

*Well-posedness and regularity for the
Elasticity equation with mixed boundary
conditions on polyhedral domains and
domains with cracks*

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Dedicated to Michael E. Taylor on occasion of His Sixtieth Birthday

Abstract

We prove a regularity result for the anisotropic linear elasticity equation $Pu := \operatorname{div}(\mathbf{C} \cdot \nabla u) = f$, with mixed (displacement and traction) boundary conditions on a curved polyhedral domain $\Omega \subset \mathbb{R}^3$ in weighted Sobolev spaces $\mathcal{K}_{a+1}^{m+1}(\Omega)$, for which the weight is given by the distance to the set of edges. In particular, we show that there is no loss of \mathcal{K}_a^m -regularity. Our curved polyhedral domains are allowed to have cracks. We establish a well-posedness result when there are no neighboring traction boundary conditions and $|a| < \eta$, for some small $\eta > 0$ that depends on P , on the boundary conditions, and on the domain Ω . Our results extend to other strongly elliptic systems and higher dimensions.

Key words. Elasticity, elliptic systems, nonsmooth domains, weighted spaces, well-posedness

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and P a uniformly strongly elliptic, second-order operator on Ω . To fix ideas, we shall assume in this introduction that $P = \operatorname{div} \mathbf{C} \cdot \nabla$, the operator of anisotropic elasticity, the case in which we are most interested. The tensor \mathbf{C} is chosen to satisfy the usual positivity and symmetry conditions (see Definition 7). Assume for the moment that the boundary of Ω is smooth. Then it is well known (see for example [1, 34, 59, 53, 86]) that the boundary value problem

$$Pu := \operatorname{div} (\mathbf{C} \cdot \nabla u) = f \in H^{m-1}(\Omega)^3, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1)$$

has a unique solution $u \in H^{m+1}(\Omega)^3$. This well-posedness result is especially useful in applications for the numerical approximation of the solution u of Equation (1), see [11, 12, 18, 81] from among the many possible references. It also gives that u will be smooth on $\overline{\Omega}$ if f is smooth on $\overline{\Omega}$.

In practice, however, it is rarely the case that Ω is smooth. Let us hence assume that Ω (or, more precisely, its boundary $\partial\Omega$) is *not smooth*. The problem is then that the smoothness of f on $\overline{\Omega}$ does not imply that the solution u of Equation (1) is also smooth on $\overline{\Omega}$. This is the well-known phenomenon of *loss of regularity* for elliptic problems on non-smooth domains, a phenomenon extensively discussed in many research papers, including [29, 41, 47, 52], among many others (see also the subsection ‘‘Earlier results’’ below, where more references are given). Moreover, Wahlbin [93] has shown that this loss of regularity leads to the inconvenience that a quasi-uniform sequence of triangulations on Ω will *not* give optimal rates of convergence for the Galerkin approximations u_h of the solution of (1). See also [10, 55, 94]. This serious inconvenience can be avoided though by considering *graded* meshes instead of quasi-uniform ones.

The estimates necessary to deal with graded meshes are based on *weighted Sobolev* spaces, whose definition we now recall. Let $f > 0$ be a smooth function on the domain Ω . Then we define the *m*th Sobolev space with weight f by

$$\mathcal{K}_a^m(\Omega; f) := \{u, f^{|\alpha|-a} \partial^\alpha u \in L^2(\Omega), \quad |\alpha| \leq m\}, \quad m \in \mathbb{Z}_+, \quad a \in \mathbb{R}. \quad (2)$$

We assume from now on that Ω is a domain with a polyhedral structure and let $\vartheta(x)$ denote the distance from x to the edges of Ω . (The ‘‘domains with a polyhedral structure’’ are a generalization of polyhedral domains [29] and allow for cracks. Their definition is recalled in Section 3.) We shall always take $f = \vartheta$, or some regularization of ϑ . The main reason we are interested in weighted Sobolev spaces with $f = \vartheta$ is that we can then extend the regularity result for smooth domains to the domain with a polyhedral structure $\Omega \subset \mathbb{R}^3$.

The introduction of domains with ‘‘a polyhedral structure’’ is motivated by our desire to consider mixed boundary value problems. For this reason, we decompose the oriented boundary $\partial^\nu \Omega$ of Ω as a disjoint union $\partial^\nu \Omega =$

$\partial_D\Omega \cup \partial_N\Omega$ such that both $\partial_N\Omega$ and $\partial_D\Omega$ consist of a union of faces of Ω , where a face in the polyhedral structure may not agree with a geometric face. (In particular, a geometric face of Ω can carry Neumann boundary conditions on part of it and Dirichlet boundary conditions on the rest of the face.) We impose Dirichlet or “displacement” boundary conditions on $\partial_D\Omega$, and Neumann or “traction” boundary conditions $D_\nu^P = \nu \cdot \mathbf{C} \cdot \nabla$ on $\partial_N\Omega$. Situations that require mixed boundary conditions on domains with cracks arise in engineering problems, when for instance only part of a face of Ω is externally loaded (traction boundary condition), while the other part is mechanically fixed (zero displacement boundary condition), see [28, 30, 49, 84] and the references therein.

Main results We study the boundary value problem

$$Pu = f \text{ in } \Omega, \quad u = g_D \text{ on } \partial_D\Omega, \quad \text{and} \quad D_\nu^P u = g_N \text{ on } \partial_N\Omega, \quad (3)$$

for which we shall prove regularity and well-posedness in $\mathcal{K}_{a+1}^{m+1}(\Omega)$. Our results include the case where different types of boundary conditions are imposed on each side of an interior face (*i. e.*, a “crack”) via a non-tangential limit. If $m = 0$ this boundary value problem (3) must be understood in a weak, or variational sense. *Throughout this paper we shall assume that the tensor \mathbf{C} satisfies the usual symmetry and positivity assumptions, namely $C_{ijkl} = C_{jikl} = C_{klij}$ and positivity $C_{ijkl}(x)E^{ij}E^{kl} \geq 0$ for any symmetric two-tensor $E = [E^{ij}]$.* See (7).

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a polyhedral structure and let $P := \operatorname{div} \mathbf{C} \cdot \nabla : \mathcal{C}^\infty(\Omega)^3 \rightarrow \mathcal{C}^\infty(\Omega)^3$ be the operator of anisotropic elasticity. Assume that $u \in \mathcal{K}_{a+1}^1(\Omega)^3$, $Pu \in \mathcal{K}_{a-1}^{m-1}(\Omega)^3$, $u|_{\partial_D\Omega} \in \mathcal{K}_{a+1/2}^{m+1/2}(\partial_D\Omega)^3$, and $D_\nu^P u \in \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N\Omega)^3$, $m \in \mathbb{Z}_+$. Then $u \in \mathcal{K}_{a+1}^{m+1}(\Omega)^3$ and we have the inequality*

$$\begin{aligned} \|u\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)} \leq C & \left(\|Pu\|_{\mathcal{K}_{a-1}^{m-1}(\Omega)} + \|u|_{\partial_D\Omega}\|_{\mathcal{K}_{a+1/2}^{m+1/2}(\partial_D\Omega)} \right. \\ & \left. + \|D_\nu^P u\|_{\mathcal{K}_{a-1/2}^{m-1/2}(\partial_N\Omega)} + \|u\|_{\mathcal{K}_{a+1}^0(\Omega)} \right). \quad (4) \end{aligned}$$

for a constant $C = C(\Omega) > 0$ independent of u .

The same result holds for $P = \Delta$ and other strongly elliptic operators. See Theorem 4.

The proof of this result employs a characterization of the weighted Sobolev spaces in terms of special partitions of unity near the boundary, and rescaling, which blows up the singularity. The proof also gives the result for $m = 0$ (using a weak formulation). Since the faces of Ω are not necessarily straight, dilations are used in the tangent bundle of $\overline{\Omega}$ near the singular points.

We would like to stress that Theorem 1 does *not* constitute a Fredholm (or “normal solvability”) result, because the inclusion $\mathcal{K}_{a+1}^{m+1}(\Omega) \rightarrow \mathcal{K}_{a+1}^0(\Omega)$

is *not compact* for all m and a [3]. For example, if Ω is a polygon, then $P = -\Delta$ with Dirichlet boundary conditions is Fredholm precisely when a is different from $k\pi/\alpha$, where $k \in \mathbb{Z}$, $k \neq 0$, and α ranges through the angles of the polygon [52, 53].

We need to impose a stricter condition on P to obtain well-posedness, that is, existence and uniqueness of solutions of the problem (3).

Theorem 2. *Define $\tilde{P}(u) = (Pu, u|_{\partial_D\Omega}, D_\nu^P u|_{\partial_N\Omega})$, with $P = \operatorname{div} \mathbf{C} \cdot \nabla$ the operator of anisotropic elasticity. Assume that no two adjacent faces are assigned Neumann boundary conditions and that $\partial_D\Omega$ is not empty. Then there exists $\eta > 0$ such that*

$$\tilde{P} : \mathcal{K}_{a+1}^{m+1}(\Omega)^3 \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega)^3 \oplus \mathcal{K}_{a+1/2}^{m+1/2}(\partial_D\Omega)^3 \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N\Omega)^3$$

is an isomorphism for $m \in \mathbb{Z}_+$ and $|a| < \eta$. Again, the same result holds for $P = -\Delta$ and other strictly positive operators.

Earlier results The literature on boundary value problems on non-smooth domains is incredibly rich. So is the literature on elasticity. Let us try however to explain where our results stand compared with other results.

Two of the most common approaches to boundary value problems are the approach through coercive estimates and the approach through integral equations on the boundary (*i. e.*, the “method of layer potentials”). Our approach uses coercive estimates, positivity, and the Lax-Milgram Lemma. Many significant results were obtained however using integral equations on the boundary. See [38, 87] for descriptions of this method in its most elementary form. See also [23, 24, 47, 46, 51, 56, 68, 70, 71, 90, 91, 95], and [88] for just a few references to applications of the method of layer potentials to non-smooth domains. The method of layer potentials was also applied successfully to the Lamé system [57, 69].

In what follows, we shall discuss only the papers based on coercivity methods. We include here also the closely related method of “singular functions.” The literature is very vast and we would like to mention [4, 6, 15, 25, 26, 29, 31, 41–43, 52, 54, 53, 61, 62, 74, 76, 79], which contain results relevant for elasticity on domains with cracks. Most of the literature on the subject is devoted to the Lamé system or to polygonal domains. The Finite Element Method provides ample motivation for our research [5, 7, 27, 29, 35, 36, 75]. Regularity results are useful also in Numerical Linear Algebra for the study of the convergence of Multigrid methods [16, 44]. A treatment of boundary value problems in the presence of cracks can be found also in [50] using the edge pseudo-differential calculus. In that paper, in contrast with our case, a crack is a smooth immersed manifold of codimension one. The observation that the elasticity operator with suitable mixed displacement-traction boundary conditions is strictly positive is, however, a crucial new element in this paper.

Let us now discuss the results most closely related to our results and how our results compare with them. The most closely related results to

ours are those of Dauge [29], Kozlov, Mazya, and Rossmann [53], Mazya and Plamenevski [61], Mazya and Rossmann [63], Nazarov and Plamenevski [74], and Bacuta, Nistor, and Zikatanov [15]. In [29], Dauge introduces a class of domains used in our paper (however, see also the class \mathfrak{D}_3 in [74]) and points out some conditions necessary to obtain well-posedness for the Dirichlet problem (essentially that the boundary should not contain components that are of codimension two). She obtains a complete theory of singular functions in three dimensions, including general mixed boundary value problems (*i. e.*, each face is allowed *both* Dirichlet and Neumann boundary conditions). However, she works in the framework of the usual Sobolev spaces for which well-posedness and higher regularity are not directly possible. The book by Kozlov, Mazya, and Rossmann [53] is a good introduction to pencils of operators and singular functions in two dimensions. It contains a discussion of the Lamé system and a well-posedness result for the Dirichlet problem in two dimensions, a result that can be found also in [74]. The paper by Mazya and Rossmann [63] deals with coercive elliptic systems in curvilinear polyhedral domains without cracks and obtains Fredholm results under the assumption that certain other operators are invertible. Their results are for the case when each face is given one choice of either Dirichlet or Neumann boundary conditions. The most relevant chapter in the Nazarov and Plamenevski [74] book is essentially a translation of the paper by Mazya and Plamenevski [61]. In that paper, Mazya and Plamenevski describe a quite general class of domains defined inductively and they consider very general boundary conditions, but do not consider the case when a face can be given both Dirichlet and Neumann boundary conditions. They state the well-posedness of the Dirichlet problem for the Laplace equation on their class of domains. Their main result is that the operator associated to the boundary value problems that they study is Fredholm if certain other operators are invertible and the Shapiro-Lopatinski conditions are satisfied. Their definition of domains was made rigorous by Dauge [29]. The paper [19] contains our result for the Dirichlet problem for the Laplace operator. Fredholmess and index properties in weighted spaces is also discussed in the papers [22, 32] on smooth manifolds with boundary, but for pseudodifferential operators and singular integrals.

