $\bar{\mathsf{C}}(X)$ *with the property that* $D*\bar{\mathsf{C}}(X)$ *remains transverse isotropic when* $a = b \neq 1$.

3. Definitions

We follow Marsden and Hughes [21] to represent an elastic body as a Riemannian manifold $(\Omega, \mathbf{G})$. Let $(\mathcal{S}, \mathbf{g})$ be a (fixed) Riemannian manifold of the same dimension representing the ambient space within which the body Ω moves. In this paper $\mathcal{S} = \mathbb{R}^3$ and for applications we take $\mathbf{g}$ to be the Euclidean metric. We write $\{X^A\}$ for coordinate systems on Ω, and $\{x^a\}$ for coordinate systems on the ambient space $\mathcal{S}$.

General tensor notation. We follow notational conventions for tensor analysis as in [21]. In particular, tensor fields are generally written in boldface and their components in standard font, e.g., the so-called deformation gradient $\mathbf{F}$ is a tensor field with components $\mathrm{F}^a{}_A$.

A two-point tensor $\mathbf{T}$ of type $\begin{pmatrix} q & l \\ p & m \end{pmatrix}$ at $X \in \Omega$ over a mapping $\phi : \Omega \to \mathcal{S}$ is a multilinear mapping into $\mathbb{R}$ from the product space consisting of p copies of $T^*_X\Omega$, q copies of $T_X\Omega$, l copies of $T^*_x\mathcal{S}$, and m copies of $T_x\mathcal{S}$, where $x = \phi(X)$. The components of $\mathbf{T}$ are denoted by $T^{A_1\ldots A_p}{}_{B_1\ldots B_q}{}^{a_1\ldots a_l}{}_{b_1\ldots b_m}$, where, in general, upper-case indices denote tensorial behavior with respect to the referential setting, lower-case indices with respect to the spatial setting, and the pullback of a two-point tensor by $\psi : \Omega \to \Omega$ interacts only with indices corresponding to the reference configuration (i.e., the upper-case indices). We use the terminology *two-point vector* for a two-point tensor of type $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The non-linear case. The deformation of an elastic body is modeled by mappings of the form $\phi(X,t) = \phi_t(X) : \Omega \times (0,T) \to \mathcal{S}$. In fact, we call a mapping $\phi : \Omega \to \mathcal{S}$ a *configuration* of Ω in $\mathcal{S}$, and we denote by $\mathcal{C}$ the set of all configurations of Ω in $\mathcal{S}$. A *motion* $\phi_t : \Omega \to \mathcal{S}$ *of the body* Ω *in* $\mathcal{S}$ is a curve in $\mathcal{C}$ parametrized by t, where t ranges over some open interval of $\mathbb{R}$. A motion ϕ_t is C^2 *regular* if ϕ_t is invertible on $\phi_t(\Omega)$ for each t, and if $\phi_t(X)$ and $\phi_t^{-1}(X)$ are C^2 as functions of t and $X \in \Omega$. We write $x = \phi_t(X)$.

For any fixed diffeomorphism $\psi \in \Psi$ of $\bar{\Omega}$ we write

$$\phi_t' = \psi^* \phi_t, \qquad X' = \psi^{-1}(X), \qquad \mathbf{G}' = \psi^* \mathbf{G},$$

for any ϕ_t that is a C^2 regular motion of Ω in $\mathcal{S}$, where $t \in [0,T]$ with $0 < T < \infty$. We remark that ϕ_t' is also a C^2 regular motion of Ω in $\mathcal{S}$.

The *Cauchy stress tensor* $\boldsymbol{\sigma}(x,t)$ is a $\binom{2}{0}$ tensor field which has the property that the *Cauchy stress vector* $\mathbf{t}(x,t,\mathbf{n}) = \boldsymbol{\sigma} \cdot \mathbf{n}$ represents the force per unit (of deformed) area exerted on a surface element that is oriented with unit normal vector $\mathbf{n}(\mathrm{x})$. The first Piola-Kirchhoff stress tensor $\mathbf{P}(X,t)$ is the Piola transform (on the second index) of the Cauchy stress tensor $\boldsymbol{\sigma}$; that is,

$$P^{aA}(X) \;=\; J(X,t)\; (\mathrm{F}^{-1})^{A}{}_{b}\; \sigma^{ab}, \tag{28}$$

where

$$\mathbf{F}(X,t) \;=\; D\phi_t(X), \quad \text{with } \mathrm{F}(X,t)^a{}_A \;=\; D\phi_t(X)^a{}_A \;=\; \frac{\partial {\phi_t}^a}{\partial X^A} \tag{29}$$

is the deformation gradient (with respect to a motion ϕ_t of Ω), and the scalar

$$J(X,t) \;=\; \frac{\sqrt{\det\, g}}{\sqrt{\det\, G}}\, \det\, (D\phi_t) \tag{30}$$

is the Jacobian of the linear transformation $\mathbf{F}$.

The second Piola-Kirchhoff stress tensor $\mathbf{S}(X,t)$ is the pull-back of the first leg of $\mathbf{P}$ via ϕ_t, i.e., $\mathrm{S^{AB}} = (\mathrm{F}^{-1})^{\mathrm{A}}{}_{\mathrm{a}}\mathrm{P^{aB}}$.

A material is called *hyperelastic* if there exists a scalar functional $W = \widehat{W}(X, \mathbf{G}, \mathbf{C}^\flat)$ such that

$$\widehat{S}^{AB} \;=\; 2\rho\, \frac{\partial \widehat{W}}{\partial {C^\flat}_{AB}},$$

where $\mathbf{C}^\flat = \phi_t^* g$ is related to the *deformation tensor* $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ by $C^A{}_B = G^{A\underline{A}} {C^\flat}_{\underline{A}B}$. W is called the *stored energy function*, and corresponds to the internal energy per unit mass $E = \widehat{E}(X, \mathbf{G}, \mathbf{C}^\flat)$ at thermal equilibrium.

It follows from the constitutive equations that the elasticity tensor $\mathsf{C} = \widehat{\mathsf{C}}(X, \mathbf{G}, \mathbf{C}^\flat)$, which measures the dependence of stress on the strain, may be written as

$$\mathsf{C}^{ABCD} \;=\; 2\rho\; \frac{\partial^2 \widehat{E}}{\partial {C^\flat}_{AB} \partial {C^\flat}_{CD}}. \tag{31}$$

That is, in this case C depends on the motion ϕ_t, and so we view $\mathbf{G}$ and $\mathcal{O} = (E, \rho)$ as the material parameters of the elastic object.

We say that ϕ_t *solves the initial-boundary-value problem for non-linear hyperelasticity* if, for given Dirichlet boundary data f,

$$F_{\mathbf{G},E,\rho}(\phi_t) = \rho\mathbf{A} - \mathrm{DIV}_{\mathbf{G}}\,\mathbf{P} = 0 \qquad \text{on } \Omega, \tag{32a}$$

$$\phi_t(X) = X + f(X,t) \qquad \text{for } X \in \partial\Omega, \tag{32b}$$

$$\phi_t(X) = 0 \qquad \text{for } t = 0, \tag{32c}$$

$$\mathbf{V}(X,t) = 0 \qquad \text{for } t = 0, \tag{32d}$$

where $\mathbf{V}(X,t) = \partial\phi_t/\partial t$ is the material velocity vector field, and $\mathbf{A}(X,t)$, the acceleration vector field, is given by

$$A^a = \partial V^a/\partial t + (\gamma^a_{bc} \circ \phi_t) V^b V^c \tag{33}$$

on Ω. Throughout, γ^a_{bc} and Γ^A_{BC} denote (respectively) the Christoffel symbols for the spatial metric $\mathbf{g}$ and referential metric $\mathbf{G}$. The divergence of a two-point tensor $\mathbf{P} = (P^{aA})$ is defined in the referential setting by

$$(\mathrm{DIV}_X\,\mathbf{P})^a = \frac{\partial P^{aA}}{\partial X^A} + \Gamma^{\underline{A}}_{\underline{A}A}\, P^{aA} + (\gamma^a_{\underline{b}\,\underline{a}} \circ \phi) \mathrm{F}^{\underline{b}}{}_A P^{\underline{a}A}, \tag{34}$$

as in [21].

We refer to [21], p. 401, 406, for a discussion of existence and uniqueness for the non-linear initial-boundary-value problem.

The surface measurements used to determine the material parameters $\mathbf{G}$, E, ρ of the elastic object are modeled by the dynamic Dirichlet-to-Neumann map $\Lambda_{\mathbf{G},E,\rho}$. The Dirichlet-to-Neumann map gives the correspondence (when there exist unique solutions of (32)) between the surface displacement and the surface traction taken over a finite period of time. Following [38], we view the Dirichlet-to-Neumann map as a vector-valued form representing the flux, given in the case of non-linear elasticity by

$$(\Lambda_{\mathbf{G},E,\rho} f)^a = \langle \mathbf{P}, \boldsymbol{\nu}_{\mathbf{G}} \rangle^a \, dS_{\mathbf{G}} = P^{aA}\, \mathbf{G}_{AB}\, \boldsymbol{\nu}^B_{\mathbf{G}}\, dS_{\mathbf{G}} \tag{35}$$

for $t \in [0,T]$, where $dS_{\mathbf{G}}$ is the area form on $\partial\Omega$ induced by $\mathbf{G}$, $\boldsymbol{\nu}_{\mathbf{G}}$ is the unit outer normal to $\partial\Omega$ with respect to $\mathbf{G}$, and the motion ϕ_t solves (32) for boundary values f. By Corollary 2 the Dirichlet-to-Neumann map may also be written with respect to the Euclidean metric as

$$(\Lambda_{\mathbf{G},E,\rho} f)^a = P^{aA}\, \delta_{AB}\, \boldsymbol{\nu}^B \sqrt{\det \mathbf{G}}\; dS, \tag{36}$$

where dS is the area form on $\partial\Omega$, and $\boldsymbol{\nu}$ is the unit outer normal vector to $\partial\Omega$, each with respect to the Euclidean metric.

The linearized case. We linearize about an arbitrary C^2-regular motion $\mathring{\phi}_t$ which solves (32a), as in [21], Section 4.2. The initial-boundary-value problem in this case is given by

$$P_{\mathbf{G},\mathring{\mathsf{A}},\mathring{\mathsf{C}},\rho,\mathring{\mathbf{S}},\mathring{\mathbf{P}}}\,\mathbf{U} = 0 \qquad \text{on } \Omega, \tag{37a}$$

$$\mathbf{U}(X,t) = f(X,t) \qquad \text{for } X \in \partial\Omega, \tag{37b}$$

$$\mathbf{U}(X,t) = 0 \qquad \text{for } t = 0, \tag{37c}$$

$$\partial\mathbf{U}(X,t)/\partial t = 0 \qquad \text{for } t = 0, \tag{37d}$$

with Dirichlet boundary data f, where the operator $P_{\mathbf{G},\mathring{\mathsf{A}},\mathring{\mathsf{C}},\rho,\mathring{\mathbf{S}},\mathring{\mathbf{P}}}$ is given by

$$P_{\mathbf{G},\mathring{\mathsf{A}},\mathring{\mathsf{C}},\rho,\mathring{\mathbf{S}},\mathring{\mathbf{P}}}\,\mathbf{U} = \rho\Big(\frac{\partial^2\mathbf{U}(X,t)}{\partial t^2} + \mathring{\mathbf{A}}\Big) - \mathrm{DIV}_{\mathbf{G}}(\mathring{\mathsf{A}} \cdot \nabla\mathbf{U} + \mathring{\mathbf{P}}). \tag{38}$$

Here $\rho(X)$ is the mass density in the reference configuration; $\mathbf{U}$, the *infinitesimal displacement*, is a vector field over the motion $\mathring{\phi}_t$ (that is, a two-point vector over $\mathring{\phi}_t$); and $\mathring{\mathsf{A}}$, the first elasticity tensor, is given by

$$\mathring{\mathsf{A}}^{aA}{}_b{}^B = 2\mathring{\mathsf{C}}^{EAFB}\mathring{\mathrm{F}}^a{}_E\mathring{\mathrm{F}}^c{}_F\mathring{g}_{bc} + \mathring{\mathrm{S}}^{AB}\delta^a{}_b.$$

In addition, $\mathring{\mathbf{A}}(X,t)$ is the material acceleration, $\mathring{\mathsf{C}}$ is the (second) elasticity tensor defined in (31), $\mathring{\mathbf{F}}$ is the deformation gradient, and $\mathring{\mathbf{P}}$ and $\mathring{\mathbf{S}}$ are the first and second Piola-Kirchhoff stress tensors, respectively, all associated with $\mathring{\phi}_t$.