The most distinctive feature of our paper is that, for the fully anisotropic Elasticity operator, we obtain invertibility (or well-posedness) of the operator associated to the boundary value problem just under the usual symmetry and positivity assumptions on the elasticity tensor. Another distinctive feature is that we consider general domains with cracks, with curved faces, and we allow Dirichlet boundary conditions on part of each face and Neumann on the rest (only Dauge [29] seems to be working in a similar generality, the only common point of [29] and our paper). In order to deal with curved faces, we introduce for each face D a desingularization ΣD so that the outer unit normal vector function ν extends by continuity to a smooth function $\nu : \Sigma D \rightarrow S^2 \subset \mathbb{R}^3$ and we point out that the Shapiro-Lopatinski conditions must be satisfied at *all points of the desingularization* (in some other papers,

these conditions are imposed only at the boundary of the domain). The set ΣD is a smooth manifold with corners and is compact if Ω is bounded. Our framework thus owes to the work of Melrose [67] and Mazzeo [64]. In case D is not curved (*i. e.*, it is contained in a plane), we can replace ΣD with \overline{D} in our considerations, at least if D itself has no cracks, while Ω could still have cracks. We also consistently consider the case $m = 0$ in our results (when the datum f has H^{-1} regularity), this allows us to prove that certain operators are self-adjoint and allows to use the Lax-Milgram lemma. Yet another distinctive feature is that we consider *anisotropic* elasticity. From a mathematical point of view, this does not require a major adjustment of our theory, but we think is significant for applications. Let us notice that the case of anisotropic elasticity is much more difficult to treat using singular functions than it is to treat the Lamé system (isotropic elasticity) using singular functions, because the singular functions are significantly affected the nature of the system.

As far as current and future work is concerned, the paper [58] contains an extension of these results for domains in the plane to the case of pure Neumann boundary conditions and for interface problems as well. There *anisotropic* weighted spaces are used to recover quasi-optimal rates of convergence for Finite Element approximations. Also, we plan to apply our results to non-linear elasticity. In that regard, it would be interesting to see what is the limiting form of our results as one of the dimensions approaches zero, as in the theory of plates and shells (see e.g. [40]).

Contents of the paper The paper is organized as follows. In Section 2, we consider the special case of convex polyhedral domains, as an introduction and warm-up for the general case of domains with polyhedral structure, which is treated at length in Section 3. Section 4 deals with the appropriate class of smooth functions and differential operators on $\overline{\Omega}$. In Section 5, we define the spaces $\mathcal{K}_a^m(\Omega)$ and their boundary counterparts $\mathcal{K}_a^m(\partial\Omega)$, we study their properties and establish a trace theorem. Section 6 discusses the boundary value problems, the assumptions on the operator and boundary conditions, and introduces the notion of regularity upon freezing the coefficients, coercivity and positivity. Section 7 is devoted to the proof of the regularity theorem, while Section 8 addresses our main result, that is, the well-posedness of the boundary value problem (3) for positive operators. Finally, Sections 9 and 10 contain applications to the Laplace and elasticity operators. The weighted Poincaré and Korn's inequalities, used to establish positivity of these operators with mixed boundary conditions, are proved there.

Notation and conventions We conclude with relevant notations and conventions that will apply throughout this paper: $\Omega \subset \mathbb{R}^3$ will be a fixed domain with a polyhedral structure; in all our results, we shall further assume either that Ω is bounded or that it is a cone (a wedge is a cone);

$B^k \subset \mathbb{R}^k$ will denote the open unit ball (disk if $k = 2$) in \mathbb{R}^k and by S^{k-1} we will denote its boundary. Ω is always assumed open.

Our Hilbert and Banach spaces are taken to be real, unless mentioned otherwise. Even when we need to consider complex Hilbert spaces and operators between such spaces, they will arise by complexification. By C we denote a generic constant that may be different at each occurrence. Finally, we denote $\mathbb{Z}_+ = \{0, 1, \dots\}$.

2. Weights and smooth functions in the case Ω convex

To familiarize the reader, in this section we shall introduce two basic constructions, the “weight function” r_Ω and the function space $\mathcal{C}^\infty(\Sigma\Omega)$, in the classical case of Ω a polytope (the convex hull of finitely many points). In particular, Ω has finitely many vertices and edges.

A central role in our analysis is played by the “canonical weight function” $r_\Omega : \overline{\Omega} \rightarrow [0, \infty)$. Such weight function has the crucial property that is comparable to the distance function ϑ , where $\vartheta(x)$ the distance from $x \in \overline{\Omega}$ to the union of all edges of Ω . More precisely, $\vartheta(x)/r_\Omega(x)$ extends to a continuous function on $\overline{\Omega}$, and hence there exists a constant $C > 0$ such that

$$C^{-1}\vartheta(x) \leq r_\Omega(x) \leq C\vartheta(x), \quad (5)$$

for all $x \in \Omega$. (This result will be proved for general domains with polyhedral structure in Lemma 4.) However, r_Ω is more regular than ϑ in the sense that r_Ω belong to the space of “smooth” functions $\mathcal{C}^\infty(\Sigma\Omega)$ discussed below, while ϑ does not.

To define r_Ω , we need to introduce first some auxiliary functions ρ_Q and \tilde{r}_e , where Q ranges through the vertices of Ω and e ranges through the edges of Ω . We first set

$$\rho_Q(x) = \text{the distance from } x \text{ to } Q. \quad (6)$$

(This definition is the same in the general case.) Let next $r_e(x)$ be the distance to the *line* containing $e = [AB]$. We then define

$$\tilde{r}_e = (\rho_A \rho_B)^{-1} r_e. \quad (7)$$

Finally, we define the canonical weight r_Ω as the product

$$r_\Omega(x) = \prod_Q \rho_Q(x) \times \prod_e \tilde{r}_e(x), \quad (8)$$

where Q ranges through the vertices of Ω and e ranges through the edges of Ω .

We now turn to the definition of the space $\mathcal{C}^\infty(\Sigma\Omega)$, which is a space of smooth functions on Ω containing $\mathcal{C}^\infty(\overline{\Omega})$. Let us choose, for each edge e of Ω , a plane \mathcal{P}_e containing one of the open faces D of Ω such that $e \subset \overline{D}$. If x is not on the line defined by e , we then define θ_e to be the angle in a

cylindrical coordinates system (r_e, θ_e, z) determined by the edge e and the plane \mathcal{P}_e . More precisely, let $q \in e$ be the foot of the perpendicular from x to e . Then $\theta_e(x)$ is the angle between xq and \mathcal{P}_e . Similarly, for each vertex Q and edge e adjacent to Q , we define $\phi_{Q,e}(p)$ to be the angle between the segment xQ and the edge e .

Since Ω is convex in this section, the functions θ_e and $\phi_{P,e}$ are defined and smooth on Ω (in this paper, by Ω we shall always mean an open subset of \mathbb{R}^3 , possibly with additional properties). We shall denote by $\theta = (\theta_{e_1}, \dots, \theta_{e_r})$ the vector variable with components θ_{e_j} . Similarly, we shall denote by $\phi = (\phi_{Q_1, e_1}, \dots, \phi_{Q_p, e_p})$ with components ϕ_{Q_k, e_k} . We then introduce the space $\mathcal{C}^\infty(\Sigma\Omega)$ as the space of functions $u : \Omega \rightarrow \mathbb{C}$ of the form

$$u(x, y, z) = f(x, y, z, \theta, \phi) = f(x, y, z, \theta_{e_1}, \dots, \theta_{e_r}, \phi_{P_1, e_1}, \dots, \phi_{P_p, e_p}),$$

$$f \in \mathcal{C}^\infty(\overline{\Omega} \times [0, \pi]^r \times [0, \pi/2]^p).$$

The point of introducing the space $\mathcal{C}^\infty(\Sigma\Omega)$ is that, for example, θ_e is a smooth function on Ω which is not in $\mathcal{C}^\infty(\overline{\Omega})$ but $\theta_e \in \mathcal{C}^\infty(\Sigma\Omega)$. By definition, $\mathcal{C}^\infty(\Sigma\Omega)$ consists of smooth, bounded functions on Ω .

One can show as in [3] that there exists a smooth manifold with corners $\Sigma\Omega$, canonically associated to Ω , such that $\mathcal{C}^\infty(\Sigma\Omega)$ consists of the smooth function on $\Sigma\Omega$. The relevant properties of the function space $\mathcal{C}^\infty(\Sigma\Omega)$, especially in relation to the Babuška–Kondratiev spaces $\mathcal{K}_a^m(\Omega)$ will be discussed in the general framework of domains with a polyhedral structure in Section 5.

3. Domains with polygonal and polyhedral structures

In this section, we shall introduce the class of domains Ω with polyhedral structure and will extend to this class the definitions of the weight functions ρ_A , \tilde{r}_e , and r_Ω , as well as the definition of the space $\mathcal{C}^\infty(\Sigma\Omega) \subset \mathcal{C}^\infty(\Omega)$.

Domains with a polyhedral structure generalize polyhedral domains. This more general class is needed in order to include domains with cracks (or slits) and to deal with mixed boundary conditions, when Dirichlet boundary conditions are used on a part of a face and Neumann (or natural) boundary conditions are used on the remaining part of that face.

3.1. Domains with a polygonal structure

The definition of a “domain with a polyhedral structure” is based on that of a “domain with a polygonal structure.” Recall that throughout this paper, we shall denote by $B^k \subset \mathbb{R}^k$ the open unit ball (disk if $k = 2$) in \mathbb{R}^k and by S^{k-1} we shall denote its boundary. Thus $S^0 = \{-1, 1\}$, S^1 is the unit circle, and S^2 is the unit sphere.

Definition 1. A domain with polygonal structure D in a two dimensional smooth manifold M is an open subset $D \subset M$ together with a distinguished finite subset $\mathcal{V} \subset \partial D$ such that, for each $x \in \partial\Omega$, we are given a neighborhood V'_x of x in M satisfying

- (i) there is an open subset $\omega'_x \subset S^1$, $\omega'_x \neq S^1$, not necessarily connected, and a diffeomorphism $\phi'_x : V'_x \rightarrow B^2$ such that, in polar coordinates (r, θ) , we have

$$\phi'_x(V'_x \cap D) = \{(r, \theta), r \in (0, 1), \theta \in \omega'_x\};$$

- (ii) $\phi'_x(x) = 0$;
- (iii) if $x \notin \mathcal{V}$, then ω'_x consists of one or two intervals of length π .

In this paper, we will mostly need the case when $M = \mathbb{R}^2$ or $M = S^2$. We continue with some simple remarks and some examples.

Remark 1. The points in \mathcal{V} are called the *vertices* of D . Note that the set D does *not* determine its polygonal structure, because we can always increase the set \mathcal{V} . However, if there is a polygonal structure on D , then the one with the minimum set of vertices is unique. These are the *true vertices* of D . The other vertices of D will be called *artificial vertices*. The true vertices are the ones for which ω'_x consists of finitely many intervals in S^1 all of length less than π . It follows from the definition that each component of $\partial D \setminus \mathcal{V}$ is a smooth curve γ without self-intersections such that $\overline{\gamma} \subset \gamma \cup \mathcal{V}$. The curves γ will be called the *open sides* of D . The artificial vertices, and polygonal structures in general, are useful for the study of mixed boundary value problems. Our domains with a polygonal structure are not required to be bounded or connected.

Here are some examples.

Example 1. A polygonal domain (in the usual sense of the term) is a typical example of a domain with a polygonal structure. A domain with smooth boundary in \mathbb{R}^2 is a domain with a polygonal structure if we set $\mathcal{V} = \emptyset$ and the sides are the connected components of the boundary. In Figure 1 are given two domains with a polygonal structure. The true vertices are represented by thick points whereas the artificial vertices are represented by the symbol \times .

In order to treat curved domains in three dimensions, we need to introduce the desingularization ΣD of a domain D with a polygonal structure. In case only straight faces are considered (*i. e.*, each face is contained in a plane) and the faces have themselves no cracks, then we can replace ΣD with \overline{D} , and the following construction of ΣD becomes unnecessary. Let us consider first $\omega = (a_1, b_1) \cup \dots \cup (a_k, b_k) \subset S^1$. We define $\Sigma\omega$ to be the *disjoint union* $[a_1, b_1] \sqcup \dots \sqcup [a_k, b_k]$, that is, the disjoint union of the closures of the intervals comprising ω (by \sqcup we denote the disjoint union). If $\omega = S^1 \setminus \{e^{it}\}$, then $\Sigma\omega := [t, t + 2\pi] \subset \mathbb{R}$.

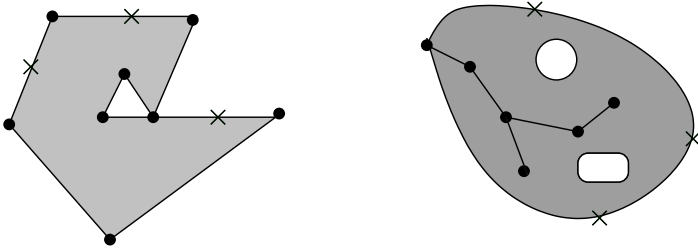


Fig. 1. Domains with polygonal structures

We now turn to the construction of ΣD . The main idea is that we separate the edges of the cracks and, for any vertex $x \in \mathcal{V} \subset \partial D$, we use the diffeomorphisms $\phi_x = \phi'_x$ of Definition 1 to attach $[0, \epsilon) \times \Sigma\omega_x$ to \overline{D} . More precisely, we first define $\partial^\nu D$ to be the set of *inner normal unit vectors*. The set $\partial^\nu D$ maps canonically to $\partial D \setminus \mathcal{V}$ and the map is one-to-one, except at the crack points, where it is two-to-one. We define $D^\nu := D \cup \partial^\nu D$. If there are no crack points, then $D^\nu = \overline{D} \setminus \mathcal{V}$ (we removed the vertices). Let S be the disjoint union of the sets D^ν and $[0, \epsilon) \times \Sigma\omega_x$ for x ranging through the set \mathcal{V} of vertices of D . We assume that ϵ is chosen small enough so that the sets $\phi_x^{-1}([0, \epsilon) \times \omega_x)$ are disjoint in D . The map ϕ_x^{-1} extends to a continuous map $(0, \epsilon) \times \Sigma\omega_x \rightarrow D^\nu$. We use ϕ_x^{-1} to identify the points of $(0, \epsilon) \times \Sigma\omega_x$ to their image in $\overline{D} \setminus \mathcal{V}$. This defines an equivalence relation on the set S . The quotient set with respect to this equivalence relation is the desired desingularization ΣD . The space ΣD has a natural structure of a manifold with corners (of codimension at most two) and we obtain a canonical map

$$\kappa : \Sigma D \rightarrow \overline{D}. \quad (9)$$

The desingularization ΣD was used already in the framework of manifolds with corners by Melrose [67]. See also [3, 4, 14].

3.2. Domains with a polyhedral structure

We now introduce domains with a polyhedral structure. The simplest example is that of a convex polyhedron, but in general we allow curved boundaries and cuts (*i. e.*, cracks).

Recall that a *smooth stratification* $S_0 \subset S_1 \subset \dots \subset X$ of a topological space X is an increasing sequence of closed sets $S_j = S_j(X)$ such that each point of X has a neighborhood that meets only finitely many of the sets S_j , S_0 is a discrete subset, $S_{j+1} \setminus S_j$, $j \geq 0$, is a disjoint union of smooth manifolds of dimension $j + 1$, and $X = \cup S_j$. A continuous map $f : X \rightarrow Y$ of stratified spaces is a continuous map satisfying $f(S_j(X)) \subset S_j(Y)$. If X is a stratified space and Y is a smooth manifold of dimension k , then $X \times Y$ is stratified canonically by $S_{j+k}(X \times Y) := S_j(X) \times Y$. For example, when $\omega \subset S^1$ is a disjoint union of finitely many open intervals, then we stratify $\overline{\omega}$ by $S_0(\overline{\omega}) = \partial\omega$ and $S_1(\overline{\omega}) = \overline{\omega}$. Similarly, in two dimension, when D is a

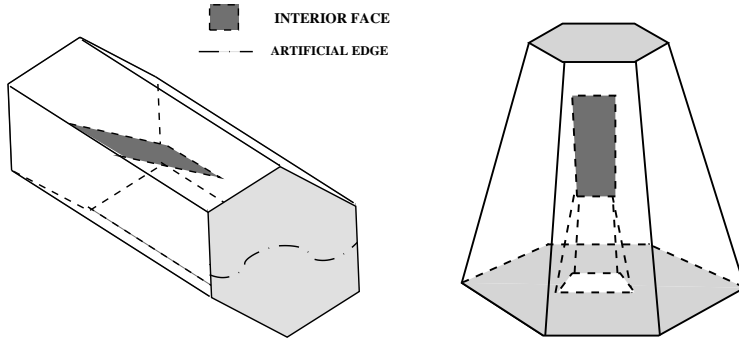


Fig. 2. Domains with polyhedral structure

domain with a polygonal structure, then \overline{D} is given the natural stratification $\mathcal{V} \subset \partial D \subset \overline{D}$.

If X is a stratified space, we define the cone over X , $CX := ([0, 1) \times X) / (\{0\} \times X)$, with stratification $S_{j+1}(CX)$ being defined as the image of $[0, 1) \times S_j(X)$ in CX and with $S_0(X)$ consisting of the point of CX that is obtained by collapsing $\{0\} \times X$.

We are now ready to introduce the “domains with a polyhedral structure.” Again, these domains do not need to be bounded nor connected. However, in view of applications to the analysis of boundary value problems, *we will assume throughout that Ω is connected.*

Definition 2. A *domain with a polyhedral structure* is an open subset $\Omega \subset \mathbb{R}^3$ together with a smooth stratification $\partial_0\Omega \subset \partial_1\Omega \subset \partial_2\Omega := \partial\Omega$ satisfying:

- (i) for each $x \in \partial\Omega$, we are given a fixed neighborhood V_x of x in \mathbb{R}^3 ;
- (ii) for each $x \in \partial_j\Omega \setminus \partial_{j-1}\Omega$, we are given an open, non-empty subset $\omega_x \subset S^{2-j}$;
- (iii) we are given diffeomorphisms $\phi_x : V_x \rightarrow B^j \times B^{3-j}$, $\phi_x(x) = 0$, such that for some $a \in (0, +\infty]$;

$$\phi_x(V_x \cap \Omega) = \{(y, rx')\}, y \in B^j, r \in (0, a), x' \in \omega_x \simeq B^j \times (0, a) \times \omega_x;$$

- (iv) if $j = 0$ (i. e., if $x \in \partial_0\Omega$), then $\omega_x \subset S^2$ is domain with a polygonal structure;
- (v) if $j = 1$ (i. e., if $x \in \partial_1\Omega \setminus \partial_0\Omega$), then ω_x is an open subset of S^1 with finitely many components;
- (vi) the inverse of ϕ_x extends to a continuous map $B^j \times C\overline{\omega}_x \rightarrow \overline{\Omega}$ of stratified spaces that is a homeomorphism onto its image.

From now on and throughout this paper, $\Omega \subset \mathbb{R}^3$ will be a fixed domain with a “polyhedral structure” that is either bounded or is an infinite cone (a wedge is a cone).

The set $\partial_0\Omega$ consists of the *vertices* of Ω . The connected components of $\partial_1\Omega \setminus \partial_0\Omega$ are the (open) *edges* of Ω and the connected components of

$\partial_2\Omega \setminus \partial_1\Omega$ are the (open) *faces* of Ω . The set $\partial_2\Omega \setminus \partial_1\Omega$ is the set of *smooth boundary points* and $\partial_{\text{sing}}\Omega = \partial_1\Omega$ is the set of *singular boundary points*. Our definition is closely related to that of Whitney stratified spaces [96].

Although Ω is not necessarily a Lipschitz domain, it can be written as a finite disjoint union of Lipschitz domains. For completeness, we include a short proof in the case Ω is bounded. The same proof applies for an infinite cone or wedge.

Lemma 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with polyhedral structure, then there exist finitely many disjoint Lipschitz domains Ω_j such that $\overline{\Omega} = \bigcup_{j=1}^N \overline{\Omega}_j$.*

Proof. Clearly, the interior of Ω can be triangulated, that is, divided into a finite number of tetrahedra with disjoint interiors. We need only cover a neighborhood of the boundary. By definition, Ω has a finite number of vertices Q and edges e . Therefore, there exists a finite number of neighborhoods V_x of Definition 2 that covers $\partial\Omega$. The neighborhoods $V_x \cap \Omega$ are diffeomorphic images (through the diffeomorphisms ϕ_x^{-1}) of

$$B^j \times [0, a) \times \omega_x, \quad j = 0, 1, 2,$$

where ω_x is a domain with polygonal structure on S^{2-j} . If x is a smooth point, then $j = 2$ and $\omega_x = \{1\}$ or $\omega_x = \{-1, 1\}$, so that $V_x \cap \Omega$ is the disjoint union of at most two connected Lipschitz domains. If x is a vertex Q , then $j = 0$ and ω_x is a domain on the sphere S^2 . We identify B^0 with $\{1\}$. Using spherical coordinates (θ, ϕ) for ω_x , it is easy to see that each connected component of $V_x \cap \Omega$ is a Lipschitz domain, provided $0 < \theta < \pi$. By splitting the interval $(0, 2\pi)$ into disjoint subintervals of length less than π , we again can write $V_x \cap \Omega$ as a finite disjoint union of Lipschitz components. Finally, if X belong to an edge e , then $j = 1$ and ω_x is an open interval in S^1 , while we identify $B^1 = [-1, 1]$. Again $V_x \cap \Omega$ can be triangulated by construction. \square

Remark 2. Let Ω be a domain with a polyhedral structure and $x \in \partial\Omega \setminus \partial_1\Omega$. Then the boundary of Ω is smooth *near* x . We distinguish two possibilities: Ω is only on one side of the boundary close to x or Ω is on both sides of the boundary close to x . We have $x \in \partial\overline{\Omega}$ in the first case and $x \notin \partial\overline{\Omega}$ in the second case. There will be *one* inward unit normal to $\partial\Omega$ at x in the first case and *two* inward unit normals to $\partial\Omega$ at x in the second case. We shall again denote by $\partial^\nu\Omega$ the *set of inward unit normal vectors* to the smooth part of the boundary of Ω and call it the *oriented boundary* of Ω with canonical projection

$$\kappa : \partial^\nu\Omega \rightarrow \partial\Omega \setminus \partial_{\text{sing}}\Omega, \quad (10)$$

from the *oriented boundary* of Ω to the boundary of Ω . This map is one-to-one at the points where Ω is on one side of the boundary and two-to-one at the points where Ω is on both sides of the boundary. Therefore,

there is a unique *outer normal unit vector* $\nu(w)$ at any point $w \in \partial^\nu \Omega$: $\nu : \partial^\nu \Omega \rightarrow S^2$, $\nu(w) = -w$. An *oriented face* of Ω is a connected component of $\partial^\nu \Omega$. Thus an oriented face of Ω is a face of Ω together with a choice of an inward unit vector. As before, we define the set

$$\Omega^\nu := \Omega \cup \partial^\nu \Omega$$

and endow it with the natural topology that makes it a smooth manifold with boundary (usually non-compact). We have $\partial^\nu \Omega = \partial \Omega \setminus \partial_1 \Omega$ if Ω has no cracks.