We emphasize that $\mathbf{U}$ does not behave like a vector field on Ω; in fact, the covariant derivative ∇ (w.r.t $\mathbf{G}$) of the two-point vector $\mathbf{U}$ over $\mathring{\phi}_t$ is given by

$$(\nabla\mathbf{U})^b{}_B = \partial\mathbf{U}^b/\partial X^B + (\mathring{\gamma}^b_{ac} \circ \mathring{\phi}_t)\,\mathbf{U}^a\,\mathrm{F}^c{}_B. \tag{39}$$

The Dirichlet-to-Neumann map is defined in this case by

$$\begin{aligned}(\Lambda_{\mathbf{G},\mathring{\mathsf{A}},\mathring{\mathsf{C}},\rho,\mathring{\mathbf{S}},\mathring{\mathbf{P}}}\,f\,)^a &= \big\langle \mathring{\mathsf{A}} \cdot \nabla\mathbf{U} + \mathring{\mathbf{P}},\ \boldsymbol{\nu}\big\rangle^a\, dS_{\mathbf{G}} \\ &= \big[\mathring{\mathsf{A}}^{aA}{}_b{}^B\,(\nabla\mathbf{U})^b{}_B + \mathring{\mathrm{P}}^{aA}\big]\,G_{AC}\,\nu^C_{\mathbf{G}}\,dS_{\mathbf{G}}\end{aligned} \tag{40}$$

for $t \in [0,T]$, where, again, $dS_{\mathbf{G}}$ is the area form on $\partial\Omega$ induced by $\mathbf{G}$, $\boldsymbol{\nu}_{\mathbf{G}}$ is the unit outer normal to $\partial\Omega$ with respect to $\mathbf{G}$, and $\mathbf{U}$ solves (37) with boundary data f.

By Corollary 2 the Dirichlet-to-Neumann map may also be written with respect to the Euclidean metric as

$$(\Lambda_{\mathbf{G},\mathring{\mathsf{A}},\mathring{\mathsf{C}},\rho,\mathring{\mathbf{S}},\mathring{\mathbf{P}}}\,f\,)^a = \big[\mathring{\mathsf{A}}^{aA}{}_b{}^B\,(\nabla\mathbf{U})^b{}_B + \mathring{\mathrm{P}}^{aA}\big]\ \delta_{AC}\ \nu^C\ \sqrt{\det\mathbf{G}}\ dS, \tag{41}$$

where, again, dS is the area form on $\partial\Omega$, and $\boldsymbol{\nu}$ is the unit outer normal vector to $\partial\Omega$, each with respect to the Euclidean metric.

The classical linear case. Linearizing about a stress-free, undeformed state $\mathring{\phi}(X,t)$, which we choose to be an embedding $\iota : \Omega \hookrightarrow \mathbb{R}^3$, results in the classical linear equations (6) and the associated Dirichlet-to-Neumann map (12). (See (8) and (14) for their representation in Euclidean coordinates.) In fact, in this case, $\mathring{\mathbf{S}}$, $\mathring{\mathbf{P}}$, and $\mathring{\mathbf{A}}$ vanish, so that (38) reduces to

$$P_{\mathbf{G},\mathsf{C},\rho}\,\mathbf{U} \;=\; \rho\frac{\partial^2 \mathbf{U}}{\partial t^2} \;-\; \mathrm{DIV}_{\mathbf{G}}(\mathsf{A}\cdot\nabla\mathbf{U}).$$

If the spatial metric $\mathbf{g}$ is Euclidean, as we take in applications, then ∇ acts on $\mathbf{U}$ as if $\mathbf{U}$ were a scalar, and the equations of motion, in local coordinates, are

$$\rho\frac{\partial^2 U^a}{\partial t^2} \;-\; \frac{1}{\sqrt{\det \mathbf{G}}}\,\frac{\partial}{\partial X^A}\left(\sqrt{\det\,\mathbf{G}}\;\; \mathsf{A}^{aA}{}_b{}^B\frac{\partial U^b}{\partial X^B}\right) \;=\; 0.$$

where A depends on the elasticity tensor C in this case simply via $\mathsf{A}^{aA}{}_b{}^B = 2\mathsf{C}^{EAFB}(D\iota)^a{}_E(D\iota)^c{}_F\delta_{bc}$.

Given any smooth diffeomorphism ψ of Ω, we observe that $\mathring{\phi}'(X,t) = \mathring{\phi}(\psi(X),t)$ is also stress-free, since the stress tensor $\mathbf{S}$ associated with a motion is covariant (see Section 4), so $\mathring{\mathbf{S}}' = \psi^*\mathring{\mathbf{S}} = 0$ since $\mathring{\mathbf{S}} = 0$.

It follows that when the equations of motion are rewritten in Euclidean coordinates, there are no “lower-order” terms arising from the Christoffel symbols, and so the equations of motion continue to describe an elastic response, but with respect to modified physical parameters. Therefore, as in the analysis of the wave equation, we may reduce anisotropic elasticity to the isotropic case by pulling back by ψ, at least for certain anisotropic media.

4. Proof of Theorem 1

We begin with **the non-linear case.** We model the behavior of the hyperelastic body $(\Omega, \mathbf{G}, E, \rho)$ by the initial-boundary-value problem (32). We will first show that the equations of motion (32a) are covariant; that is, given a sufficiently regular motion ϕ_t and any diffeomorphism $\psi : \bar{\Omega} \to \bar{\Omega}$, we show that

$$\psi^*(F_{\mathbf{G},E,\rho}\,\phi_t) \;=\; F_{\mathbf{G}',E',\rho'}\,\phi_t', \tag{42}$$

where $X' = \psi^{-1}(X)$, $\phi'(X',t) = \phi(X,t)$, $\mathbf{G}' = \psi^*\mathbf{G}$, $E' = \psi^*E = E\circ\psi$, and $\rho' = \psi^*\rho = \rho\circ\psi$.

It is established [21] that the equations of elastodynamics are covariant in Eulerian coordinates. Below we extend the argument to Lagrangian coordinates, with special consideration taken for the anisotropy of the energy E in Lagrangian coordinates.

To prove (42) we begin by writing the material acceleration, the first Piola-Kirchhoff stress tensor, and the second Piola-Kirchhoff stress tensor associated with $(\psi^*\mathbf{G}, \psi^*E, \psi^*\rho)$ and ${\phi_t}'$ as $\mathbf{A}'$, $\mathbf{P}'$, and $\mathbf{S}'$, respectively.

Note that the representation (33) of $\mathbf{A}$ in terms of the velocity $\mathbf{V}$, which is given in terms of ϕ_t, implies that $\mathbf{A}'(X,t) = \mathbf{A}\big(\psi(X),t\big) = \psi^*\mathbf{A}$. (In particular, $\mathbf{A}$ behaves like a scalar with respect to ψ.) In addition, $\mathbf{S}'^{AB} = 2\rho' \left(\partial(E\circ\psi)/\partial(\psi^*\mathbf{C}^\flat)\right)^{AB}$ since $\mathbf{C}^{\flat\prime} = \phi_t'^*\mathbf{e} = (\phi_t\circ\psi)^*\mathbf{e} = \psi^*\mathbf{C}^\flat$.

Then $\psi^*(F_{\mathbf{G},E,\rho}\,\phi_t) = \rho'\mathbf{A}' - \psi^*(\mathrm{DIV}_{\mathbf{G}}\mathbf{P}) = \rho'\mathbf{A}' - \mathrm{DIV}_{\psi^*\mathbf{G}}\psi^*\mathbf{P}$ by the Piola transform ([21], pp. 116–118). We conclude that (42) follows if $\mathbf{P}' = \psi^*\mathbf{P}$.

Now $\mathbf{P}^{aB} = \mathrm{F}^a{}_A S^{AB}$, and $\mathrm{F}'^a{}_C = \partial\phi'^a(X',t)/\partial X'^C = (\mathrm{F}^a{}_A \circ \psi)(D\psi)^A{}_C = (\psi^*\mathbf{F})^a{}_C$, so $(\psi^*\mathbf{P})^{aB'} = (\psi^*\mathbf{F})^a{}_C(\psi^*\mathbf{S})^{CB'} = \mathrm{F}'^a{}_C(\psi^*S)^{CB'}$. Therefore, to show $\psi^*\mathbf{P} = \mathbf{P}'$ we will show $\psi^*\mathbf{S} = \mathbf{S}'$.

Under the constitutive hypothesis for hyperelastic media, the free energy function $E = \overline{E}(X,\mathbf{G},\mathbf{C}^\flat)$ is related to $S = \bar{S}(X,\mathbf{G},\mathbf{C}^\flat)$ by

$$S^{AB} \;=\; 2\rho\,\frac{\partial\overline{E}}{\partial\mathbf{C}^\flat{}_{AB}}. \tag{43}$$

(See [21], Proposition 4.6.) In addition, the balance laws (conservation of mass, balance of momentum, balance of moment of momentum, and balance of energy) that describe the behavior of hyperelastic media are equivalent to the covariance of the energy [21], Theorem 4.13. But the referential energy E is covariant, in the case that E depends functionally just on the point X and the Cauchy-Green deformation tensor $\mathbf{C}$, only if the elastic medium is isotropic.

To include anisotropic media in the discussion we first recall that any particular type of material anisotropy may be characterized by its material symmetry group. (See Section 2.2.) We may associate with any material symmetry group $\mathcal{G}$ a finite collection of *structural tensors* $\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_m$ which form a basis for the space of invariant tensors under the action of $\mathcal{G}$; that is, the structural tensors $\boldsymbol{\zeta}_i$ (of order k_i), $i = 1,\ldots,m$, satisfy

$$\mathbf{Q}\in\mathcal{G} \qquad \text{iff} \qquad \langle\mathbf{Q}\rangle_{k_1}\boldsymbol{\zeta}_1 \;=\; \boldsymbol{\zeta}_1,\quad \ldots,\quad \langle\mathbf{Q}\rangle_{k_m}\boldsymbol{\zeta}_m \;=\; \boldsymbol{\zeta}_m,$$

where the Kronecker product $\langle\mathbf{Q}\rangle_k$ of an orthogonal transformation $\mathbf{Q}$ is defined by

$$(\langle\mathbf{Q}\rangle_k\zeta)^{i\ldots j} \;=\; Q^i{}_{\underline{i}}\cdots Q^j{}_{\underline{j}}\zeta^{\underline{i}\ldots\underline{j}} \tag{44}$$

for any tensor $\boldsymbol{\zeta}$ of order k, with $Q^i{}_j$, $\zeta^{i\ldots j}$ the Cartesian components of $\mathbf{Q}$ and $\boldsymbol{\zeta}$.

Therefore, we write, given [21], p. 220, [19], and [42], the energy E as a function $E = \widehat{E}(\mathbf{G},\mathbf{C}^\flat,\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_m)$ of $\mathbf{G}$, $\mathbf{C}^\flat$, and the structural tensors $\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_m$. It follows from (43) that $S^{AB} = 2\rho\left(\partial\widehat{E}/\partial\mathbf{C}^\flat\right)^{AB} = 2\rho\,\partial\widehat{E}/\partial\mathbf{C}^\flat{}_{AB}$.

Given any diffeomorphism $\psi:\bar{\Omega}\to\bar{\Omega}$, we define the function $\check{E}$ of $\bar{\mathbf{G}}$, $\bar{\mathbf{C}}^\flat,\bar{\boldsymbol{\zeta}}_1,\ldots,\bar{\boldsymbol{\zeta}}_m$ by $\check{E}(\bar{\mathbf{G}},\bar{\mathbf{C}}^\flat,\bar{\boldsymbol{\zeta}}_1,\ldots,\bar{\boldsymbol{\zeta}}_m) = [\widehat{E}(\psi_*\bar{\mathbf{G}},\psi_*\bar{\mathbf{C}}^\flat,\psi_*\bar{\boldsymbol{\zeta}}_1,\ldots,\psi_*\bar{\boldsymbol{\zeta}}_m)]\circ\psi$. Then, due to the covariance of the energy, $\psi^*E = E\circ\psi =$

$\check{E}(\psi^*\mathbf{G}, \psi^*\mathbf{C}^\flat, \psi^*\boldsymbol{\zeta}_1, \ldots, \psi^*\boldsymbol{\zeta}_m)$. It follows that

$$\psi^*\Big(\frac{\partial \widehat{E}}{\partial \mathbf{C}^\flat}\Big)^{AB} \;=\; (D\psi^{-1})^A{}_{\underline{A}}(D\psi^{-1})^B{}_{\underline{B}} \cdot \Big(\frac{\partial \widehat{E}}{\partial \mathbf{C}^\flat}\Big)^{\underline{A}\,\underline{B}} \circ \psi \tag{45}$$

and

$$\begin{aligned} \Big(\frac{\partial \widehat{E}}{\partial \mathbf{C}^\flat}\Big)^{\underline{A}\,\underline{B}} \circ \psi \;&=\; \Big(\frac{\partial \check{E}}{\partial(\psi^*\mathbf{C}^\flat)}\Big)^{A'B'} \cdot \frac{\partial(\psi^*\mathbf{C}^\flat \circ \psi^{-1})_{A'B'}}{\partial \mathbf{C}^\flat_{\underline{A}\,\underline{B}}} \\ &=\; \Big(\frac{\partial \check{E}}{\partial(\psi^*\mathbf{C}^\flat)}\Big)^{A'B'} \cdot (D\psi)^{\underline{A}}{}_{A'}(D\psi)^{\underline{B}}{}_{B'}. \end{aligned} \tag{46}$$

Consequently, $\psi^*\mathbf{S} = \mathbf{S}'$ since

$$\psi^*\mathbf{S} \;=\; 2(\rho\circ\psi)\,\psi^*\big(\frac{\partial \widehat{E}}{\partial \mathbf{C}^\flat}\big) \;=\; 2\rho'\,\frac{\partial \check{E}}{\partial(\psi^*\mathbf{C}^\flat)} \;=\; \mathbf{S}'. \tag{47}$$

The Dirichlet-to-Neumann maps agree in this case because $\psi \in \Psi$ is the identity at the boundary to first order, which implies that $\mathbf{G}' = \psi^*\mathbf{G}$ is $\mathbf{G}$ at $\partial\Omega$ and $\mathbf{P}' = \psi^*\mathbf{P}$ is $\mathbf{P}$ at $\partial\Omega$. That is, $\Lambda_{\mathbf{G},E,\rho} f \;=\; \langle \mathbf{P}, \boldsymbol{\nu}\rangle \sqrt{\det \mathbf{G}}\; dS$ agrees with $\Lambda_{\mathbf{G}',E',\rho'} f \;=\; \langle \mathbf{P}', \boldsymbol{\nu}\rangle \sqrt{\det \mathbf{G}'}\; dS$ at $\partial\Omega$.