Let D be an oriented (open) face of our fixed domain with a polyhedral structure Ω . It is not true in general that D is a domain with a polygonal structure, because there may be no two dimensional smooth manifold containing the closure of D , for example when $\partial \Omega$ has an isolated conical point. However, it is not difficult to check that there exists a domain with a polygonal structure $D' \subset M$, for some manifold M and a diffeomorphism $D \rightarrow D'$. This observation allows to define $\Sigma D = \Sigma D'$ up to a diffeomorphism. Again in the case of an isolated conical point, we can take $M = S^2$ and $D' = \Sigma D'$ to be S^2 with a small cap removed.

We need to define ΣD so that the outer unit normal $\nu : D \rightarrow S^2$ extends by continuity to a smooth function on ΣD . (It does not extend to a continuous function on \overline{D} , though.)

4. Differential operators and $\mathcal{C}^\infty(\Sigma \Omega)$

In this section we introduce the relevant space of functions and differential operators on Ω . The natural function space will contain functions smooth on Ω , but not necessarily on $\bar{\Omega}$ such that, however, it still contains the analogues of the polar, spherical, and cylindrical coordinate functions.

4.1. The space $\mathcal{C}^\infty(\Sigma \Omega)$

We first recall the diffeomorphisms $\phi_x : V_x \rightarrow B^j \times B^{3-j}$ of Definition 2 satisfying

$$\phi_x(V_x \cap \Omega) = \{(y, rx'), y \in B^j, r \in (0, a), x' \in \omega_x\} \simeq B^j \times (0, a) \times \omega_x.$$

Also we recall that for $j = 0, 1$, the sets $\Sigma \omega_x$ were already defined ($\omega_x \subset S^{2-j}$). For $j = 2$, the set ω_x consists of one or two points, and then we define $\Sigma \omega_x = \omega_x$. Finally, given a smooth function $f : \Omega \rightarrow \mathbb{R}$, we say that $f \in \mathcal{C}^\infty(\Sigma D)$ if, and only if, $f \circ \phi_x^{-1}$ extends to a smooth function on $B^j \times [0, a) \times \Sigma \omega_x$, for all x .

To better explain this definition, we consider a vertex Q of Ω . Let $\omega_Q \subset S^2$ be the domain with a polygonal structure introduced in Definition 2 so that $\phi_Q : V_Q \cap \Omega \simeq (0, \epsilon) \times \omega_Q$. We concentrate on the part

of the neighborhood $U_Q := V_Q \cap \Omega$ close to a fixed edge e that contains Q . Let A be the vertex of ω_Q corresponding to the fixed edge e . Since ω_Q is a domain with polygonal structure by assumption, there exists a diffeomorphism $\phi_A : V_A \cap \omega_Q \rightarrow (0, \epsilon) \times \omega_A$, $\omega_A \subset S^1$. We denote by $(\phi, \theta) \in (0, \epsilon) \times \omega_A$ the corresponding coordinates. Then, in the open set corresponding to $(0, \epsilon) \times (0, \epsilon) \times \omega_A$, the condition that $f \in \mathcal{C}^\infty(\Sigma\Omega)$ is equivalent to the fact that $f(x, y, z)$, when written in spherical coordinates ($x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$), extends to a smooth function on $(\rho, \phi, \theta) \in [0, \epsilon) \times [0, \epsilon) \times \Sigma\omega_A$. By definition, $\mathcal{C}^\infty(\Sigma\Omega)$ is closed under addition and multiplication (*i. e.*, it is an algebra).

Let us also notice that if f is smooth in the neighborhood V_x of some $x \in \partial\Omega$, then $f \circ \phi_x^{-1}$ is automatically smooth on the closure of $(0, a) \times \omega_x$, that is, $\mathcal{C}^\infty(\overline{\Omega}) \subset \mathcal{C}^\infty(\Sigma\Omega)$.

If $U \subset \Omega$ is an open set and $f : U \rightarrow \mathbb{R}$ is smooth, we shall say that $f \in \mathcal{C}^\infty(\Sigma U)$ if it is the restriction of a function in $\mathcal{C}^\infty(\Sigma\Omega)$. The definition of $\mathcal{C}^\infty(\Sigma U)$ depends on Ω , although this is not shown in this notation.

4.2. The canonical weight function

As mentioned earlier, we shall need a canonical weight function $r_\Omega \in \mathcal{C}^\infty(\Sigma\Omega)$ that has the same type of growth as the function ϑ given by

$$\vartheta(x) := \text{the distance from } x \in \Omega \text{ to the singular points of } \partial\Omega. \quad (11)$$

The reason we want to replace in some reasonings the function ϑ with r_Ω is that the function ϑ is not regular enough, *i. e.*, $\vartheta \notin \mathcal{C}^\infty(\Sigma\Omega)$. (This is the case both in two and three dimensions.)

Let us treat first the case of a domain $D \subset \mathbb{R}^2$ with a polygonal structure, which is easier. In that case $\vartheta(x)$ is the distance from x to the set of vertices of D . The smoothed version is then defined by

$$r_D(y) = \prod_{Q \in \mathcal{V}} \rho_Q(y), \quad (12)$$

where Q ranges through the vertices of D . In particular, if D is an angle with vertex A , then $r_D = \rho_A$. Also, the definition of r_D shows that r_D/ϑ extends to a continuous function $\overline{D} \rightarrow (0, \infty)$.

We now turn to the case of a domain $\Omega \subset \mathbb{R}^3$, which we recall is either bounded or an infinite cone. Also, we recall that $\rho_Q(y)$ denotes the distance from y to Q . The constructions below are significantly simpler in the case of straight polyhedral domains, which is the case typically considered in the literature.

Let e be an open edge of Ω . We first extend the definition of the function \tilde{r}_e in Equation (7) from the case of a polytope to the general case.

If $\Omega = \{rx', x' \in \omega, 0 < r \leq \infty\}$, for some domain $\omega \subset S^2$ with a polygonal structure, $\omega \neq \emptyset$, $\omega \neq S^2$, that is, if Ω is a cone that is *not* a wedge, let A be the vertex of ω to which e corresponds and let $\rho_A^\omega(x')$ be

the Euclidean distance from $x' \in S^2$ to A (measured using straight lines). Then we define

$$\tilde{r}_e(rx') := \rho_A^\omega(x') \quad (e \text{ corresponds to } A), \quad x' \in S^2. \quad (13)$$

Note that we do not require ω to be connected. On the other hand, if $\Omega = \{(r, \theta, z), 0 < \theta < \alpha\}$ in polar coordinates, that is, if Ω is a wedge with edge e , then we define $\tilde{r}_e = r$, the distance to e .

For a bounded Ω , we want \tilde{r}_e to be defined near each vertex as done above for a cone. Away from the vertices but close to the edge instead, we want \tilde{r}_e to be equivalent to the distance to the edge. Otherwise, we want \tilde{r}_e to be smooth and bounded away from zero. This construction can be achieved easily using a partition of unity.

We can now define the *canonical weight function* r_Ω by the same formula as for a convex polyhedral domain, namely

$$r_\Omega(y) = \prod_Q \rho_Q(y) \times \prod_e \tilde{r}_e(y), \quad (14)$$

where Q ranges through the vertices of Ω and e ranges through the edges of Ω . These products are well-defined since we have assumed that Ω is either an infinite cone or a bounded set.

We next prove that the various functions that we have introduced are in $\mathcal{C}^\infty(\Sigma\Omega)$.

Lemma 2. *Let Q be a vertex of Ω , then $\rho_Q \in \mathcal{C}^\infty(\Sigma\Omega)$.*

Proof. Let (ρ_Q, x') , $x'(y) = y/\rho_Q(y) \in S^2$, be the generalized polar coordinates around Q . Let $\phi_Q : V_Q \rightarrow B^j \times B^{3-j} = B^3$ be the given diffeomorphism of Definition 2 (in this case $j = 0$, because Q is a vertex). Then ϕ_Q^{-1} extends to a differentiable map $S^2 \times [0, \epsilon) \rightarrow \Sigma(V_Q \cap \Omega)$ for some small $\epsilon > 0$, where $\Sigma((V_Q \cap \Omega))$ is the desingularization of the domain $V_Q \cap \Omega$ constructed similarly to ΣD . Consequently, $\rho_Q \circ \phi_Q^{-1}$ is differentiable. Finally, let $b \in \overline{\Omega}$, with $b \neq Q$. Then Q will not belong to V_b and hence ρ_Q is smooth on V_b . Consequently, $\rho_Q \circ \phi_b^{-1}$ extends to differentiable function on $[0, a) \times \overline{\omega_Q}$. \square

We also have the following result.

Lemma 3. *Let e be an edge of Ω , then $\tilde{r}_e \in \mathcal{C}^\infty(\Sigma\Omega)$. In particular, $r_\Omega := \prod_e \tilde{r}_e \times \prod_Q \rho_Q \in \mathcal{C}^\infty(\Sigma\Omega)$.*

For the purpose of the following proof, we will say that two functions $f_1, f_2 : U \rightarrow [0, \infty)$, defined on some open subset $U \subset \Omega$ are *equivalent* if the quotients f_1/f_2 and f_2/f_1 extend to functions in $\mathcal{C}^\infty(\Sigma U)$, i. e., are restrictions of some functions in $\mathcal{C}^\infty(\Sigma\Omega)$. Furthermore, for the purpose of this and later proofs, the set defined by x' close enough to the vertices will be denoted W_1 , while the set defined by x' far from the vertices will be denoted W_2 .

Proof. We introduce ideas and notation that will be used in other proofs as well. Let $\phi_x : V_x \rightarrow B^j \times B^{3-j}$ be as in definition 2. Let $U_x := V_x \cap \Omega$. By Lemma 2 it is enough to show what $\tilde{r}_e \in \mathcal{C}^\infty(\Sigma U_x)$ for all x . (Recall that $\mathcal{C}^\infty(\Sigma U)$ denotes the set of restrictions to U of functions in $\mathcal{C}^\infty(\Sigma \Omega)$.)

Assume first that $x = Q$ is a vertex. Fix spherical coordinates (ρ, θ, ϕ) in a neighborhood of an edge e of Ω corresponding to a vertex A of ω_x . The vertex Q then corresponds to $\rho = 0$ and the edge e corresponds to $\phi = 0$. For x' in a sufficiently small neighborhood of A , $r = \rho \sin \phi$ is equivalent to r_e , as functions of (ρ, θ, ϕ) , for ϕ small enough.

For x' away from the edges r_Ω and ρ are equivalent in the same sense. Similarly, ρ_x and ρ are equivalent. It follows that r_e/ρ_A and $r/\rho = \sin \phi$ are equivalent. Since $\sin \phi$ is in $\mathcal{C}^\infty(\Sigma U_x)$, r_e/ρ_A is in $\mathcal{C}^\infty(V_x)$. If A is the only end point of e , then $\tilde{r}_e = r_e/\rho_A$ is in $\mathcal{C}^\infty(V_x)$. Otherwise, let B be the other end point of e . Then ρ_B is smooth on V_x . Consequently, $\tilde{r}_e := (r_e/\rho_A)/\rho_B$ is in $\mathcal{C}^\infty(V_x)$ as well.

Assume now that x is an edge point, a slightly easier case. Again, let us chose $\phi_x : V_x \rightarrow B^1 \times B^2$ as in definition 2. Let (z, r, θ) be the corresponding cylindrical coordinates (z parameterizes $B^1 = (-1, 1)$). Then r_e is equivalent to r on V_x , as functions of (z, r, θ) . Since ρ_A and ρ_B are smooth function on V_x , $\tilde{r}_e := r_e/(\rho_A \rho_B)$ is in $\mathcal{C}^\infty(\Sigma U_x)$. The case when there are fewer end points is completely similar.

The proof is therefore complete, since there are no restrictions at the smooth boundary points of $\partial \Omega$. \square

Lemma 4. *Let $\vartheta(y)$ be the distance from y to the union of the edges of Ω . Then there exists $C > 0$ such that*

$$C^{-1}\vartheta(y) \leq r_\Omega(y) \leq C\vartheta(y)$$

for all $y \in \Omega$.

Proof. We use the notation of the previous proof and set x to be a vertex. Then, as shown above, we can replace r_Ω with $\rho_x r_{\omega_x}$ in each of the sets V_x . In the new coordinate system given by the diffeomorphism ϕ_x , the function ϑ is replaced with a function ϑ' such that the quotients ϑ/ϑ' and ϑ'/ϑ are bounded. We can therefore replace ϑ with ϑ' . But the quotient $\rho_x r_{\omega_x}/\vartheta'$ is homogeneous of degree zero, so that it is enough to prove the lemma in the domain with a polygonal structure ω_x . It is clear then from the definitions that the quotients r_D/ϑ' and ϑ'/r_D are bounded on $\omega_x \subset S^2$.

The case of an edge point is completely similar. Since away from the edges the quotients r_Ω/ϑ and ϑ/r_Ω are continuous and $\overline{\Omega}$ can be covered with finitely many sets of the form V_x , the proof is complete. \square

4.3. Rescaled tangent bundle

For $x \in \overline{\Omega}$, we shall denote by

$$\alpha_x(z) = x + r_\Omega(x)(z - x) \tag{15}$$

the dilation of center x and ratio $r_\Omega(x)$, where r_Ω is the canonical weight. Recall that $\partial^\nu \Omega$ denotes the set of *inner normal unit vectors* y to Ω and we have a canonical projection $\kappa : \partial^\nu \Omega \rightarrow \partial\Omega$, given in Equation (10).

Let us fix a connected component $D \subset \partial^\nu \Omega$ (i. e., an oriented face of Ω). Let also $T_y D$ be tangent space to D at y , which we identify with $T_x \partial\Omega$, the tangent space of $\partial\Omega$ at $x = \kappa(y)$. Then the dilation α_x^{-1} of Equation (15) maps $T_y D$ to itself. We consider the orthogonal projection $\tilde{\pi}_y$ of $\kappa(D)$ onto $T_y D$.

By $\pi_y : D \rightarrow T_y D \simeq \mathbb{R}^2$ we shall denote the composition $\alpha_x^{-1} \circ \tilde{\pi}_y \circ \kappa$, the *rescaled projection* of D onto its tangent plane (where $x = \kappa(y)$ as before). We consider these maps for all faces D of Ω .

Lemma 5. *Assume $\overline{\Omega}$ is compact. Then there exists $r > 0$ such that, for all faces D and all $y \in D$, the rescaled projection $\pi_y := \alpha_x^{-1} \circ \tilde{\pi}_y \circ \kappa : D \rightarrow T_y D$, $x = \kappa(y)$, is a diffeomorphism from a small neighborhood of y in D to $B_y^2(r)$, the ball of radius r and center 0 in $T_y D$.*

Proof. For every $y \in D$, there exists a small $r_y > 0$ such that the rescaled projection π_y is a diffeomorphism from a small neighborhood of x in D to $B_y^2(r_y)$. Let us choose for every $y \in D$ the largest r_y with this property. We need to show that there exists $r > 0$ such that $r_y > r$. By continuity, r_y is lower semi-continuous. It is therefore enough to show that r_y is bounded from below close to the boundary of D .

To this end, we notice that the property that r_y be bounded from below is invariant under any change of coordinates, even non-linear change of coordinates (they can change however the lower bound), because $\overline{\Omega}$ is compact. Using the diffeomorphisms ϕ_x of Definition 2, we can therefore reduce to the case Ω is the dihedral angle

$$D_\alpha := \{(r, \theta, z), 0 < \theta < \alpha\}$$

or the cone

$$\mathcal{C}_\omega := \{tx', t \in (0, \infty), x' \in \omega\}, \quad \omega \subset S^2.$$

By Lemma 4, we can replace r_Ω with ϑ . Then $r_y = 1$ and hence it is bounded away from 0. This completes the proof. \square

The following lemma is very important for our treatment of curved faces.

Corollary 1. *Assume that $\overline{\Omega}$ is compact. Let $\pi_y^{-1} : B_r^2(y) \rightarrow D$ be the inverse of the diffeomorphism of Lemma 5. Then, after decreasing $r > 0$, if needed, the map $\chi_y : B_y^2(r) \times [0, r) \rightarrow \Omega^\nu := \Omega \cup \partial^\nu \Omega$*

$$\chi_y(z, t) := \pi_y^{-1}(z) + t r_\Omega(x)y = \pi_y^{-1}(z) - t r_\Omega(x)\nu(y), \quad x = \kappa(y),$$

defines a diffeomorphism onto a neighborhood W_y of y in Ω^ν such that $B_y^2(r) \times \{0\}$ maps to the boundary $\partial^\nu \Omega$ of Ω^ν and $B_y^2(r) \times (0, r)$ maps to Ω .

Let \mathbb{H}_y be the half-space determined by the unit vector $y \in \partial^\nu \Omega$. Let D be the oriented face of Ω containing y . Then $\partial \mathbb{H}_y = T_y D$. The set $B_y^2(r) \times [0, r)$ will be regarded, in what follows, as a subset of the (closure of) \mathbb{H}_y .

4.4. Differential operators

In this subsection we introduce the differential operators that will be used in this paper. We continue to denote by r_Ω the canonical weight function introduced in (14). These constructions are a direct extension of the results in [3,4], so we shall omit the proofs that are very close to those in these two papers.

In this section, x , y , and z shall denote the Euclidean coordinates of the point $(x, y, z) \in \mathbb{R}^3$. We will need the following differentiability properties.

Lemma 6. *The functions $\partial_x r_\Omega$, $\partial_y r_\Omega$, and $\partial_z r_\Omega$ are in $\mathcal{C}^\infty(\Sigma\Omega)$. If $u \in \mathcal{C}^\infty(\Sigma\Omega)$, then the functions $r_\Omega \partial_x u$, $r_\Omega \partial_y u$, and $r_\Omega \partial_z u$ are also in $\mathcal{C}^\infty(\Sigma\Omega)$.*

Proof. Using the notation introduced in the proof of Lemma 3, it is enough to prove that the resulting functions are smooth functions of (ρ, θ, ϕ) on a set of the form W_1 , that they are smooth functions of (ρ, x') , $x' \in S^2$, on a set of the form W_2 , and that they are smooth functions of $(r, \theta, z) \in [0, a) \times \omega_x \times B^1$ on a set of the form V_p with p an edge point.

Assume first that we are on a set of the form W_1 . Writing $\partial_x r$, $\partial_y r$, $\partial_z r$, $r \partial_x u$, $r \partial_y u$, and $r \partial_z u$ in spherical coordinates (ρ, θ, ϕ) immediately show that these functions are also in $\mathcal{C}^\infty(\Sigma W_1)$. The result for sets of form W_1 then follows from the fact that r and r_Ω are equivalent on W_1 .

Assume next that we are on a set of the form W_2 . Let p be an arbitrary point in W_2 . Our statement is independent of linear changes of coordinates, so we can assume that p is on the positive Oz semi-axis. Let us assume first that $r_\Omega = z$. Then $\partial_x z$, $\partial_y z$, and $\partial_z z$ are smooth functions, which take care of the first part of our statement. Next, we have $u \in \mathcal{C}^\infty(\Sigma W)$ if, and only if, $u(x, y, z) = \tilde{u}(x/z, y/z, z)$ for some smooth function \tilde{u} (smooth in the usual sense). Therefore $z \partial_x u$, $z \partial_y u$, and $z \partial_z u$ are in $\mathcal{C}^\infty(\Sigma W)$ as well. The result then follows from the fact that ρ and z are equivalent on W , because then we can replace z with r_Ω . Since the point p was chosen arbitrarily, this completes the proof on a set of the form W_2 .

For a set of the form V_x , with x an edge point, the proof is completely similar and easier, writing $r_\Omega \partial_x$ and $r_\Omega \partial_y$ in cylindrical coordinates (r, θ, z) and using that r_Ω is equivalent to r in V_x . \square

Let us denote by $\text{Diff}_0^m(\Omega)$ the space of order m differential operators on Ω linearly generated over $\mathcal{C}^\infty(\Sigma\Omega)$ by

$$(r_\Omega \partial)^\alpha := (r_\Omega \partial_x)^{\alpha_1} (r_\Omega \partial_y)^{\alpha_2} (r_\Omega \partial_z)^{\alpha_3}, \quad |\alpha| := \alpha_1 + \alpha_2 + \alpha_3 \leq m,$$

We agree that $\text{Diff}_0^0(\Omega) := \mathcal{C}^\infty(\Sigma\Omega)$ and denote $\text{Diff}_0^\infty(\Omega) := \bigcup_m \text{Diff}_0^m(\Omega)$.

We have the following simple, but fundamental result.

Lemma 7. *Let $\lambda \in \mathbb{R}$ and let ∂_j and ∂_k stand for either of ∂_x , ∂_y , or ∂_z . Then $r_\Omega^{-\lambda} (r_\Omega \partial_j) r_\Omega^\lambda - r_\Omega \partial_j = \lambda \partial_j (r_\Omega) \in \mathcal{C}^\infty(\Sigma\Omega)$, and $[r_\Omega \partial_j, r_\Omega \partial_k] \in \text{Diff}_0^1(\Omega)$.*

The proof of this lemma and of the following proposition are as in [4].

Proposition 1. *We have $\text{Diff}_0^k(\Omega) \text{Diff}_0^m(\Omega) \subset \text{Diff}_0^{k+m}(\Omega)$ and hence $\text{Diff}_0^\infty(\Omega)$ is an algebra. If P is a differential operator of order m with smooth coefficients, then $r_\Omega^m P \in \text{Diff}_0^m(\Omega)$.*

In particular, for $|\alpha| = m$

$$r_\Omega^m \partial_x^{\alpha_1} \partial_y^{\alpha_2} \partial_z^{\alpha_3} - (r_\Omega \partial_x)^{\alpha_1} (r_\Omega \partial_y)^{\alpha_2} (r_\Omega \partial_z)^{\alpha_3} \in \text{Diff}_0^{m-1}(\Omega). \quad (16)$$

5. Function spaces on Ω

In this section, we shall introduce and study the weighted spaces $\mathcal{K}_a^m(\Omega)$ and $\mathcal{K}_a^m(\partial^\nu \Omega) := \mathcal{K}_a^m(\partial^\nu \Omega; r_\Omega) = \mathcal{K}_a^m(\partial^\nu \Omega; \vartheta)$. These spaces are weighted Sobolev spaces with weight given by the canonical weight r_Ω or, equivalently, the distance to the set of edges of Ω , as in Equation (2). Therefore these spaces depend on the choice of polyhedral structure on Ω , a fact significant for the study of mixed boundary value problems.

5.1. Definitions and main properties

For $m \in \mathbb{Z}_+$, we consider weighted Sobolev spaces

$$W_{BK}^{m,p,a}(\Omega) = \{u : \Omega \rightarrow \mathbb{R}, r_\Omega^{|\alpha|-a} \partial^\alpha u \in L^p(\Omega), \text{ for all } |\alpha| \leq m\}. \quad (17)$$

If $p = 2$, we denote $\mathcal{K}_a^k(\Omega) := W_{BK}^{k,2,a}(\Omega)$. These are the spaces discussed in Equation (2) of the Introduction. In this article, we refer to $W^{M,p,a}(\Omega)$, in particular to $\mathcal{K}_a^m(\Omega)$, as *Babuška-Kondratiev spaces*, though we recognize that other authors have contributed to their theory, including the essential work of Mazy'a and Plamenevskii.

We let $\mathcal{K}_a^m(\partial^\nu \Omega) := \bigoplus W_{BK}^{m,2,a}(D_k)$ for all faces D_k of Ω , where, if D is an oriented face of Ω and $m \in \mathbb{Z}_+$,

$$W_{BK}^{m,p,a}(D) = \{u : D \rightarrow \mathbb{R}, r_\Omega^{k-a} P u \in L^p(D), P \text{ a differential operator of order } k \leq m \text{ on } D\}.$$

Note that we require *no compatibility conditions* for the resulting functions on the faces D_k .

Lemma 4, Lemma 7, and Equation 16 give easily the following lemmas.

Lemma 8. *We have $\mathcal{K}_a^m(\Omega) = \{u, \vartheta^{-a} P u \in L^2(\Omega), \text{ for all } P \in \text{Diff}_0^k(\Omega)\}$. A similar result holds for $\mathcal{K}_a^m(\partial^\nu \Omega)$ and for $W_{BK}^{k,p,a}(\Omega)$.*

Lemma 9. *The multiplication map $W_{BK}^{m,\infty,b} \times \mathcal{K}_a^m(\Omega) \ni (u, f) \rightarrow u f \in \mathcal{K}_{a+b}^m(\Omega)$ is continuous. We also have $\mathcal{C}^\infty(\Sigma \Omega) \subset W_{BK}^{m,\infty,0}(\Omega)$ and $r_\Omega^b \in W_{BK}^{m,\infty,b}(\Omega)$, and hence the map $\mathcal{K}_a^m(\Omega) \ni u \rightarrow r_\Omega^b u \in \mathcal{K}_{a+b}^m(\Omega)$ is a continuous isomorphism of Banach spaces. The same result is true if we replace Ω with $\partial^\nu \Omega$.*

From this lemma we obtain immediately the following result.