The linearized case. In this case we model the displacement of the hyperelastic body by the initial-boundary-value problem (37). To check that the equations of motion (37a) are covariant, we consider a motion $\mathring{\phi}_t : (\Omega, \mathbf{G}) \to (\mathcal{S}, \mathbf{g})$ of an elastic body $(\Omega, \mathbf{G}, \mathring{E}, \mathring{\rho})$. Given a diffeomorphism $\psi \in \Psi$, we define $\mathring{\phi}'(X', t) : (\Omega, \psi^*\mathbf{G}) \to (\mathcal{S}, \mathbf{g})$ by $\mathring{\phi}'(X', t) = \mathring{\phi}(\psi(X'), t)$, where $X' = \psi^{-1}(X)$. It follows from the proof of covariance for the non-linear case that $\mathring{\phi}'$ is a motion of the transformed elastic body $(\Omega, \mathring{\mathbf{G}}', \mathring{E}', \mathring{\rho}')$, where $\mathring{\mathbf{G}}' = \psi^*\mathring{\mathbf{G}}$, $\mathring{\rho}' = \mathring{\rho}\circ\psi$, and $\mathring{E}' = \mathring{E}\circ\psi$. The same proof shows that $\mathring{\mathbf{F}}' = \psi^*\mathbf{F}$, $\mathring{\mathbf{P}}' = \psi^*\mathring{\mathbf{P}}$, and $\mathring{\mathbf{S}}' = \psi^*\mathring{\mathbf{S}}$. Also, $\mathring{\mathsf{C}}' = \psi^*\mathring{\mathsf{C}}$ by (31) and the reasoning in (45) and (47).

It follows that $\mathring{\mathsf{A}}' = \psi^*\mathring{\mathsf{A}}$ since A depends tensorially on C, $\mathbf{F}$, and the metric. Explicitly, in components:

$$\begin{aligned} \mathring{\mathsf{A}}'^{aA'}{}_b{}^{B'} \;&=\; 2\,\mathring{\mathsf{C}}'^{E'A'F'B'}\mathring{\mathrm{F}}'^a{}_{E'}\mathring{\mathrm{F}}'^c{}_{F'}(\mathring{g}_{bc}\circ\phi') \;+\; \mathring{\mathbf{S}}'^{A'B'}(\delta^a{}_b\circ\phi') \\ &=\; 2(D\psi^{-1})^{E'}{}_E(D\psi^{-1})^{A'}{}_A(D\psi^{-1})^{F'}{}_F(D\psi^{-1})^{B'}{}_B(\mathring{\mathsf{C}}^{EAFB}\circ\psi) \\ &\quad\cdot\; (\mathring{\mathrm{F}}^a{}_E\mathring{\mathrm{F}}^c{}_F)\circ\psi(D\psi)^E{}_{E'}(D\psi)^F{}_{F'}(\mathring{g}_{bc}\circ\phi)\circ\psi \\ &\quad+\; (D\psi^{-1})^{A'}{}_A(D\psi^{-1})^{B'}{}_B(\mathring{\mathbf{S}}^{AB}\circ\psi)\;(\delta^a{}_b\circ\phi)\circ\psi \\ &=\; (D\psi^{-1})^{A'}{}_A(D\psi^{-1})^{B'}{}_B\;(\mathring{\mathsf{A}}^{aA}{}_b{}^B\circ\psi). \end{aligned} \tag{48}$$

We conclude that

$$\begin{aligned} [\nabla_{\psi^*\mathbf{G}}(\mathbf{U}\circ\psi)]^b{}_{B'} \;&=\; \frac{\partial(U^b\circ\psi)}{\partial X'^{B'}} \;+\; (\gamma^b_{\underline{a}c}\circ\phi')(U^{\underline{a}}\circ\psi)\mathrm{F}'^c{}_{B'} \\ &=\; [(\nabla_{\mathbf{G}}\mathbf{U})^b{}_B\circ\psi]\;(D\psi)^B{}_{B'}, \end{aligned} \tag{49}$$

where $[\nabla_{\mathbf{G}}\mathbf{U}]^b{}_B = \partial\mathbf{U}^b/\partial X^B + (\gamma^b_{ac}\circ\phi)\,\mathbf{U}^a\,\mathrm{F}^c{}_B$ is the covariant derivative of the two-point vector $\mathbf{U}$. Therefore, the linearized stress $[\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{aA} = \mathring{\mathrm{A}}^{aA}{}_b{}^B[\nabla_{\mathbf{G}}\mathbf{U}]^b{}_B$ satisfies $[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}']^{aA'} = \mathring{\mathrm{A}}'^{aA'}{}_b{}^{B'}[\nabla_{\psi^*\mathbf{G}}(\mathbf{U}\circ\psi)]^b{}_{B'} = (D\psi^{-1})^{A'}{}_A([\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{aA}\circ\psi) = (\psi^*[\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}U])^{aA'}$, where $\mathbf{U}' = \mathbf{U}\circ\psi$.

For completeness we check below that $(\mathrm{DIV}_{X'}[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}'])^a = (\mathrm{DIV}_X[\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}])^a\circ\psi$. (See (34) for the definition of the divergence of a two-point tensor.) We first define $\epsilon_{ABC} = 1$ if A, B, C is an even permutation of $1,2,3$, $\epsilon_{ABC} = -1$ if A,B,C is an odd permutation of $1,2,3$, and $\epsilon_{ABC} = 0$ otherwise. Then the cofactor matrix Cof M of a 3×3 matrix M is given by $(\mathrm{Cof}\ M)^{A'}{}_A = (1/2)\epsilon_{ABC}\ \epsilon^{A'B'C'}\ M^B{}_{B'}\,M^C{}_{C'}$. It follows that $\partial/\partial X'^{A'}(\mathrm{Cof}\ D\psi)^{A'}{}_A = \sum_{B,C,C'}\epsilon_{ABC}\ \cdot\ 0\ \cdot\partial X^C/\partial X'^{C'} = 0$. Also, $\sum_A \Gamma^A_{AB} = \partial/\partial X^B(\log\sqrt{\det\mathbf{G}})$.

Writing $M^{-1} = \mathrm{Cof}\ M/\det M$ and $\partial/\partial X'^{A'} = (D\psi)^A{}_{A'}(\partial/\partial X^A)$, we may therefore conclude that covariance holds, since

$$
\begin{aligned}
&(\mathrm{DIV}_{X'}[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}'])^a\\
&= \frac{\partial}{\partial X'^{A'}}[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}']^{aA'} + \Gamma'^{\underline{A}}_{\underline{A}A'}[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}']^{aA'}\\
&\qquad + (\gamma^a_{\underline{b}\,\underline{a}}\circ\phi')\mathring{\mathrm{F}}'^{\underline{b}}{}_{A'}[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}']^{\underline{a}A'}
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial X'^{A'}}\big[(D\psi^{-1})^{A'}{}_A([\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{aA}\circ\psi)\big]\\
&\quad + \Big(\frac{\partial}{\partial X'^{A'}}\log\sqrt{\det\ \psi^*\mathbf{G}}\Big)(D\psi^{-1})^{A'}{}_A([\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{aA}\circ\psi)\\
&\quad + \big((\gamma^a_{\underline{b}\,\underline{a}}\circ\phi)\circ\psi\big)\,(\mathring{\mathrm{F}}^{\underline{b}}{}_A\circ\psi)\,([\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{\underline{a}A}\circ\psi)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial X'^{A'}}\big[(\det\ D\psi^{-1})\,\mathrm{Cof}\ (D\psi)^{A'}{}_A\ ([\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{aA}\circ\psi)\big]\\
&\quad + \Big(\frac{\partial}{\partial X'^{A'}}\log(\det\ D\psi) + (D\psi)^A{}_{A'}\Gamma^{\underline{A}}_{\underline{A}A}\Big)\\
&\qquad\qquad \cdot(D\psi^{-1})^{A'}{}_A([\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{aA}\circ\psi)\\
&\quad + \Big((\gamma^a_{\underline{b}\,\underline{a}}\circ\phi)\,\mathring{\mathrm{F}}^{\underline{b}}{}_A\,[\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{\underline{a}A}\Big)\circ\psi\\
&= (\mathrm{DIV}_X[\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}])^a\circ\psi.
\end{aligned}
\tag{51}
$$

It follows, since $\psi^*(\mathrm{DIV}_{\mathbf{G}}\mathring{\mathbf{P}}) = \mathrm{DIV}_{\psi^*\mathbf{G}}\psi^*\mathring{\mathbf{P}}$, that

$$
\begin{aligned}
&\psi^*(P_{\mathbf{G},\mathring{\mathrm{A}},\mathring{E},\mathring{\rho},\mathring{\mathbf{S}},\mathring{\mathbf{P}}}\ \mathbf{U})\\
&= (\mathring{\rho}\circ\psi)\Big(\frac{\partial^2\mathbf{U}(X,t)\circ\psi}{\partial t^2} + \psi^*\mathring{\mathbf{A}}\Big) - \psi^*\Big(\mathrm{DIV}_{\mathbf{G}}(\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U} + \mathring{\mathbf{P}})\Big)\\
&= P_{\mathbf{G}',\mathring{\mathrm{A}}',\mathring{E}',\mathring{\rho}',\mathring{\mathbf{S}}',\mathring{\mathbf{P}}'}\ \mathbf{U}'.
\end{aligned}
\tag{52}
$$

Again, the Dirichlet-to-Neumann maps agree because $\psi\in\Psi$ is the identity at the boundary, which implies that $\mathbf{G}' = \psi^*\mathbf{G}$ is $\mathbf{G}$ at $\partial\Omega$,

$\mathbf{P}' = \psi^*\mathbf{P}$ is $\mathbf{P}$ at $\partial\Omega$, and $\mathring{\mathbf{A}}' \cdot \nabla_{\mathbf{G}'}\mathbf{U}' = \psi^*[\mathring{\mathbf{A}} \cdot \nabla_{\mathbf{G}}U]$ is $\mathring{\mathbf{A}} \cdot \nabla_{\mathbf{G}}\mathbf{U}$ at $\partial\Omega$. That is, $\Lambda_{\mathbf{G}',\mathring{\mathrm{A}}',\mathring{E}',\mathring{\rho}',\mathring{\mathbf{S}}',\mathring{\mathbf{P}}'}\ f = \langle[\mathring{\mathbf{A}}' \cdot \nabla_{\mathbf{G}'}\mathbf{U}'] + \mathring{\mathbf{P}}', \boldsymbol{\nu}\rangle \sqrt{\det \mathbf{G}'}\, dS$ agrees with $\Lambda_{\mathbf{G},\mathring{\mathrm{A}},\mathring{E},\mathring{\rho},\mathring{\mathbf{S}},\mathring{\mathbf{P}}}\ f = \langle[\mathring{\mathbf{A}} \cdot \nabla_{\mathbf{G}}\mathbf{U}] + \mathring{\mathbf{P}}, \boldsymbol{\nu}\rangle \sqrt{\det \mathbf{G}}\ dS$ where $\langle \mathbf{P}, \boldsymbol{\nu}\rangle^a = \mathbf{P}^{aA}\delta_{AB}\boldsymbol{\nu}^B$.

The classical linear case. Here we take $\mathring{\phi}_t$ to be a stress-free, undeformed state $\mathring{\phi}_t(X,t) = \iota(X)$, where ι is an embedding $(\Omega, \mathbf{G}) \to (\mathcal{S}, \mathbf{g})$. Therefore, $\mathring{\mathbf{S}}, \mathring{\mathbf{P}} = 0$, and $\mathring{\mathbf{A}} = 0$. We observe that $\mathring{\phi}'(X,t) = \mathring{\phi}(\psi(X),t)$ is also stress-free since the stress tensor $\mathbf{S}$ associated with a motion is covariant, so $\mathring{\mathbf{S}} = 0$ implies $\mathring{\mathbf{S}}' = \psi^*\mathring{\mathbf{S}} = 0$.

In applications, we take $\mathbf{g} = \mathbf{e}$ so that $\gamma^a_{bc} = 0$. Then, a calculation similar to (50) along with (39) establishes the local-coordinate representation (8) of the equations of motion. □

Remark 16. For classical linear elasticity, a weak formulation of the equations of motion, in terms of Dirichlet integrals, holds (see [21]). In this case, as it happens with the scalar wave equation, it may be possible to use the weak formulation to obtain obstruction to uniqueness for diffeomorphisms ψ that fix the boundary to zeroth-order only; i.e., for which $D\psi$ is not necessarily the identity at the boundary.

5. Proof of Theorem 11

5.1. **The harmonic-Cartan decomposition of $\bar{\mathsf{C}}$.** We write $\bar{\mathsf{C}}$ (at any point $X \in \Omega$ and with respect to any orthogonal basis e_1, e_2, e_3, here the standard basis for Euclidean space) in terms of its harmonic decomposition (as described in [8])

$$\bar{\mathsf{C}} \;=\; \bar{\alpha} I \otimes I \;+\; \bar{\beta} I \circledast I \;+\; 2I \circledS \bar{A} \;+\; 2I \circledast \bar{B} \;+\; \bar{\mathsf{H}}, \tag{53}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are scalars, $\bar{A}$ and $\bar{B}$ are symmetric, traceless 3×3 matrices (i.e., with trace $\bar{A}^{11} + \bar{A}^{22} + \bar{A}^{33}$ equal to zero), and $\bar{\mathsf{H}}$ is a totally symmetric rank-four tensor (i.e., $\bar{\mathsf{H}}^{ijkl} = \bar{\mathsf{H}}^{\sigma(ijkl)}$, where σ is any permutation of the (i,j,k,l) indices). In addition, $\bar{\mathsf{H}}$ is harmonic; that is, for $\mathbf{r} = xe_1 + ye_2 + ze_3$,

$$\bar{\mathsf{H}}[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] = \sum_{i,j,k,l=1}^{3} \bar{\mathsf{H}}^{ijkl} x^{e(1)} y^{e(2)} z^{e(3)} \tag{54}$$

is a harmonic polynomial in x, y, z, i.e., $(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)\bar{\mathsf{H}}[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] = 0$. Here δ_{ij} is the Kronecker delta, and we use the notation

$$
\begin{aligned}
e(m) &= \delta_{im} + \delta_{jm} + \delta_{km} + \delta_{lm} \\
(A \circledS B)^{ijkl} &= \frac{1}{2}(A \otimes B \ + \ B \otimes A)^{ijkl} \ = \ \frac{1}{2}(A^{ij}B^{kl} \ + \ B^{ij}A^{kl}) \\
A \diamondsuit B &= \frac{1}{2}(A \diamondsuit B \ + \ B \diamondsuit A) \\
(A \diamondsuit B)^{ijkl} &= \quad A^{ik}B^{jl} \ + \ A^{il}B^{jk}.
\end{aligned} \tag{55}
$$

The harmonic decomposition (53) of $\bar{\mathsf{C}}$ may be written in terms of the components of $\bar{\mathsf{C}}$ as follows:

$$
\begin{aligned}
\bar{\alpha} &= \frac{1}{15}\big(2\bar{\mathsf{C}}^{ppkk} \ - \ \bar{\mathsf{C}}^{pkpk}\big) \\
\bar{\beta} &= \frac{1}{30}\big(-\bar{\mathsf{C}}^{ppkk} \ + \ 3\bar{\mathsf{C}}^{pkpk}\big) \\
\bar{A}^{ij} &= \frac{1}{21}\big(15\bar{\mathsf{C}}^{ijkk} \ - \ 12\bar{\mathsf{C}}^{ikjk} \ - \ 5\delta^{ij}\bar{\mathsf{C}}^{ppkk} \ + \ 4\delta^{ij}\bar{\mathsf{C}}^{pkpk}\big) \\
\bar{B}^{ij} &= \frac{-1}{21}\big(6\bar{\mathsf{C}}^{ijkk} \ - \ 9\bar{\mathsf{C}}^{ikjk} \ - \ 2\delta^{ij}\bar{\mathsf{C}}^{ppkk} \ + \ 3\delta^{ij}\bar{\mathsf{C}}^{pkpk}\big) \\
\bar{H}^{ijkl} &= \frac{1}{3}\big(\bar{\mathsf{C}}^{ijkl} \ + \ \bar{\mathsf{C}}^{ikjl} \ + \ \bar{\mathsf{C}}^{iljk}\big) \\
&\quad - \frac{1}{21}\Big[(\bar{\mathsf{C}}^{ijmm} \ + \ 2\bar{\mathsf{C}}^{imjm})\delta^{kl} \ + \ (\bar{\mathsf{C}}^{klmm} \ + \ 2\bar{\mathsf{C}}^{kmlm})\delta^{ij} \\
&\qquad + (\bar{\mathsf{C}}^{ikmm} \ + \ 2\bar{\mathsf{C}}^{imkm})\delta^{jl} \ + \ (\bar{\mathsf{C}}^{jlmm} \ + \ 2\bar{\mathsf{C}}^{imjm})\delta^{ik} \\
&\qquad + (\bar{\mathsf{C}}^{ilmm} \ + \ 2\bar{\mathsf{C}}^{imlm})\delta^{jk} \ + \ (\bar{\mathsf{C}}^{jkmm} \ + \ 2\bar{\mathsf{C}}^{imjm})\delta^{il}\Big] \\
&\quad + \frac{1}{105}\big(\bar{\mathsf{C}}^{ppmm} + 2\bar{\mathsf{C}}^{pmpm}\big)\big(\delta^{ij}\delta^{kl} \ + \ \delta^{ik}\delta^{jl} \ + \ \delta^{il}\delta^{jk}\big).
\end{aligned} \tag{56}
$$

(Note the typographical error in [8] in the expression for $\bar{\beta}$.)

We then write $\bar{\mathbf{H}}$ in terms of the Cartan decomposition (as described in [8]):

$$
\bar{\mathbf{H}} \ = \ \bar{\alpha}_0\mathbf{U} \ + \ \sum_{m=1}^{4}\bar{\alpha}_m\boldsymbol{\mathcal{S}}_m \ + \ \bar{\beta}_m\boldsymbol{\mathcal{T}}_m, \tag{57}
$$

where $\bar{\alpha}_0, \bar{\alpha}_i, \bar{\beta}_i, \ i = 1...4,$ are scalars, and the harmonic tensors $\mathbf{U}, \boldsymbol{\mathcal{S}}_i, \boldsymbol{\mathcal{T}}_i, \ i = 1...4,$ have polynomial form

$$
\begin{aligned}
\mathbf{U}[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= u(x,y,z) \ = \ 8z^4 + 3(x^2+y^2)(x^2+y^2-8z^2) \\
\boldsymbol{\mathcal{S}}_1[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= s_1(x,y,z) \ = \ zx\big(4z^2 - 3[x^2+y^2]\big) \\
\boldsymbol{\mathcal{T}}_1[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= t_1(x,y,z) \ = \ zy\big(4z^2 - 3[x^2+y^2]\big) \\
\boldsymbol{\mathcal{S}}_2[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= s_2(x,y,z) \ = \ (x^2-y^2)\big(6z^2 - [x^2+y^2]\big) \\
\boldsymbol{\mathcal{T}}_2[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= t_2(x,y,z) \ = \ 2xy\big(6z^2 - [x^2+y^2]\big)
\end{aligned} \tag{58}
$$

$$
(59)\qquad
\begin{array}{rcccl}
\boldsymbol{\mathcal{S}}_3[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] & = & s_3(x,y,z) & = & zx(x^2-3y^2)\\
\boldsymbol{\mathcal{T}}_3[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] & = & t_3(x,y,z) & = & zy(3x^2-y^2)\\
\boldsymbol{\mathcal{S}}_4[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] & = & s_4(x,y,z) & = & x^4+y^4-6x^2y^2\\
\boldsymbol{\mathcal{T}}_4[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] & = & t_4(x,y,z) & = & 4xy(x^2-y^2).
\end{array}
$$

We refer to their polynomial form (54) to compute the components of U, for example:
(60)

$$
U^{3,3,3,3}=8,\qquad U^{1,1,1,1}=3,\qquad U^{2,2,2,2}=3,\qquad \sum_{\sigma\in S_4} U^{\sigma(1,1,3,3)}=-24,
$$

$$
\sum_{\sigma\in S_4} U^{\sigma(2,2,3,3)}=-24,\qquad\qquad \sum_{\sigma\in S_4} U^{\sigma(1,1,2,2)}=6,
$$

where S_4 is the symmetric group of permutations of four elements.

It follows that the components $\bar{\mathsf{H}}^{ijkl}$ of $\bar{\mathsf{\mathbf{H}}}$ are given by the following:

$$
(61)\qquad
\begin{array}{r|c|c|c}
ij\backslash kl & 11 & 22 & 33\\
\hline
11 & 3\bar{\alpha}_0-\bar{\alpha}_2+\bar{\alpha}_4 & \bar{\alpha}_0-\bar{\alpha}_4 & -4\bar{\alpha}_0+\bar{\alpha}_2\\
22 & & 3\bar{\alpha}_0+\bar{\alpha}_2+\bar{\alpha}_4 & -4\bar{\alpha}_0-\bar{\alpha}_2\\
33 & & & 8\bar{\alpha}_0
\end{array}
$$

$$
(62)\qquad
\begin{array}{r|c|c|c}
ij\backslash kl & 12 & 23 & 13\\
\hline
11 & -\bar{\beta}_2/2+\bar{\beta}_4 & -[\bar{\beta}_1-\bar{\beta}_3]/4 & -[3\bar{\alpha}_1-\bar{\alpha}_3]/4\\
22 & -\bar{\beta}_2/2-\bar{\beta}_4 & -[3\bar{\beta}_1+\bar{\beta}_3]/4 & -[\bar{\alpha}_1+\bar{\alpha}_3]/4\\
33 & \bar{\beta}_2 & \bar{\beta}_1 & \bar{\alpha}_1\\
12 & \bar{\alpha}_0-\bar{\alpha}_4 & -[\bar{\alpha}_1+\bar{\alpha}_3]/4 & -[\bar{\beta}_1-\bar{\beta}_3]/4\\
23 & & -4\bar{\alpha}_0-\bar{\alpha}_2 & \bar{\beta}_2\\
13 & & & -4\bar{\alpha}_0+\bar{\alpha}_2
\end{array}
$$

while, writing $M=(\bar{\alpha}+2\bar{\beta})I+2(\bar{A}+2\bar{B})$, $\bar{\mathsf{\mathbf{C}}}-\bar{\mathsf{\mathbf{H}}}$ is given by:

$$
(63)\qquad
\begin{array}{r|c|c|c}
ij\backslash kl & 11 & 22 & 33\\
\hline
11 & M^{11} & \bar{\alpha}+\bar{A}^{11}+\bar{A}^{22} & \bar{\alpha}-\bar{A}^{22}\\
22 & & M^{22} & \bar{\alpha}-\bar{A}^{11}\\
33 & & & M^{33}
\end{array}
$$

$$
(64)\qquad
\begin{array}{r|c|c|c}
ij\backslash kl & 12 & 23 & 13\\
\hline
11 & (\bar{A}+2\bar{B})^{12} & \bar{A}^{23} & (\bar{A}+2\bar{B})^{13}\\
22 & (\bar{A}+2\bar{B})^{12} & (\bar{A}+2\bar{B})^{23} & \bar{A}^{13}\\
33 & \bar{A}^{12} & (\bar{A}+2\bar{B})^{23} & (\bar{A}+2\bar{B})^{13}\\
12 & \bar{\beta}+\bar{B}^{11}+\bar{B}^{22} & \bar{B}^{13} & \bar{B}^{23}\\
23 & & \bar{\beta}-\bar{B}^{11} & \bar{B}^{12}\\
13 & & & \bar{\beta}-\bar{B}^{22}
\end{array}
$$

5.2. **Describing $\mathrm{Sym}_e\bar{\mathbf{C}}$ in terms of $\bar{A}, \bar{B}, \bar{\alpha}_i, \bar{\beta}_i$.** The symmetry group of $\bar{\mathbf{C}}$ is given [8] in terms of the symmetry groups of $\bar{A}$, $\bar{B}$, and $\bar{\mathbf{H}}$ by

$$\mathrm{Sym}_e(\bar{\mathbf{C}}) \;=\; \mathrm{Sym}_e(\bar{A}) \;\cap\; \mathrm{Sym}_e(\bar{B}) \;\cap\; \mathrm{Sym}_e(\bar{\mathbf{H}}). \tag{65}$$

In addition, following the reasoning given in [8] we present in Table 2 a more precise description of $\mathrm{Sym}_e\bar{\mathbf{H}}$, almost entirely in terms of the vanishing of $\bar{\alpha}_0, \bar{\alpha}_i, \bar{\beta}_i,\ i = 1..4$. Note that each of the entries (i.e., sets of rows) of Table 2 is numbered as in Table 1 to indicate its type of (an)isotropy. In Table 2 and below we denote by $\mathcal{Z}_n, \mathcal{D}_n, \mathcal{O}, SO(n), O(n)$ the cyclic, dihedral, octahedral, special orthogonal, and orthogonal groups. We emphasize that the entries in Table 2 depend on a choice of orthogonal basis $\{e_1, e_2, e_3\}$ since the $\bar{\alpha}_0, \bar{\alpha}_i, \bar{\beta}_i$ may change with a change of basis.