Proposition 2. *Let $k \geq m$. Each $P_0 \in \text{Diff}_0^p(\Omega)$ defines a continuous map $P_0 : \mathcal{K}_a^k(\Omega) \rightarrow \mathcal{K}_a^{k-m}(\Omega)$. The family $r_\Omega^{-\lambda} P_0 r_\Omega^\lambda$ is a family of bounded operators $\mathcal{K}_a^k(\Omega) \rightarrow \mathcal{K}_a^{k-m}(\Omega)$ depending continuously on λ .*

Similarly, if P is a differential operator with smooth coefficients on Ω , then $r_\Omega^{-\lambda} P r_\Omega^\lambda$ defines a continuous family of bounded operators $\mathcal{K}_a^k(\Omega) \rightarrow \mathcal{K}_{a-m}^{k-m}(\Omega)$.

We define the spaces $\mathcal{K}_a^{-k}(\Omega)$, $k \in \mathbb{Z}_+$, by duality. More precisely, let $\mathring{\mathcal{K}}_a^k(\Omega)$ be the closure of $\mathcal{C}_c^\infty(\Omega)$ in $\mathcal{K}_a^k(\Omega)$. Then we define $\mathcal{K}_{-a}^{-k}(\Omega)$ to be the dual of $\mathring{\mathcal{K}}_a^k(\Omega)$, the duality pairing being an extension of the bilinear form $(u, v) \mapsto \int_\Omega uv \, dx$. With this definition, we can drop the requirement that $k \geq m$ in Proposition 2.

We define $\mathcal{K}_{-a}^{-k}(\partial^\nu \Omega)$ as the dual of $\mathcal{K}_a^k(\partial^\nu \Omega)$. The spaces $\mathcal{K}_a^s(\partial^\nu \Omega)$, $s \notin \mathbb{Z}$ will be defined using partitions of unity, but could be defined as with the standard spaces by real interpolation as well.

5.2. Partitions of unity

We now introduce a smooth partition of unity \mathcal{P}_Ω on $\Omega^\nu := \Omega \cup \partial^\nu \Omega$, where either Ω is a cone or $\overline{\Omega}$ is compact. This partition is a particular case of a construction that, in the case of non-compact manifolds, goes back to Aubin [8, 9]. It was subsequently used by Gromov [82] and later in [3, 83, 89] and other papers.

The partition of unity \mathcal{P}_Ω will be constructed out of a uniformly locally finite family of non-negative functions \mathcal{F}_Ω by the usual procedure: let $g := \sum f$, $f \in \mathcal{F}_\Omega$, which is well defined since the family \mathcal{F}_Ω is locally finite. Then we consider the family

$$\mathcal{P}_\Omega := \{f/g, f \in \mathcal{F}_\Omega\}, \quad (18)$$

which will be also well defined as $g > 0$ in our case.

We need to consider first the same construction for the case of a domain $D \subset S^2$ with a *polygonal* structure, yielding a uniformly locally finite family \mathcal{F}_D and then, by the procedure of Equation (18), a partition of unity \mathcal{P}_D . Let $\mathcal{V} \subset \partial D$ be the set of vertices of D . For each $A \in \mathcal{V}$, let $\phi_A : V_A \cap D \rightarrow (0, \epsilon) \times \omega_A$ be the diffeomorphism of Definition 1 and ρ_A be the distance function to A . Let $\psi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that $\psi = 1$ on $[0, 1]$ and ψ has support in $[-1, 2]$. Then for each vertex we consider the family of functions $\psi_n(y) := \psi(\log_2 \rho_A(y) - n)$, for all values of n for which these functions are supported in V_A . We also consider a smooth partition of unity ξ_k of ω_A consisting of at least two functions. Let $\mathcal{F}_A := \{(\xi_k \circ \theta \circ \phi_A) \psi_n\}$.

If D is in fact an angle with vertex A , then the family \mathcal{F}_A will suffice. Otherwise, we complete the union of \mathcal{F}_A for all vertices A by adding functions in a finite partition of unity \mathcal{F}_0 on D^ν with sufficiently small support

that are zero in the neighborhood of all vertices. We denote \mathcal{F}_D the resulting family.

The next lemma follows directly from the construction of \mathcal{F}_D by refining the partition of unity ξ_k on the components of $\omega_A \subset S^1$ if necessary.

Lemma 10. *Assume that \overline{D} is compact. Then there exists a constant $\kappa_D > 0$ such that no point $y \in \overline{D}$ belongs to the support of more than κ_D of the functions $\psi \in \mathcal{F}_D$.*

We will denote by \mathcal{P}_D the partition of unity associated to the family \mathcal{F}_D as explained in the beginning of this subsection (see Equation (18) and the discussion there). We recall r_D is the canonical weight function of D given in Equation (12).

Lemma 11. *There exists a constant $C > 0$ such that*

$$|r_D^{|\alpha|} \partial^\alpha \psi(y)| \leq C \text{ for all } y \in D \text{ and all } \psi \in \mathcal{F}_D \text{ or } \psi \in \mathcal{P}_D. \quad (19)$$

A point $y \in D$ belongs to at most κ_D of the supports of the functions ψ in \mathcal{P}_D . Moreover, for any $\psi \in \mathcal{P}_D$, the support of ψ contains no vertex, intersects at most one oriented side of D , and has diameter $\leq Cr_D(y)$, for any y in the support of ψ and a constant C independent of ψ .

Proof. Only the estimate (19) needs a proof. In view of Lemma 10, it is enough to prove the statement for $\psi \in \mathcal{F}_D$. We need only check that (19) holds close to a vertex Q . Let us fix polar coordinates (r, θ) in a neighborhood W of Q . The functions r_D and r are equivalent in this neighborhood of Q in the usual sense that their quotients are in $C^\infty(\Sigma W)$, and hence we can assume that $r_D = r$. Then we use the fact that $(r\partial_r)^i \partial_\theta^j \psi$ are bounded, by construction, and employ local polar coordinates to express $r^{|\alpha|} \partial^\alpha \psi$ as a combination of $(r\partial_r)^i \partial_\theta^j \psi$ with coefficients that are smooth functions in r and θ (up to $r = 0$). \square

For a polyhedral domain, we proceed in a similar way. On a set of the form $B^j \times (0, a) \times \omega_x$, we consider a product type family, coming from a partition of unity ζ_k on a neighborhood of B^j , a dyadic (infinite) partition of unity $\psi_n = \psi(\log_2 r - n)$ for $r \in (0, a)$, and a partition of unity ξ_l of ω_x . The partition of unity on ω_x is as before: at least two function if ω_x is an interval, and a partition of unity of the form \mathcal{F}_{ω_x} if ω_x is a domain with a polygonal structure. Only the former case must be considered if Ω is a dihedral angle and only the later case must be considered if Ω is a cone. Let $\phi_x : V_x \rightarrow B^j \times (0, a) \times \omega_x$ be as in Definition 2. The family $\mathcal{F}_x := \{(\zeta_k \otimes \phi_n \otimes \xi_l) \circ \phi_x\}$ is then a partition of unity subordinated to V_x . By slightly decreasing the sets V_x , we can assume that all derivatives of ϕ_x are bounded on the closure of V_x .

Assume now that $\overline{\Omega}$ is compact (this is always the case unless Ω is an infinite cone or a dihedral angle), then $\partial^\nu \Omega$ can be covered by finitely many open sets of the form V_x . We fix such a covering corresponding to a finite set of points $x_j \in \partial^\nu \Omega$.

As in the case of a polygonal domain D discussed earlier, we complete the union of \mathcal{F}_{x_j} for all the chosen points $x_j \in \partial\Omega$ above, by adding functions in a finite partition of unity \mathcal{F}_0 on Ω^ν with sufficiently small support that are zero in the neighborhood of all edges. We denote \mathcal{F}_Ω the resulting family.

Lemma 12. *Assume that $\overline{\Omega}$ is compact. Then there exists a constant $\kappa_\Omega > 0$ such that no point $y \in \overline{\Omega}$ belongs to the support of more than κ_Ω of the functions $\psi \in \mathcal{F}_\Omega$.*

Proof. It is enough to prove the result for each of the sets \mathcal{F}_x , with $x \in \partial^\nu\Omega$ one of the finitely many points x_j . Assume x is a vertex and let κ_x be the constant bounding the multiplicity of the partition of unity associated to ω_x used in the construction of \mathcal{F}_x . Such a constant exists by Lemma 10. Then we can take κ (for V_x) to be $3\kappa_x$. The proof when x is an edge point is completely similar (in fact, slightly easier). \square

We again see that the family \mathcal{F}_Ω is locally finite and, hence, there is an associated partition of unity \mathcal{P}_Ω as recalled above (we can assume $g > 0$ by decreasing the supports of the functions in \mathcal{F}_0 , if necessary). We recall r_Ω be the canonical weight on Ω given in Equation (14).

Lemma 13. *There exists a constant $C > 0$ such that*

$$|r_\Omega^{|\alpha|} \partial^\alpha \psi(y)| \leq C \text{ for all } y \in \Omega \text{ and all } \psi \in \mathcal{F}_\Omega \text{ or } \psi \in \mathcal{P}_\Omega. \quad (20)$$

A point $y \in \Omega$ belongs to at most κ_Ω of the supports of the functions ψ in \mathcal{P}_Ω . Moreover, for any $\psi \in \mathcal{P}_\Omega$, the support of ψ intersects no edge, intersects at most one oriented face of Ω , and has diameter $\leq Cr_\Omega(y)$, for any y in the support of ψ and a constant C independent of ψ .

Proof. Again, we need only prove the estimate (20). In view of Lemma 12, it is enough to prove the statement for $\psi \in \mathcal{F}_\Omega$. Let $x \in \partial^\nu\Omega$ be one of the x_j . If x is not a vertex, the proof is the same as that of Lemma 11.

Assume then that x is a vertex. Let $\phi_x : V_x \rightarrow B^t \times B^{3-t}$ be the diffeomorphism of Definition 2 such that $V_x \cap \Omega \simeq (0, a) \times \omega_x$. Since all the derivatives of ϕ_x (for all x) are bounded on the closure of V_{x_j} , we can replace the function ψ with $\psi \circ \phi_x^{-1} = \zeta_k \otimes \phi_n \otimes \xi_l$. Moreover, on $W := (0, a) \times \omega_x$, the function $r_\Omega \circ \phi_x^{-1}$ and $\rho r_{\omega_x}(x')$ are equivalent (*i. e.*, their quotients are in $\mathcal{C}^\infty(\Sigma W)$). We can therefore replace the first function with the later and assume that $r(tx') = tr_D(x')$. We can also assume that Ω is the cone $\mathcal{C} := \{rx', r \in (0, \infty), x' \in \omega_x\}$.

The result then follows by proceeding as in the proof of Lemma 6, expressing $r\partial_x$, $r\partial_y$, and $r\partial_z$ in spherical coordinates if $y \in W_1$ or cylindrical coordinates if $y \in W_2$, using the notation introduced there. \square

5.3. Definition of Sobolev spaces using partitions of unity

As in [3], it is important to define the spaces $\mathcal{K}_m^a(\Omega)$ using partitions of unity. Similar constructions were used in [20, 82, 83, 89]. This construction is possible because the spaces $\mathcal{K}_{3/2}^m(\Omega)$ are the usual Sobolev spaces associated to the metric $r_\Omega^{-2}g_E$, where g_E is the Euclidean metric.

We will need to use the partition of unity \mathcal{P}_Ω defined in the previous subsection, as well as the diffeomorphisms χ_y defined in Corollary 1 for any $y \in \partial^\nu \Omega$.