We observe that the polynomial forms $\mathbf{r}^T\bar{A}\mathbf{r} = \bar{A}_{11}x^2 + 2\bar{A}_{12}xy + \ldots$ and $\mathbf{r}^T\bar{B}\mathbf{r}$ of the harmonic 2-tensors $\bar{A}$ and $\bar{B}$ may also be decomposed in terms of the basis

$$\begin{aligned}\tilde{u} \;&=\; z^2 - \frac{1}{2}(x^2+y^2), \qquad \tilde{s}_1 \;=\; zx, \qquad \tilde{t}_1 \;=\; zy,\\ \tilde{s}_2 \;&=\; x^2 - y^2, \qquad \tilde{t}_2 \;=\; 2xy\end{aligned}$$

of homogeneous, degree-2 polynomials in x, y, z. In fact,

$$\begin{aligned}\mathbf{r}^T A'\mathbf{r} \;&=\; \tilde{\alpha}_0\tilde{u} \;+\; \tilde{\alpha_1}\tilde{s}_1 \;+\; \tilde{\beta}_1\tilde{t}_1 \;+\; \tilde{\alpha}_2\tilde{s}_2 \;+\; \tilde{\beta}_2\tilde{t}_2\\ &=\; A'_{33}\tilde{u} \;+\; 2A'_{13}\tilde{s}_1 \;+\; 2A'_{23}\tilde{t}_1 \;+\; \frac{1}{2}(A'_{11} - A'_{22})\tilde{s}_2 \;+\; A'_{12}\tilde{t}_2\end{aligned}$$

for any symmetric, traceless, rank-2 tensor A'. It follows that the symmetry groups of $\bar{A}$ and $\bar{B}$ may be described in terms of the vanishing of their components, as summarized in Table 3.

In addition, as shown in [8],

$$\mathrm{Sym}_e(\bar{A}) \;\cap\; \mathrm{Sym}_e(\bar{B}) \;\cong\; \{I\},\ \mathcal{Z}_2,\ \mathcal{D}_2,\ O(2),\ \text{or}\ SO(3). \tag{66}$$

We apply this result, together with (24), (65), Table 2, and Table 3 below.

For orthotropic $\bar{\mathbf{C}}$ with $\mathrm{Sym}_e\bar{\mathbf{C}} = \{Q^\pi_{e_1}, Q^\pi_{e_2}, Q^\pi_{e_3}\}$ the possibilities for $(\mathrm{Sym}_e\bar{A} \cap \mathrm{Sym}_e\bar{B}) \cap \mathrm{Sym}_e\bar{\mathbf{H}}$, and the corresponding restrictions on the parameters describing $\bar{A}, \bar{B}$ and $\bar{\mathbf{H}}$ (in addition to the requirements that $\bar{A}$, $\bar{B}$ be diagonal, and $\bar{\alpha}_1,\ \bar{\beta}_1,\ \bar{\alpha}_3,\ \bar{\beta}_3,\ \bar{\beta}_2,\ \bar{\beta}_4 = 0$), are:

$$\begin{aligned}
&i) \quad \{Q^\pi_{e_1}, Q^\pi_{e_2}, Q^\pi_{e_3}\} \cap \{Q^\pi_{e_1}, Q^\pi_{e_2}, Q^\pi_{e_3}\},\\
&\qquad \bar{A}^{11} \neq \bar{A}^{22},\ \bar{B}^{11} \neq \bar{B}^{22},\ \ \bar{\alpha}_2 \neq 0\\
&ii) \quad \{Q^\pi_{e_1}, Q^\pi_{e_2}, Q^\phi_{e_3}\} \cap \{Q^\pi_{e_1}, Q^\pi_{e_2}, Q^\pi_{e_3}\},\\
&\qquad \bar{A}^{11} = \bar{A}^{22},\ \bar{B}^{11} = \bar{B}^{22},\ \ \bar{\alpha}_2 \neq 0\\
&iii) \quad O(3) \cap \{Q^\pi_{e_1}, Q^\pi_{e_2}, Q^\pi_{e_3}\},\\
&\qquad \bar{A}^{11},\ \bar{A}^{22},\ \bar{B}^{11},\ \bar{B}^{22} \;=\; 0,\ \ \bar{\alpha}_2 \neq 0
\end{aligned} \tag{67}$$

$$
(68)\qquad
\begin{aligned}
iv)\quad & \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi/2}_{e_3}\},\\
& \qquad \bar{A}^{11} \neq \bar{A}^{22},\ \bar{B}^{11} \neq \bar{B}^{22},\ \text{and } \bar{\alpha}_2 = 0,\ \bar{\alpha}_4 \neq 0\\
v)\quad & \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \cap \{Q^{\pi/2}_{e_1}, Q^{\pi/2}_{e_2}, Q^{\pi/2}_{e_3}\},\\
& \qquad \bar{A}^{11} \neq \bar{A}^{22},\ \bar{B}^{11} \neq \bar{B}^{22},\ \text{and } \bar{\alpha}_2 = 0,\ \bar{\alpha}_4 = 5\bar{\alpha}_0 \neq 0\\
vi)\quad & \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\},\\
& \qquad \bar{A}^{11} \neq \bar{A}^{22},\ \bar{B}^{11} \neq \bar{B}^{22},\ \text{and } \bar{\alpha}_2,\ \bar{\alpha}_4 = 0\\
vii)\quad & \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \cap O(3)\\
& \qquad \bar{A}^{11} \neq \bar{A}^{22},\ \bar{B}^{11} \neq \bar{B}^{22},\ \text{and } \bar{\alpha}_0,\ \bar{\alpha}_2,\ \bar{\alpha}_4 = 0.
\end{aligned}
$$

For trigonal $\bar{\mathbf{C}}$ with $\mathrm{Sym}_e\bar{\mathbf{C}} = \{Q^{\pi}_{e_1}, Q^{2\pi/3}_{e_3}\}$ the possibilities for $(\mathrm{Sym}_e\bar{A} \cap \mathrm{Sym}_e\bar{B}) \cap \mathrm{Sym}_e\bar{\mathbf{H}}$, and the corresponding restrictions on the parameters describing $\bar{A}, \bar{B}$ and $\bar{\mathbf{H}}$ (in addition to the requirements that $\bar{A},\ \bar{B}$ be diagonal, $\bar{A}^{11} = \bar{A}^{22},\ \bar{B}^{11} = \bar{B}^{22},\ \bar{\alpha}_1,\ \bar{\beta}_1,\ \bar{\alpha}_2,\ \bar{\beta}_2,\ \bar{\alpha}_3,\ \bar{\alpha}_4 = 0,\ \bar{\beta}_4,\ \bar{\beta}_3 \neq 0$), are:

$$
(69)\qquad
\begin{aligned}
i)\quad & \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\} \cap \{Q^{\pi}_{e_1}, Q^{2\pi/3}_{e_3}\},\\
ii)\quad & O(3) \cap \{Q^{\pi}_{e_1}, Q^{2\pi/3}_{e_3}\},\quad \bar{A}^{11},\ \bar{A}^{22},\ \bar{B}^{11},\ \bar{B}^{22} = 0.
\end{aligned}
$$

For tetragonal $\bar{\mathbf{C}}$ with $\mathrm{Sym}_e\bar{\mathbf{C}} = \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi/2}_{e_3}\}$ the possibilities for $(\mathrm{Sym}_e\bar{A} \cap \mathrm{Sym}_e\bar{B}) \cap \mathrm{Sym}_e\bar{\mathbf{H}}$, and the corresponding restrictions on the parameters describing $\bar{A}, \bar{B}$ and $\bar{\mathbf{H}}$ (in addition to the requirements that $\bar{A},\ \bar{B}$ be diagonal, $\bar{A}^{11} = \bar{A}^{22},\ \bar{B}^{11} = \bar{B}^{22},\ \bar{\alpha}_1,\ \bar{\beta}_1,\ \bar{\alpha}_2,\ \bar{\beta}_2,\ \bar{\alpha}_3,\ \bar{\beta}_3,\ \bar{\beta}_4 = 0,\ \bar{\alpha}_4 \neq 0$), are:

$$
(70)\qquad
\begin{aligned}
i)\quad & \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\} \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi/2}_{e_3}\},\\
ii)\quad & O(3) \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi/2}_{e_3}\},\quad \bar{A}^{11},\ \bar{A}^{22},\ \bar{B}^{11},\ \bar{B}^{22} = 0.
\end{aligned}
$$

For cubic $\bar{\mathbf{C}}$ with $\mathrm{Sym}_e\bar{\mathbf{C}} = \{Q^{\pi/2}_{e_1}, Q^{\pi/2}_{e_2}, Q^{\pi/2}_{e_3}\}$ the only possibility for $(\mathrm{Sym}_e\bar{A} \cap \mathrm{Sym}_e\bar{B}) \cap \mathrm{Sym}_e\bar{\mathbf{H}}$ is $O(3) \cap \{Q^{\pi/2}_{e_1}, Q^{\pi/2}_{e_2}, Q^{\pi/2}_{e_3}\}$, and the corresponding restrictions on the parameters describing $\bar{A}, \bar{B}$ and $\bar{\mathbf{H}}$ are that $\bar{A}, \bar{B} = 0,\ \bar{\alpha}_1,\ \bar{\beta}_1,\ \bar{\alpha}_2,\ \bar{\beta}_2,\ \bar{\alpha}_3,\ \bar{\beta}_3,\ \bar{\beta}_4 = 0,\ \bar{\alpha}_4 = 5\bar{\alpha}_0 \neq 0$.

For transversely isotropic $\bar{\mathbf{C}}$ with $\mathrm{Sym}_e\bar{\mathbf{C}} = \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\}$ the possibilities for $(\mathrm{Sym}_e\bar{A} \cap \mathrm{Sym}_e\bar{B}) \cap \mathrm{Sym}_e\bar{\mathbf{H}}$, and the corresponding restrictions on the parameters describing $\bar{A}, \bar{B}$ and $\bar{\mathbf{H}}$ (in addition to the requirements that $\bar{A},\ \bar{B}$ be diagonal, $\bar{A}^{11} = \bar{A}^{22},\ \bar{B}^{11} = \bar{B}^{22},\ \bar{\alpha}_i,\ \bar{\beta}_i = 0,\ i = 1, \ldots, 4$), are:

$$
(71)\qquad
\begin{aligned}
i)\quad & \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\} \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\},\\
& \qquad \bar{A}^{11},\ \bar{B}^{11} \neq 0,\quad \bar{\alpha}_0 \neq 0
\end{aligned}
$$

$$
\begin{array}{ll}
ii) & O(3)\cap\{Q^{\pi}_{e_1},Q^{\pi}_{e_2},Q^{\phi}_{e_3}\},\\
& \quad \bar A,\ \bar B=0,\quad \bar\alpha_0\neq 0\\
iii) & \{Q^{\pi}_{e_1},Q^{\pi}_{e_2},Q^{\phi}_{e_3}\}\cap O(3)\\
& \quad \bar A^{11},\ \bar B^{11}\neq 0,\quad \bar\alpha_0=0.
\end{array}
\tag{72}
$$

For isotropic $\bar{\mathsf{C}}$ both $\mathrm{Sym}_e\bar A\cap\mathrm{Sym}_e\bar B$ and $\mathrm{Sym}_e\bar{\mathsf{H}}$ must be $O(3)$, and so $\bar A,\bar B=0$ and $\bar\alpha_0,\ \bar\alpha_i,\ \bar\beta_i=0,\ i=1,\dots,4$.

5.3. **Computing $\hat{\hat{\mathsf{C}}}$.** We now compute $\hat{\hat{\mathsf{C}}}=D*\bar{\mathsf{C}}$ in terms of $\bar{\mathsf{C}}$. We observe that $(D*\bar{\mathsf{C}})^{ijkl}=\bar{\mathsf{C}}^{ijkl}\,a^{e(1)}\,b^{e(2)}\,c^{e(3)}$, where in this formula $e(m)=\delta_{im}+\delta_{jm}+\delta_{km}+\delta_{lm}$ and there is no summation over i,j,k,l. It follows that $D*\bar{\mathsf{H}}$ has components given by

$$
\begin{array}{c|c|c|c}
ij\backslash kl & 11 & 22 & 33\\
\hline
11 & a^4(3\bar\alpha_0-\bar\alpha_2+\bar\alpha_4) & a^2b^2(\bar\alpha_0-\bar\alpha_4) & -a^2c^2(4\bar\alpha_0-\bar\alpha_2)\\
22 & & b^4(3\bar\alpha_0+\bar\alpha_2+\bar\alpha_4) & -b^2c^2(4\bar\alpha_0+\bar\alpha_2)\\
33 & & & 8c^4\bar\alpha_0
\end{array}
\tag{73}
$$

$$
\begin{array}{c|c|c|c}
ij\backslash kl & 12 & 23 & 13\\
\hline
11 & -a^3b(\bar\beta_2/2-\bar\beta_4) & -a^2bc[\bar\beta_1-\bar\beta_3]/4 & -a^3c[3\bar\alpha_1-\bar\alpha_3]/4\\
22 & -ab^3(\bar\beta_2/2+\bar\beta_4) & -b^3c[3\bar\beta_1+\bar\beta_3]/4 & -ab^2c[\bar\alpha_1+\bar\alpha_3]/4\\
33 & abc^2\bar\beta_2 & bc^3\bar\beta_1 & ac^3\bar\alpha_1\\
12 & a^2b^2[\bar\alpha_0-\bar\alpha_4] & -ab^2c[\bar\alpha_1+\bar\alpha_3]/4 & -a^2bc[\bar\beta_1-\bar\beta_3]/4\\
23 & & -b^2c^2(4\bar\alpha_0+\bar\alpha_2) & abc^2\bar\beta_2\\
13 & & & -a^2c^2(4\bar\alpha_0-\bar\alpha_2)
\end{array}
\tag{74}
$$

while, writing $M=(\bar\alpha+2\bar\beta)I+2(\bar A+2\bar B)$, $\quad\hat{\hat{\mathsf{C}}}-(D*\bar{\mathsf{H}})$ is given by:

$$
\begin{array}{c|c|c|c}
ij\backslash kl & 11 & 22 & 33\\
\hline
11 & a^4M^{11} & a^2b^2[\bar\alpha+\bar A^{11}+\bar A^{22}] & a^2c^2[\bar\alpha-\bar A^{22}]\\
22 & & b^4M^{22} & b^2c^2[\bar\alpha-\bar A^{11}]\\
33 & & & c^4M^{33}
\end{array}
\tag{75}
$$

$$
\begin{array}{c|c|c|c}
ij\backslash kl & 12 & 23 & 13\\
\hline
11 & a^3b(\bar A+2\bar B)^{12} & a^2bc\bar A^{23} & a^3c(\bar A+2\bar B)^{13}\\
22 & ab^3(\bar A+2\bar B)^{12} & b^3c(\bar A+2\bar B)^{23} & ab^2c\bar A^{13}\\
33 & abc^2\bar A^{12} & bc^3(\bar A+2\bar B)^{23} & ac^3(\bar A+2\bar B)^{13}\\
12 & a^2b^2[\bar\beta+\bar B^{11}+\bar B^{22}] & ab^2c\bar B^{13} & a^2bc\bar B^{23}\\
23 & & b^2c^2[\bar\beta-\bar B^{11}] & abc^2\bar B^{12}\\
13 & & & a^2c^2[\bar\beta-\bar B^{22}]
\end{array}
\tag{76}
$$

To compute the harmonic-Cartan decomposition of $\hat{\hat{\mathsf{C}}}$ we first compute $\hat{\hat\alpha}=\alpha_{\hat{\hat{\mathsf{C}}}}$, $\hat{\hat\beta}=\beta_{\hat{\hat{\mathsf{C}}}}$, $\hat{\hat A}=A_{\hat{\hat{\mathsf{C}}}}$, $\hat{\hat B}=B_{\hat{\hat{\mathsf{C}}}}$, and $\hat{\hat{\mathsf{H}}}=\mathsf{H}_{\hat{\hat{\mathsf{C}}}}$, given (53) and (56).

We then determine $\widehat{\widehat{\alpha}}_0 = (\alpha_0)_{\widehat{\mathbf{C}}}$, $\widehat{\widehat{\alpha}}_i = (\alpha_i)_{\widehat{\mathbf{C}}}$, $\widehat{\widehat{\beta}}_i = (\beta_i)_{\widehat{\mathbf{C}}}$, $i = 1, \ldots, 4$ from the Cartan decomposition (57) for $\widehat{\widehat{\mathbf{H}}}$.

It follows by a calculation, given $\bar{A}^{33} = -\bar{A}^{11} - \bar{A}^{22}$ and $\bar{B}^{33} = -\bar{B}^{11} - \bar{B}^{22}$, that we may write the linear dependence of

$$\begin{aligned} \widehat{\widehat{k}}_{1313} &= (\widehat{\widehat{A}}^{13}, \widehat{\widehat{B}}^{13}, \widehat{\widehat{\alpha}}_1, \widehat{\widehat{\alpha}}_3) \quad \text{on} \quad \bar{k}_{1313} = (\bar{A}^{13}, \bar{B}^{13}, \bar{\alpha}_1, \bar{\alpha}_3) \\ \text{as} \quad \widehat{\widehat{k}}_{1313} &= K_{1313} \cdot \bar{k}_{1313}, \end{aligned}$$

the linear dependence of

$$\begin{aligned} \widehat{\widehat{k}}_{2313} &= (\widehat{\widehat{A}}^{23}, \widehat{\widehat{B}}^{23}, \widehat{\widehat{\beta}}_1, \widehat{\widehat{\beta}}_3) \quad \text{on} \quad \bar{k}_{2313} = (\bar{A}^{23}, \bar{B}^{23}, \bar{\beta}_1, \bar{\beta}_3) \\ \text{as} \quad \widehat{\widehat{k}}_{2313} &= K_{2313} \cdot \bar{k}_{2313}, \end{aligned}$$

and the linear dependence of

$$\begin{aligned} \widehat{\widehat{k}}_{1224} &= (\widehat{\widehat{A}}^{12}, \widehat{\widehat{B}}^{12}, \widehat{\widehat{\beta}}_2, \widehat{\widehat{\beta}}_4) \quad \text{on} \quad \bar{k}_{1224} = (\bar{A}^{12}, \bar{B}^{12}, \bar{\beta}_2, \bar{\beta}_4) \\ \text{as} \quad \widehat{\widehat{k}}_{1224} &= K_{1224} \cdot \bar{k}_{1224}. \end{aligned}$$

Finally, we write the linear dependence of

$$\begin{aligned} \widehat{\widehat{k}}_{024} &= (15\widehat{\widehat{\alpha}}, 15\widehat{\widehat{\beta}}, 21\widehat{\widehat{A}}^{11}, 21\widehat{\widehat{A}}^{22}, 21\widehat{\widehat{B}}^{11}, 21\widehat{\widehat{B}}^{22}, 280\widehat{\widehat{\alpha}}_0, 14\widehat{\widehat{\alpha}}_2, 8\widehat{\widehat{\alpha}}_4) \\ \text{on} \quad \bar{k}_{024} &= (\bar{\alpha}, \bar{\beta}, \bar{A}^{11}, \bar{A}^{22}, \bar{B}^{11}, \bar{B}^{22}, \bar{\alpha}_0, \bar{\alpha}_2, \bar{\alpha}_4) \\ \text{as} \quad \widehat{\widehat{k}}_{024} &= K_{024} \cdot \bar{k}_{024}. \end{aligned}$$

We observe that the 4×4 matrices K_{1313}, K_{2313}, K_{1224} are invertible. In fact, K_{1313} is given by $ac/28$ times

$$\begin{pmatrix} 4(a^2+5b^2+c^2) & 8(a^2-2b^2+c^2) & -(3a^2+b^2-4c^2) & (a^2-b^2) \\ 4(a^2-2b^2+c^2) & 4(2a^2+3b^2+2c^2) & -(3a^2+b^2-4c^2) & (a^2-b^2) \\ -4(3a^2+b^2-4c^2) & -8(3a^2+b^2-4c^2) & (9a^2+3b^2+16c^2) & -3(a^2-b^2) \\ 28(a^2-b^2) & 56(a^2-b^2) & -21(a^2-b^2) & 7(a^2+3b^2) \end{pmatrix},$$

and has determinant $a^2b^4c^2 \cdot (ac)^4$, up to a positive constant, so does not vanish since $a, b, c > 0$. Similarly, K_{2313} is given by $bc/28$ times

$$\begin{pmatrix} 4(5a^2+b^2+c^2) & 8(-2a^2+b^2+c^2) & -(a^2+3b^2-4c^2) & (a^2-b^2) \\ 4(-2a^2+b^2+c^2) & 4(3a^2+2b^2+2c^2) & -(a^2+3b^2-4c^2) & (a^2-b^2) \\ -4(a^2+3b^2-4c^2) & -8(a^2+3b^2-4c^2) & (3a^2+9b^2+16c^2) & -3(a^2-b^2) \\ 28(a^2-b^2) & 56(a^2-b^2) & -21(a^2-b^2) & 7(3a^2+b^2) \end{pmatrix},$$

and has determinant $a^4b^2c^2 \cdot (bc)^4$, up to a positive constant. In addition, K_{1224} is given by $ab/28$ times

$$\begin{pmatrix} 4(a^2{+}b^2{+}5c^2) & 8(a^2{+}b^2{-}2c^2) & -2(a^2{+}b^2{-}2c^2) & 4(a^2-b^2) \\ 4(a^2{+}b^2{-}2c^2) & 4(2a^2{+}2b^2{+}3c^2) & -2(a^2{+}b^2{-}2c^2) & 4(a^2-b^2) \\ -8(a^2{+}b^2{-}2c^2) & -16(a^2{+}b^2{-}2c^2) & 4(a^2{+}b^2{+}12c^2) & -8(a^2{-}b^2) \\ 14(a^2-b^2) & 28(a^2-b^2) & -7(a^2-b^2) & 14(a^2{+}b^2) \end{pmatrix},$$

and has determinant $a^2b^2c^4 \cdot (ab)^4$, up to a positive constant.

For $\bar{\mathbf{C}}$ isotropic, transversely isotropic, cubic, or tetragonal having parameters $(\bar{\alpha},\ \bar{\beta},\ \bar{A}^{11},\ \bar{B}^{11},\ \bar{\alpha}_0,\ \bar{\alpha}_4)$ it follows by the reasoning of Lemma 17 (Section 5.4) that $\widehat{\bar{\mathbf{C}}} = D * \bar{\mathbf{C}}$ may be written in terms of the parameters $\widehat{\bar{\alpha}}$, $\widehat{\bar{\beta}}$, $\widehat{\bar{A}}^{11}$, $\widehat{\bar{A}}^{22}$, $\widehat{\bar{B}}^{11}$, $\widehat{\bar{B}}^{22}$, $\widehat{\bar{\alpha}}_0$, $\widehat{\bar{\alpha}}_2$, $\widehat{\bar{\alpha}}_4$.