Let us fix y and denote by D the oriented face of Ω containing y . We agree to rotate $T_y D$ to identify it with \mathbb{R}^2 when computing the Sobolev norms. The same applies to the half space \mathbb{H}_y defined by y (this half space has $T_y D$ as a boundary and y as an *inner* unit normal vector). These identifications are not unique, but differ by orthogonal transformations that do not change the Sobolev norms.

Let J be the set of indices j such that the support of ψ_j intersects $\partial^\nu \Omega$. Let us pick a point y_j in the support of ψ_j for each j . If $j \in J$, then we shall choose $y_j \in \partial^\nu \Omega$. We can assume that the supports of the functions ψ_j are small enough so that they are contained in the range W_{y_j} of the diffeomorphism χ_{y_j} of Corollary 1.

We let $x_j = \kappa(y_j)$. We shall denote $\theta_j := \vartheta(x_j)$ and define $\alpha_j(x) = \alpha_{x_j}(x) := \theta_j(x - x_j) + x_j$ the dilation of ratio θ_j and center x_j , where ϑ is the distance to the edges of Ω . Also, we let $\chi_j = \chi_{y_j}$ and $\mathbb{H}_j = \mathbb{H}_{y_j}$.

We are now in a position to characterize the spaces \mathcal{K}_a^m via partition of unity. (Note the half spaces \mathbb{H}_j below.) For $m \in \mathbb{Z}_+$, $a \in \mathbb{R}$, we set

$$\begin{aligned} \nu_{m,a}(u)^2 &:= \sum_j \theta_j^{3-2a} \|(\psi_j u) \circ \alpha_j\|_{H^m}^2 \\ &:= \sum_{j \notin J} \theta_j^{3-2a} \|(\psi_j u) \circ \alpha_j\|_{H^m(\mathbb{R}^3)}^2 + \sum_{j \in J} \theta_j^{3-2a} \|(\psi_j u) \circ \chi_j^{-1}\|_{H^m(\mathbb{H}_j)}^2, \end{aligned} \quad (21)$$

where we agree that $\|(\psi_j u) \circ \alpha_j\|_{H^m} = \infty$ if $(\psi_j u) \circ \alpha_j \notin H^m(\mathbb{R}^3)$ (or if $(\psi_j u) \circ \chi_j^{-1} \notin H^m(\mathbb{H}_j)$, respectively).

Proposition 3. *We have $u \in \mathcal{K}_a^m(\Omega)$, $m \in \mathbb{Z}$, if, and only if, $\nu_{m,a}(u) < \infty$. Moreover, $\nu_{m,a}(u)$ defines an equivalent norm on $\mathcal{K}_a^m(\Omega)$.*

For polyhedral domains, this characterization is well-known and the proof is standard (see [13, Lemma 2.4], [3, 4], or [89]); for $m < 0$ one also has to check that both definitions are compatible with duality. We sketch a proof for general domains with polyhedral structure.

We proceed in the same way to study the spaces $\mathcal{K}_a^s(\partial^\nu \Omega)$, $s, a \in \mathbb{R}$. Let again J be the set of indices j for which $x_j \in \partial \Omega$. We set

$$\mu_{s,a}(u)^2 := \sum_{j \in J} \theta_j^{2-2a} \|(\psi_j u) \circ \chi_j^{-1}\|_{H^s(\partial \mathbb{H}_j)}^2, \quad s \in \mathbb{R}_+. \quad (22)$$

Then we have an analogous description of the spaces $\mathcal{K}_a^s(\partial^\nu\Omega)$, $s \in \mathbb{Z}$. We have $u \in \mathcal{K}_a^s(\partial^\nu\Omega)$ if, and only if, $\mu_{s,a}(u) < \infty$. Moreover, $\mu_{s,a}(u)$ defines an equivalent norm on $\mathcal{K}_a^s(\partial^\nu\Omega)$, $s \in \mathbb{Z}$. Therefore, we define

$$\mathcal{K}_a^s(\partial^\nu\Omega) := \{u, \mu_{s,a}(u) < \infty\}, \quad s \in \mathbb{R}, \quad (23)$$

with the induced norm.

Recall that $\partial^\nu\Omega$ is the set of inward unit normal vectors. It is an at most two-to-one cover of the smooth part of $\partial\Omega$. Its connected components are called the *oriented faces* of Ω . Recall that the set $\Omega^\nu := \Omega \cup \partial^\nu\Omega$ is naturally endowed with the structure of a smooth manifold with boundary $\partial^\nu\Omega$. However, a smooth function on Ω^ν may not extend to a smooth function on $\overline{\Omega} \setminus \partial_{\text{sing}}\Omega$, because a smooth function on Ω^ν may have different limit values at the crack points, depending on the side from which we approach these points.

If $S \subset \partial\Omega$ is a union of oriented faces D of Ω , we then let

$$\mathcal{K}_a^m(S) := \oplus_{D \subset S} \mathcal{K}_a^m(D), \quad D \subset S, \quad D \text{ an oriented face.}$$

In particular, we let $\partial_N\Omega \subset \partial^\nu\Omega$ and $\partial_D\Omega \subset \partial^\nu\Omega$ be unions of *oriented faces* of Ω such that $\partial_N\Omega \cap \partial_D\Omega = \emptyset$. The proof of the next result is as in [3, 4].

Theorem 3. *The space $\mathcal{C}_c^\infty(\Omega^\nu)$ is dense in $\mathcal{K}_a^m(\Omega)$, $m \in \mathbb{Z}_+$. Then the restriction to $\partial_D\Omega$ extend to a continuous, surjective map*

$$\mathcal{K}_a^m(\Omega) \ni u \rightarrow u|_{\partial_D\Omega} \in \mathcal{K}_{a-1/2}^{m-1/2}(\partial_D\Omega)$$

for $m \geq 1$. The kernel of this map is the closure of $\mathcal{C}_c^\infty(\Omega^\nu \setminus \partial_D\Omega)$ in $\mathcal{K}_a^m(\Omega)$.

6. The boundary value problem

Let P be a second order operator on Ω and let L_k be a boundary operator on the face D_k of Ω , which is either the restriction to D_k (in the case of Dirichlet boundary conditions) or the conormal derivative (in the case of Neumann boundary conditions) D_ν^P defined for each k in Equation (27) below. We assume that P have smooth coefficients that extend to smooth functions in a neighborhood of $\overline{\Omega}$.

We will prove several regularity and well-posedness results for the boundary value problem

$$Pu = f \text{ in } \Omega, \quad L_k u = g_k \text{ on } D_k.$$

These results hold under different types of additional assumptions on P and on the boundary operators L_k . We begin this section by formulating these in increasing generality. As usual, we concentrate on domains $\Omega \subset \mathbb{R}^3$ with polyhedral structure.

We assume in this section $\overline{\Omega}$ is bounded. Some of these result extend to the more general case when $\overline{\Omega}$ is unbounded, but has a *finite* covering with neighborhoods of the form V_p (e.g. Ω is a cone).

6.1. The three types of assumptions on P and L_k : an informal statement

In addition to assuming that P is a uniformly strongly elliptic, second-order operator with smooth coefficients on $\overline{\Omega}$ and the boundary operator L_k is Dirichlet if the face $D_k \subset \partial_D \Omega$, or Neumann if $D_k \subset \partial_N \Omega$, we will consider increasingly stringent assumptions on P and L_k . We formulate these conditions informally first.

- The *regularity upon freezing the coefficients* assumption: at any point of ΣD_k the pair $(P; L_k)$ satisfies an H^{m+1} -regularity estimate on the half-space determined by that point;
- The *coercivity* assumption: the Dirichlet form B_P associated to P satisfies the following Gårding inequality

$$B_P(u, u) \geq C_1 \|u\|_{\mathcal{K}_1^1(\Omega)}^2 - C_2 \|r_\Omega^{-1} u\|_{L^2(\Omega)}^2 \quad (24)$$

for all u satisfying homogeneous Dirichlet boundary conditions on $\partial_D \Omega$ and some constants $C_1 > 0$, C_2 independent of u ;

- The *“positivity assumption”*: the operators P and L_k satisfy the same conditions as in the “coercivity assumption” and, in addition, the form B_P is positive-definite, that is, (24) holds with $C_2 = 0$.

In the first condition, L_k can be a boundary operator of order $k \leq 1$ not necessarily associated to the bilinear form B_P . It is well-known that the regularity upon freezing the coefficients is satisfied if $[P, L_k]$ satisfies the so-called Shapiro-Lopatinski condition or if $[P, L_k]$ is an elliptic system in the sense of Agmon-Douglis-Nirenberg [2]. In practice, it is difficult for systems, in particular for anisotropic elasticity and mixed boundary conditions, to check the Shapiro-Lopatinski condition. We therefore will use coercivity instead.

We will prove full regularity in the space $\mathcal{K}_a^m(\Omega)$, $m \geq 1$, $m \in \mathbb{Z}_+$, and any $a \in \mathbb{R}$, under the assumption of regularity upon freezing the coefficients. Then we will show that the coercivity assumption implies the regularity upon freezing the coefficients assumption. Clearly, the positivity assumption is then the strongest, and we will establish a well-posedness result under this assumption of positivity, provided $|a| < \eta$ for some small enough $\eta > 0$.

In the next subsection, we shall explain in detail these assumptions. We begin by fixing notation and by making some preliminary remarks.

6.2. The form of the operator

We shall consider a differential operator $P : \mathcal{C}_c^\infty(\Omega)^\mu \rightarrow \mathcal{C}_c^\infty(\Omega)^\mu$. Thus, for $\mu > 1$, we have a system. Let $P = [P_{pq}]$, $1 \leq p, q \leq \mu$, be the matrix notation for P , with

$$P_{pq} = - \sum_{i,j=1}^3 \partial_j a_{pq}^{ij} \partial_i + \sum_{i=1}^3 b_{pq}^i \partial_i + c_{pq}. \quad (25)$$

We assume that all the coefficients of the differential operators P_{pq} are smooth *real valued* functions in $x \in \overline{\Omega}$ (that is, these coefficients extend to smooth functions in a neighborhood of $\overline{\Omega}$). Our results also extend to some operators with complex coefficients, but this extension is not needed for the case of the elasticity operator, which our main interest. We also assume, for simplicity, that $a_{pq}^{ij} = a_{pq}^{ji}$. We shall denote by $B_P(u, v)$ the bilinear form:

$$B_P(u, v) := \int_{\Omega} a_{pq}^{ij} \partial_i u_p \partial_j u_q dx + \int_{\Omega} b_{pq}^i \partial_i u_p u_q dx + \int_{\Omega} c_{pq} u_p u_q dx, \quad (26)$$

$1 \leq p, q \leq \mu$, $1 \leq i, j \leq 3$, where Einstein's summation convention (summation over repeated indices), was used. We will also denote by D_{ν}^P the Neumann operator or conormal derivative associated to B_P :

$$(D_{\nu}^P u)_p := \nu_i a_{pq}^{ij} \partial_j u_q, \quad (27)$$

where $\nu = (\nu_i)$ is the *unit outer normal vector* to $\partial\Omega$. D_{ν}^P is well defined on each oriented face $D_k \subset \partial^{\nu}\Omega$ and extend with smooth coefficients to each ΣD_k . Then, P , B_P , and D_{ν}^P are related as usual by the following divergence formula.

Lemma 14. *We have $(Pu, v)_{L^2(\Omega)} = B_P(u, v) - (D_{\nu}^P u, v)_{L^2(\partial^{\nu}\Omega)}$ for any real valued $u \in H^2(\Omega)^{\mu}$ and $v \in H^1(\Omega)^{\mu}$.*

Proof. We partition Ω into a finite number of disjoint Lipschitz domains by Lemma 1, to which the usual Gauss-Green formula applies [65,97]. \square

A typical example is $P = -\Delta$, in which case $B_P(u, v) := \int_{\Omega} \nabla u \cdot \nabla v dx$ and $D_{\nu}^P u = \partial_{\nu} u$ (normal derivative).

6.3. The boundary conditions

We assume that, for any oriented face D_k of our polyhedral structure on Ω , we are given $m_k \in \{0, 1\}$. We then let

$$L_k : \mathcal{C}^{\infty}(\overline{\Omega})^{\mu} \rightarrow \mathcal{C}^{\infty}(D_k)^{\mu} \quad (28)$$

to be $L_k(u) = u|_{D_k}$, if $m_k = 0$, and $L_k(u) = D_{\nu}^P u$, if $m_k = 1$. For notational simplicity, we shall assume that $m_1 = \dots = m_r = 1$ and $m_{r+1} = \dots = m_N = 0$.