Finally, we compute K_{024} to be the following. We write $\widehat{\bar{k}}_{024} = (15\widehat{\bar{\alpha}},\ 15\widehat{\bar{\beta}},\ 21\widehat{\bar{A}}^{11},\ 21\widehat{\bar{A}}^{22},\ 21\widehat{\bar{B}}^{11},\ 21\widehat{\bar{B}}^{22},\ 280\widehat{\bar{\alpha}}_0,\ 14\widehat{\bar{\alpha}}_2,\ 8\widehat{\bar{\alpha}}_4)$ as $M \cdot v$, where

$$v = (\bar{\alpha}, \bar{\beta}, \bar{A}^{11}, \bar{B}^{11}, \bar{\alpha}_0, \bar{\alpha}_4, \bar{\alpha}_0, \bar{\alpha}_0, \bar{\alpha}_0)$$

and M is given by the product of the following two invertible matrices:

$$\begin{pmatrix} 4a^2b^2 & -2a^2b^2 & 4a^2 & -2a^2 & a^4 & 1 & b^4 & -2b^2 & 4b^2 \\ -a^2b^2 & 3a^2b^2 & -a^2 & 3a^2 & a^4 & 1 & b^4 & 3b^2 & -b^2 \\ 5a^2b^2 & -4a^2b^2 & 5a^2 & -4a^2 & 2a^4 & -1 & -b^4 & 8b^2 & -10b^2 \\ 5a^2b^2 & -4a^2b^2 & -10a^2 & 8a^2 & -a^4 & -1 & 2b^4 & -4b^2 & 5b^2 \\ -2a^2b^2 & 3a^2b^2 & -2a^2 & 3a^2 & 2a^4 & -1 & -b^4 & -6b^2 & 4b^2 \\ -2a^2b^2 & 3a^2b^2 & 4a^2 & -6a^2 & -a^4 & -1 & 2b^4 & 3b^2 & -2b^2 \\ 2a^2b^2 & 4a^2b^2 & -8a^2 & -16a^2 & 3a^4 & 8 & 3b^4 & -16b^2 & -8b^2 \\ 0 & 0 & 2a^2 & 4a^2 & -a^4 & 0 & b^4 & -4b^2 & -2b^2 \\ -2a^2b^2 & -4a^2b^2 & 0 & 0 & a^4 & 0 & b^4 & 0 & 0 \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -4 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 4 & 3 & 1 & 0 & 0 & 0 \\ 1 & 2 & -4 & -8 & 8 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 4 & 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & -4 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix}.$$

By collecting terms in the above expression we conclude that the (scaled) parameters $(15\widehat{\bar\alpha}, 15\widehat{\bar\beta}, 21\widehat{\bar A}^{11}, 21\widehat{\bar A}^{22}, 21\widehat{\bar B}^{11}, 21\widehat{\bar B}^{22}, 280\widehat{\bar\alpha}_0, 14\widehat{\bar\alpha}_2, 8\widehat{\bar\alpha}_4)$ have the form (26), where the $P(a,b)$ are linearly independent for fixed a, b and are given by

(77)

$$
\begin{aligned}
P_{\bar\alpha} = \Big(& 6a^4 - 2[3(a^2-b^2)-4]a^2 + [1+(a^2-b^2)^2 - 4(a^2-b^2)],\\
& a^4 - [(a^2-b^2)+2]a^2 + [1+(a^2-b^2)^2 + (a^2-b^2)],\\
& 6a^4 - [3(a^2-b^2)+5]a^2 - [1+(a^2-b^2)^2 - 10(a^2-b^2)],\\
& 6a^4 - [9(a^2-b^2)+5]a^2 - [1-2(a^2-b^2)^2 + 5(a^2-b^2)],\\
& -a^4 + 2[2(a^2-b^2)+1]a^2 - [1+(a^2-b^2)^2 + 4(a^2-b^2)],\\
& -a^4 - 2[(a^2-b^2)-1]a^2 - [1-2(a^2-b^2)^2 - 2(a^2-b^2)],\\
& 8a^4 - 8[(a^2-b^2)+2]a^2 + [8+3(a^2-b^2)^2 + 8(a^2-b^2)],\\
& - [2(a^2-b^2)]a^2 + [(a^2-b^2)^2 + 2(a^2-b^2)],\\
& + [(a^2-b^2)^2]\Big),
\end{aligned}
$$

$$
\begin{aligned}
P_{\bar\beta} = \Big(& 2a^4 - 2[(a^2-b^2)+2]a^2 + 2[1+(a^2-b^2)^2 + (a^2-b^2)],\\
& 7a^4 - [7(a^2-b^2)-6]a^2 + [2+2(a^2-b^2)^2 - 3(a^2-b^2)],\\
& -2a^4 + 4[2(a^2-b^2)+1]a^2 - 2[1+(a^2-b^2)^2 + 4(a^2-b^2)],\\
& -2a^4 - 4[(a^2-b^2)-1]a^2 - 2[1-2(a^2-b^2)^2 - 2(a^2-b^2)],\\
& 5a^4 + [(a^2-b^2)-3]a^2 - 2[1+(a^2-b^2)^2 - 3(a^2-b^2)],\\
& 5a^4 - [11(a^2-b^2)+3]a^2 - [2-4(a^2-b^2)^2 + 3(a^2-b^2)],\\
& 16a^4 - 16[(a^2-b^2)+2]a^2 + 2[8+3(a^2-b^2)^2 + 8(a^2-b^2)],\\
& - [4(a^2-b^2)]a^2 + 2[(a^2-b^2)^2 + 2(a^2-b^2)],\\
& [2(a^2-b^2)^2]\Big),
\end{aligned}
$$

$$
\begin{aligned}
P_{\bar A^{11}} = \Big(& 12a^4 - 4[3(a^2-b^2)+2]a^2 - 2[2-(a^2-b^2)^2 - 2(a^2-b^2)],\\
& 2a^4 - -2[(a^2-b^2)+2]a^2 + [4-2(a^2-b^2)^2 - (a^2-b^2)],\\
& 12a^4 - [6(a^2-b^2)-5]a^2 + 2[2-(a^2-b^2)^2 - 5(a^2-b^2)],\\
& 12a^4 - [18(a^2-b^2)-5]a^2 + [4+4(a^2-b^2)^2 + 5(a^2-b^2)],\\
& -2a^4 + 2[4(a^2-b^2)-1]a^2 + 2[2-(a^2-b^2)^2 + 2(a^2-b^2)],\\
& -2a^4 - 2[2(a^2-b^2)+1]a^2 + 2[2+2(a^2-b^2)^2 - (a^2-b^2)],\\
& 16a^4 - 16[(a^2-b^2)-1]a^2 + 2[16-3(a^2-b^2)^2 - 4(a^2-b^2)],\\
& - [4(a^2-b^2)]a^2 + 2[(a^2-b^2)^2 - (a^2-b^2)],\\
& [2(a^2-b^2)^2]\Big),
\end{aligned}
$$

(78)

$$
\begin{aligned}
P_{\bar{B}^{11}} = \Big(&4a^4 - 4[(a^2-b^2)-1]a^2 - [8-4(a^2-b^2)^2+2(a^2-b^2)],\\
&14a^4 - 2[7(a^2-b^2)+3]a^2 - [8-4(a^2-b^2)^2-3(a^2-b^2)],\\
&-4a^4 + 4[4(a^2-b^2)-1]a^2 + 4[2-(a^2-b^2)^2+2(a^2-b^2)],\\
&-4a^4 - 4[2(a^2-b^2)+1]a^2 + 4[2+2(a^2-b^2)^2-(a^2-b^2)],\\
&10a^4 + [2(a^2-b^2)+3]a^2 + 2[4-2(a^2-b^2)^2-3(a^2-b^2)],\\
&10a^4 - [22(a^2-b^2)-3]a^2 + [8+8(a^2-b^2)^2+3(a^2-b^2)],\\
&32a^4 + 32[(a^2-b^2)-1]a^2 - 4[16-3(a^2-b^2)^2+4(a^2-b^2)],\\
&- [8(a^2-b^2)a^2] + 4[(a^2-b^2)^2-(a^2-b^2)],\\
&4(a^2-b^2)^2\Big),
\end{aligned}
$$

$$
\begin{aligned}
P_{\bar{\alpha}_0} = \Big(&8a^4 - 8[(a^2-b^2)+2]a^2 + [8+3(a^2-b^2)^2+8(a^2-b^2)],\\
&8a^4 - 8[(a^2-b^2)+2]a^2 + [8+3(a^2-b^2)^2+8(a^2-b^2)],\\
&4a^4 + [5(a^2-b^2)+4]a^2 - [8+3(a^2-b^2)^2+8(a^2-b^2)],\\
&4a^4 - [13(a^2-b^2)-4]a^2 - 2[4-3(a^2-b^2)^2-2(a^2-b^2)],\\
&4a^4 + [5(a^2-b^2)+4]a^2 - [8+3(a^2-b^2)^2+8(a^2-b^2)],\\
&4a^4 - [13(a^2-b^2)-4]a^2 - 2[4-2(a^2-b^2)-3(a^2-b^2)^2],\\
&24a^4 - 24[(a^2-b^2)-8]a^2 + [64+9(a^2-b^2)^2-96(a^2-b^2)],\\
&- [6(a^2-b^2)a^2] + 3[(a^2-b^2)^2-8(a^2-b^2)],\\
&[3(a^2-b^2)^2]\Big),
\end{aligned}
\tag{79}
$$

$$
\begin{aligned}
P_{\bar{\alpha}_4} = \Big(&(a^2-b^2)^2, \quad (a^2-b^2)^2, \quad (a^2-b^2)(2a^2+b^2),\\
&-(a^2-b^2)(a^2+2b^2), \quad (a^2-b^2)(2a^2+b^2),\\
&-(a^2-b^2)(a^2+2b^2), \quad 3(a^2-b^2)^2,\\
&-(a^2-b^2)(a^2+b^2), \quad 8a^4-(a^2-b^2)(7a^2+b^2)\Big).
\end{aligned}
$$

We remark that, due to the fact that D is invertible, if $\bar{\mathbf{C}}$ is strong elliptic, then by the definition of strong ellipticity $\hat{\bar{\mathbf{C}}}$ is also.

5.4. **Characterizing the range of the pullback by D of isotropic, transversely isotropic, cubic, and tetragonal $\bar{\mathbf{C}}$.** We now analyze the possibilities for $\hat{\bar{\mathbf{C}}}$.

Lemma 17. *If* $\mathrm{Sym}_e\bar{\mathbf{C}}$ *contains* $\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\}$, *then* $\mathrm{Sym}_e\hat{\bar{\mathbf{C}}}$ *does also.*

Proof. If $\mathrm{Sym}_e\bar{\mathbf{C}}$ contains $\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\}$, (i.e., $\bar{\mathbf{C}}$ is orthotropic, tetragonal, cubic, transversely isotropic, or isotropic), then $\hat{\bar{k}}_{1313} = K_{1313}\bar{k}_{1313} = 0$, $\hat{\bar{k}}_{2313} = K_{2313}\bar{k}_{2313} = 0$, $\hat{\bar{k}}_{1224} = K_{1224}\bar{k}_{1224} = 0$; that is, the off-diagonal terms of $\widehat{\bar{A}}$ and $\widehat{\bar{B}}$ vanish, and $\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \subseteq \mathrm{Sym}_e\widehat{\bar{\mathbf{H}}}$. The result follows from (65) and Table 3. $\square$

Proof of Proposition 14. It is given that there is a polar decomposition of $D\psi^{-1} = SV$ and diagonalization of the stretch factor $S = UDU^T$, $D = \mathrm{diag}\,(a,b,c)$ with $a,b,c > 0$, so that, with respect to the standard basis for Euclidean space, the generators of $\mathrm{Sym}_e\bar{\mathbf{C}}(X') = \mathrm{Sym}_e U^T V * (C \circ \psi)(X')$ are listed in Table 1. It follows that a polar decomposition of $D\psi$ and diagonalization of the stretch factor is given by $V^T U D^{-1} U^T$. Therefore, $(\psi^{-1})^*(\psi^*\mathbf{C}) = (D\psi) * (\psi^*\mathbf{C} \circ \psi^{-1}) = V^T U D^{-1} U^T * (\psi^*\mathbf{C} \circ \psi^{-1})$; $\overline{\psi^*\mathbf{C}}$ in this case is $U^T * (\psi^*\mathbf{C} \circ \psi^{-1})$; and $\mathrm{Sym}_e\overline{\psi^*\mathbf{C}}(X) = \mathrm{Sym}_e U^T * \big(UDU^T V * (\mathbf{C} \circ \psi)\big) \circ \psi^{-1}(X) = \mathrm{Sym}_e D * \bar{\mathbf{C}}(X')$. By Lemma 17 $\mathrm{Sym}_e D * \bar{\mathbf{C}}(X')$ contains $Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}$ since $\mathrm{Sym}_e\bar{\mathbf{C}}(X')$ does, where here $\{e_1, e_2, e_3\}$ is the standard basis for Euclidean space. It follows by the presentation of the symmetry groups for anisotropic elastic media given in [4], pp. 2489-2490, for example, that $\mathrm{Sym}_e D * \bar{\mathbf{C}}(X')$ has generators listed in Table 1. $\square$

It follows for $\bar{\mathbf{C}}$ orthotropic, tetragonal, cubic, transversely isotropic, or isotropic that $\hat{\bar{\beta}}_3 = 0$, and so $\widehat{\bar{\mathbf{H}}}$ and $\hat{\bar{\mathbf{C}}}$ are not trigonal.

For $\bar{\mathbf{C}}$ tetragonal, cubic, transversely isotropic, or isotropic we conclude from the representation (26) of $\hat{\bar{k}}_{024}$ that $a = b$ implies $\widehat{\bar{A}}^{11} = \widehat{\bar{A}}^{22}$, $\widehat{\bar{B}}^{11} = \widehat{\bar{B}}^{22}$, $\widehat{\bar{\alpha}}_2 = 0$, and $\widehat{\bar{\alpha}}_4 = a^4\bar{\alpha}_4$. It follows that $\hat{\bar{\mathbf{C}}}$ is not orthotropic. For $\bar{\mathbf{C}}$ isotropic or transversely isotropic $\hat{\bar{\mathbf{C}}}$ is not cubic or tetragonal since $\widehat{\bar{\alpha}}_4 = 0$. In fact, for $\bar{\mathbf{C}}$ isotropic $\hat{\bar{\mathbf{C}}}$ is transversely isotropic since $\widehat{\bar{\alpha}}_0 = (\bar{\alpha} + 2\bar{\beta})(a^2-1)^2/35 \neq 0$ by strong ellipticity. For $\bar{\mathbf{C}}$ cubic (so $\bar{\alpha}_4 = 5\bar{\alpha}_0 \neq 0$) $\hat{\bar{\mathbf{C}}}$ may be tetragonal, depending on the choice of a.

If $\bar{\mathbf{C}}$ is isotropic and $a \neq b$, then $\widehat{\bar{A}}^{11} \neq \widehat{\bar{A}}^{22}$, $\widehat{\bar{B}}^{11} \neq \widehat{\bar{B}}^{22}$. Also, $\widehat{\bar{\alpha}}_2 = 0$ if and only if $a^2 + b^2$ is $2c^2$. (If $a^2 + b^2 = 2c^2 = 2$, then $\widehat{\bar{\alpha}}_4 = (\bar{\alpha} + 2\bar{\beta})(a^2-1)^2/2$, which is not zero by strong ellipticity.) It follows that $\hat{\bar{\mathbf{C}}}$ is orthotropic of type *i*) or *iv*).

If $\bar{\mathbf{C}}$ is transversely isotropic, cubic, or tetragonal, and $a \neq b$, then $\widehat{\bar{A}}^{11} \neq \widehat{\bar{A}}^{22}$, $\widehat{\bar{B}}^{11} \neq \widehat{\bar{B}}^{22}$, or $\widehat{\bar{\alpha}}_2 \neq 0$. If $\widehat{\bar{A}}^{11} \neq \widehat{\bar{A}}^{22}$ or $\widehat{\bar{B}}^{11} \neq \widehat{\bar{B}}^{22}$, then $\hat{\bar{\mathbf{C}}}$ is orthotropic of type *i*), or *iv*) $-$ *vii*). On the other hand, if $\widehat{\bar{A}}^{11} = \widehat{\bar{A}}^{22}$, $\widehat{\bar{B}}^{11} = \widehat{\bar{B}}^{22}$, $\widehat{\bar{\alpha}}_2 \neq 0$, then $\hat{\bar{\mathbf{C}}}$ is orthotropic of type *ii*) or *iii*).

In conclusion, for $\bar{\mathbf{C}}$ isotropic or transversely isotropic, $\hat{\bar{\mathbf{C}}}$ is orthotropic, transversely isotropic, or isotropic (depending on $\bar{\mathbf{C}}$ and a, b), but will never be tetragonal or cubic; for $\bar{\mathbf{C}}$ tetragonal or cubic, $\hat{\bar{\mathbf{C}}}$ is (only) orthotropic, tetragonal, or cubic; and for $\bar{\mathbf{C}}$ orthotropic, $\hat{\bar{\mathbf{C}}}$ is isotropic, transverse isotropic, cubic, tetragonal, or orthotropic, depending on the values of $\bar{\mathbf{C}}$ and a, b.

In fact, in the case that $\bar{\mathbf{C}}$ is isotropic and $a = b$ the range of $D * \bar{\mathbf{C}}$ is a 3-parameter family in the 5-parameter family of transverse isotropic elasticity tensors. In the case that $\bar{\mathbf{C}}$ is isotropic and $a \neq b$ the range of $D * \bar{\mathbf{C}}$ is a 4-parameter family in the 9-parameter family of orthotropic elasticity tensors of type i). In the case that $\bar{\mathbf{C}}$ is cubic and $a = b$ the range of $D * \bar{\mathbf{C}}$ is a 4-parameter family in the 5-parameter family of tetragonal elasticity tensors.

In the case that $\bar{\mathbf{C}}$ is cubic and $a \neq b$ the range of $D * \bar{\mathbf{C}}$ is a 5-parameter family in the 9-parameter family of orthotropic elasticity tensors. In particular, in this case the range of $D * \bar{\mathbf{C}}$ is a 5-parameter family in the 9-parameter family of orthotropic elasticity tensors of type i); is a 4-parameter family in the 7-parameter family of orthotropic elasticity tensors of type ii); is a 4-parameter family in the 8-parameter family of orthotropic elasticity tensors of type iv); is a 3-parameter family in the 7-parameter family of orthotropic elasticity tensors of type v); is a 3-parameter family in the 7-parameter family of orthotropic elasticity tensors of type vi); and is a 2-parameter family in the 6-parameter family of orthotropic elasticity tensors of type vii).

The same reasoning may be applied in the case that $\bar{\mathbf{C}}$ is tetragonal. In general, if $a \neq b$, then the range of $D * \bar{\mathbf{C}}$ is a 6-parameter family in the 9-parameter family of orthotropic elasticity tensors. □

6. Concluding remarks

There are several challenging problems that the analysis in this paper naturally suggests. First, the pointwise characterization of the orbits of elasticity tensors in the non-aligned case remains an open problem. A new approach, independent of the choice of basis, seems advisable, and polynomial invariants (under the action of the orthogonal group $O(3)$) may prove a feasible tool (cf. [3]). Furthermore, the pointwise characterization of the orbits, while interesting in its own right, must be followed, eventually, by a description of the orbits under global transformations. We are inspired here by work of Ebin [6], and Ebin and Marsden [7], in the case of equivalence classes of Riemannian metrics.

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List of Tables

Table 1. **Classification of Anisotropic Elastic Media**

i	Symmetry Class	Generators of $SO(3) \cap$ the symmetry group $\mathcal{G}_i$
1	triclinic	I
2	monoclinic	$Q^{\pi}_{e_3}$ (cyclic group $\mathcal{Z}_2$)
3	orthotropic	$Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}$ (dihedral group $\mathcal{D}_2$)
4	trigonal	$Q^{\pi}_{e_1}, Q^{2\pi/3}_{e_3}$ (dihedral group $\mathcal{D}_3$)
5	tetragonal	$Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi/2}_{e_3}$ (dihedral group $\mathcal{D}_4$)
6	cubic	$Q^{\pi/2}_{e_1}, Q^{\pi/2}_{e_2}, Q^{\pi/2}_{e_3}$ (octahedral group $\mathcal{O}$)
7	transversely isotropic	$Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}$ (orthogonal group $O(2)$)
8	isotropic	(special orthogonal group $SO(3)$)

TABLE 2. $\mathrm{Sym}_e\bar{\mathbf{H}}$ **is described by the vanishing of** $\bar{\alpha}_0, \bar{\alpha}_i, \bar{\beta}_i$

	Properties of $\bar{\alpha}_0, \bar{\alpha}_i, \bar{\beta}_i$	Restrictions on $\mathrm{Sym}_e\bar{\mathbf{H}}$
1	$\bar{\alpha}_1 \neq 0$ or $\bar{\alpha}_3 \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^{\pi}_{e_1}\}$, $\mathrm{sg}\{Q^{\pi}_{e_3}\} \nsubseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
1	$\bar{\beta}_1 \neq 0$ or $\bar{\beta}_3 \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^{\pi}_{e_2}\}$, $\mathrm{sg}\{Q^{\pi}_{e_3}\} \nsubseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
1	$\bar{\beta}_2 \neq 0$ or $\bar{\beta}_4 \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^{\pi}_{e_1}\}$, $\mathrm{sg}\{Q^{\pi}_{e_2}\} \nsubseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
2	$\bar{\alpha}_1, \bar{\beta}_1, \bar{\alpha}_3, \bar{\beta}_3 = 0$ with $\bar{\beta}_2 \neq 0$ or $\bar{\beta}_4 \neq 0$ with $\bar{\alpha}_2 \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^{\pi}_{e_3}\} \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $\mathcal{D}_2$, $\mathrm{sg}\{Q^{\pi}_{e_1}\}$, $\mathrm{sg}\{Q^{\pi}_{e_2}\} \nsubseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $\mathcal{Z}_3 \cong \mathrm{sg}\{Q^{2\pi/3}_{e_i}\} \nsubseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $\mathcal{Z}_4 \cong \mathrm{sg}\{Q^{\pi/2}_{e_i}\} \nsubseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
3	$\bar{\alpha}_1, \bar{\beta}_1, \bar{\beta}_2, \bar{\alpha}_3, \bar{\beta}_3, \bar{\beta}_4 = 0$ with $\bar{\alpha}_2 \neq 0$	$\mathcal{D}_2 \cong \mathrm{sg}\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $\mathcal{Z}_3 \cong \mathrm{sg}\{Q^{2\pi/3}_{e_i}\} \nsubseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $\mathcal{Z}_4 \cong \mathrm{sg}\{Q^{\pi/2}_{e_i}\} \nsubseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
4	$\bar{\alpha}_1, \bar{\beta}_1, \bar{\alpha}_2, \bar{\beta}_2, \bar{\alpha}_3, \bar{\alpha}_4, \bar{\beta}_4 = 0$ with $\bar{\beta}_3 \neq 0$	$\mathcal{D}_3 \cong \mathrm{sg}\{Q^{\pi}_{e_1}, Q^{2\pi/3}_{e_3}\} \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $O(2)$, $\mathrm{sg}\{Q^{2\pi/3}_{e_{j\neq 3}}\}$, $\mathrm{sg}\{Q^{\pi}_{e_3}\} \nsubseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
5	$\bar{\alpha}_1, \bar{\beta}_1, \bar{\alpha}_2, \bar{\beta}_2, \bar{\alpha}_3, \bar{\beta}_3, \bar{\beta}_4 = 0$ with $\bar{\alpha}_4 \neq 0$	$\mathcal{D}_4 \cong \mathrm{sg}\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi/2}_{e_3}\} \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $\mathcal{O}$, $\mathrm{sg}\{Q^{2\pi/3}_{e_i}\}$, $\mathrm{sg}\{Q^{\pi/2}_{e_{j\neq 3}}\} \nsubseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
6	$\bar{\alpha}_1, \bar{\beta}_1, \bar{\alpha}_2, \bar{\beta}_2, \bar{\alpha}_3, \bar{\beta}_3, \bar{\beta}_4 = 0$ with $\bar{\alpha}_4 = 5\bar{\alpha}_0 \neq 0$	$\mathcal{O} \cong \mathrm{sg}\{Q^{\pi/2}_{e_1}, Q^{\pi/2}_{e_2}, Q^{\pi/2}_{e_3}\} \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $O(2) \nsubseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
7	$\bar{\alpha}_i, \bar{\beta}_i = 0$, $i = 1...4$ with $\bar{\alpha}_0 \neq 0$	$O(2) \cong \mathrm{sg}\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\} \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $O(3) \nsubseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
8	$\bar{\alpha}_0, \bar{\alpha}_i, \bar{\beta}_i = 0$, $i = 1...4$	$O(3) \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$

TABLE 3. $\mathrm{Sym}_e A'$ **is described by the vanishing of its components**

	Properties of A'_{ij}	Restrictions on $\mathrm{Sym}_e A'$
1	$A'_{13} \neq 0$ or $A'_{12} \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^{\pi}_{e_1}\} \not\subseteq \mathrm{Sym}_e A'$
1	$A'_{23} \neq 0$ or $A'_{12} \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^{\pi}_{e_2}\} \not\subseteq \mathrm{Sym}_e A'$
1	$A'_{13} \neq 0$ or $A'_{23} \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^{\pi}_{e_3}\} \not\subseteq \mathrm{Sym}_e A'$
3	$A'_{13}, A'_{23}, A'_{12} = 0$ with $A'_{11} - A'_{22} \neq 0$	$D(2) \cong \mathrm{sg}\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \subseteq \mathrm{Sym}_e A'$ $\mathrm{sg}\{Q^{\pi/2}_{e_i}\}$, $\mathrm{sg}\{Q^{2\pi/3}_{e_i}\} \not\subseteq \mathrm{Sym}_e A'$
7	$A'_{13}, A'_{23}, A'_{11} - A'_{22} = 0$ $A'_{12} = 0$, with $A'_{33} \neq 0$	$O(2) \cong \mathrm{sg}\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\} \subseteq \mathrm{Sym}_e A'$ $O(3) \not\subseteq \mathrm{Sym}_e A'$
8	$A'_{33}, A'_{13}, A'_{23} = 0$ $A'_{12}, A'_{11} - A'_{22} = 0$	$O(3) \subseteq \mathrm{Sym}_e A'$