We shall denote by $\partial_N \Omega$ the union of the open oriented faces D_k for which $m_k = 1$ and by $\partial_D \Omega$ the union of the open faces D_k for which $m_k = 0$. Then we shall denote $\mathcal{K}_a^s(\partial_N \Omega) := \oplus_{m_k=1} \mathcal{K}_a^s(D_k)$ and, similarly, $\mathcal{K}_a^s(\partial_D \Omega) := \oplus_{m_k=0} \mathcal{K}_a^s(D_k)$.

The operator P and the boundary conditions L_k give rise, for any $m \geq 1$ and any $a \in \mathbb{R}$, to continuous, linear maps

$$\begin{aligned} [P; L_k]_m &: \mathcal{K}_{a+1}^{m+1}(\Omega)^\mu \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega)^\mu \oplus \left(\bigoplus_k \mathcal{K}_{a+1/2-m_k}^{m+1/2-m_k}(D_k)^\mu \right) \\ &=: \mathcal{K}_{a-1}^{m-1}(\Omega)^\mu \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N \Omega)^\mu \oplus \mathcal{K}_{a+1/2}^{m+1/2}(\partial_D \Omega)^\mu, \\ [P; L_k]_m &:= (Pu, L_1 u, \dots, L_N u) = (Pu, D_\nu^P u|_{\partial_N \Omega}, u|_{\partial_D \Omega}). \end{aligned} \quad (29)$$

In order to deal also with the weak formulation of our boundary value problems, we will need to consider the map $[P; L_k]_m$ for $m = 0$ as well. This is achieved as follows. Let

$$\mathcal{H}_a := \{u \in \mathcal{K}_{a+1}^1(\Omega)^\mu, u = 0 \text{ on } \partial_D \Omega\}. \quad (30)$$

The bilinear form B_P defines a continuous, linear map $B_P^* : \mathcal{K}_{a+1}^1(\Omega)^\mu \rightarrow \mathcal{H}_{-a}^*$ by the equation

$$\langle B_P^* u, v \rangle := B_P(u, v). \quad (31)$$

This observation allows us to define

$$\begin{aligned} [P; L_k]_0 &: \mathcal{K}_{a+1}^1(\Omega)^\mu \rightarrow \mathcal{H}_{-a}^* \oplus \mathcal{K}_{a+1/2}^{m+1/2}(\partial_D \Omega)^\mu, \\ [P; L_k]_0(u) &:= (B_P^*(u), u|_{\partial_D \Omega}), \quad a \in \mathbb{R}. \end{aligned} \quad (32)$$

Then $[P; L_k]_m$ extends $[P; L_k]_{m+1}$ for all $m \geq 0$. For $m \geq 1$ this fact is directly seen from the definition. For $m = 0$, the meaning in which $[P; L_k]_0$ extends $[P; L_k]_1$ is the following. Let us define

$$nat : \mathcal{K}_{a+1}^0(\Omega)^\mu \oplus \mathcal{K}_{a-1/2}^{-1/2}(\partial_N \Omega)^\mu \rightarrow \mathcal{H}_{-a}^*$$

by $nat(f, u)(v) = \int_\Omega f v dV + \int_{\partial \Omega} u v dS$ for any $v \in \mathcal{H}_{-a}$. The extension of nat to a map $\mathcal{K}_{a+1}^0(\Omega)^\mu \oplus \mathcal{K}_{a-1/2}^{-1/2}(\partial_N \Omega)^\mu \oplus \mathcal{K}_{a+1/2}^{1/2}(\partial_D \Omega)^\mu \rightarrow \mathcal{H}_{-a}^* \oplus \mathcal{K}_{a+1/2}^{1/2}(\partial_D \Omega)^\mu$ will still be denoted nat (instead of $nat \oplus id$). Then nat is continuous and Lemma 14 gives

$$nat \circ [P; L_k]_1 = [P; L_k]_0.$$

If $u \in \mathcal{K}_a^1(\Omega)$ is such that $B_P^*(u) = nat(f, g)$, then we shall write $Pu = f$, $D_\nu^P u = g$.

We shall write (P, L_k) when we refer to the pair consisting of P and a fixed L_k (for instance, in the regularity condition upon freezing the coefficients). On the other hand, we shall write $\{P; L_k\}$ when we refer to the family consisting of P and *all* boundary conditions L_k (for instance, in Equation (29) and in the following definition).

Definition 3. We say that $\{P; L_k\}$ satisfies a regularity estimate on $\mathcal{K}_{a+1}^{m+1}(\Omega)$ (or, simply, that $\{P; L_k\}$ is \mathcal{K}_{a+1}^{m+1} -regular), $m \geq 1$, if there exists $C > 0$ such that

$$\|u\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)} \leq C(\|Pu\|_{\mathcal{K}_{a-1}^{m-1}(\Omega)} + \sum_k \|L_k u\|_{\mathcal{K}_{a+1/2-m_k}^{m+1/2-m_k}(D_k)} + \|u\|_{\mathcal{K}_{a+1}^m(\Omega)}), \quad (33)$$

for all $u \in \mathcal{K}_{a+1}^1(\Omega)$. For $m = 0$, we require instead

$$\|u\|_{\mathcal{K}_{a+1}^1(\Omega)} \leq C(\|B_P^*(u)\|_{\mathcal{H}_{-a}^*} + \|u|_{\partial_D \Omega}\|_{\mathcal{K}_{a+1/2}^{1/2}(\partial_D \Omega)} + \|u\|_{\mathcal{K}_{a+1}^0(\Omega)}). \quad (34)$$

Remark 3. This definition shall be understood in the sense that the undefined terms are taken to be ∞ . More precisely, assume that $u \in \mathcal{K}_{a+1}^1(\Omega)$, $Pu \in \mathcal{K}_{a-1}^{m-1}(\Omega)$, and $L_k u \in \mathcal{K}_{a+1/2-m_k}^{m+1/2-m_k}(D_k)$ for each oriented face D_k of Ω . Our definition then states that $u \in \mathcal{K}_{a+1}^{m+1}(\Omega)$ and satisfies Equation (33). All the relations similar to Equation (33) will be understood in this sense in what follows.

Remark 4. Suppose that $\{P; L_k\}$ satisfies a regularity estimate on $\mathcal{K}_{a+1}^{j+1}(\Omega)$, $0 \leq j \leq m$, then Equation (33) together with its analogue for $m = 0$ and the continuity of the map *nat* show, by induction, that there exists $C > 0$ such that

$$\|u\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)} \leq C(\|Pu\|_{\mathcal{K}_{a-1}^{m-1}(\Omega)} + \sum_k \|L_k u\|_{\mathcal{K}_{a+1/2-m_k}^{m+1/2-m_k}(D_k)} + \|u\|_{\mathcal{K}_{a+1}^0(\Omega)}),$$

for all $u \in \mathcal{K}_{a+1}^1(\Omega)$ (the last term was replaced with a weaker one).

If Ω is a smooth domain and the family L_k reduces to a single boundary condition L (with only Dirichlet or only Neumann boundary conditions), we shall simply say that $(P, L) = (P, L_k)$ satisfies a regularity estimate on $H^{m+1}(\Omega)$ if $\{P; L_k\}$ satisfies a regularity estimate on $\mathcal{K}_a^{m+1}(\Omega) = H^{m+1}(\Omega)$. (In this case the weight is 1, so the spaces $\mathcal{K}_a^{m+1}(\Omega)$ are independent of a .) Then (P, L) satisfies a regularity estimate on $H^{m+1}(\Omega)$ if, and only if, there exists a constant $C > 0$ such that

$$\|u\|_{H^{m+1}(\Omega)} \leq C(\|Pu\|_{H^{m-1}(\Omega)} + \|Lu\|_{H^{m+1/2-m_L}(\partial^\nu \Omega)} + \|u\|_{H^m(\Omega)}), \quad (35)$$

for all $u \in H^{m+1}(\Omega)$ [59, 86]. In this paper, this condition will be used for $\Omega = \mathbb{R}_+^3$ with the usual (Euclidean) metric. (The term ‘‘coercive estimate’’ is often used instead of our term, ‘‘regularity estimate.’’ We feel however that the term ‘‘coercive’’ is over used, and hence we prefer ‘‘regularity estimate’’ for our paper.)

Definition 3 and Proposition 2 allow us to make the following simple remark.

Remark 5. The condition that $\{P, L_k\}$ satisfy a regularity estimate on $\mathcal{K}_{a+1}^{m+1}(\Omega)$ is independent of perturbations of P of the form $r_\Omega^{2-k}Q$, with Q a differential operator with smooth coefficients of order $k = 0, 1$. In particular, the condition that $\{P, L_k\}$ satisfy a regularity estimate on $\mathcal{K}_{a+1}^{m+1}(\Omega)$ is independent of a .

6.4. *The assumption of “regularity upon freezing the coefficients”*

Recall that $\partial^\nu \Omega$, the oriented boundary of Ω , is the set of *inner* normal unit vectors to the smooth part of the boundary of Ω . In particular, we have the natural map $\kappa : \partial^\nu \Omega \rightarrow \partial\Omega$ of Equation (10), which assigns to a normal unit vector y its starting point. This map is one-to-one at the points where Ω is on only one side of the boundary, and two-to-one otherwise. The identity map $\nu : \partial^\nu \Omega \rightarrow S^2$, $\nu(y) = -y$, extends to a smooth map $\nu : \Sigma D \rightarrow \mathbb{R}^3$ for any oriented face D of Ω . The map κ also extends to the continuous map $\kappa : \Sigma D \rightarrow \overline{D} \subset \overline{\Omega}$ defined in Equation (9).

The extension of the map κ allows us to associate to each $y \in \Sigma D_k$ the half-space $\mathbb{H}_y \subset \mathbb{R}^3$ with outer unit normal vector $\nu(y) = -y$ and the Neumann operator $D_{\nu(y)}^P$ by the formula (27). We then obtain the boundary differential operator

$$L_{ky} : H^{m+1}(\mathbb{H}_y) \rightarrow H^{m-1/2-m_k}(\partial\mathbb{H}_y),$$

$L_{ky}u = u|_{\partial\mathbb{H}_y}$, if $m_k = 0$, and $L_{ky}u = D_y^P = D_{\nu(y)}^P u$ if $m_k = 1$. The norms on the Sobolev spaces on \mathbb{H}_y and on $\partial\mathbb{H}_y$ are determined by the Euclidean structure on $\mathbb{R}^3 \supset \mathbb{H}_y$. Let $\kappa : \Sigma D_k \rightarrow \overline{D}$ be the canonical map as above and $x = \kappa(y)$. By freezing the coefficients of P at x and dropping the lower order terms, we obtain the operators $P_y = P_x$, $x = \kappa(y)$, for any $y \in \Sigma D_k$. (Unlike \mathbb{H}_y and L_{ky} , the operators P_y depend only on $x = \kappa(y)$.)

Recall that P is called *uniformly elliptic* if there exists a constant $C_P > 0$ such that

$$\|(\zeta_p)\| \geq C_P \|(\eta_q)\|, \quad (36)$$

where $\zeta_p := a_{pq}^{ij}(x)\xi_i\xi_j\eta_p$, $x \in \overline{\Omega}$, $\|\cdot\|$ is the norm on \mathbb{R}^μ , (ξ_i) is an arbitrary vector in \mathbb{R}^3 and (η_p) is an arbitrary vector in \mathbb{R}^μ . Similarly, P is called *uniformly strongly elliptic* if there exists a constant $C_P > 0$ such that

$$a_{pq}^{ij}(x)\xi_i\xi_j\eta_q\eta_p \geq C_P \sum_q \eta_q^2, \quad (37)$$

where $x \in \overline{\Omega}$, (ξ_i) is an arbitrary vector in \mathbb{R}^3 and (η_p) is an arbitrary vector in \mathbb{R}^μ .

Definition 4. Let Ω be a domain with a polyhedral structure. We say that $\{P; L_k\}$ satisfies the *regularity assumption upon freezing the coefficients* if P is uniformly elliptic and, for any $m \in \mathbb{Z}_+$, any k , and any $y \in \Sigma D_k$, the pair

(P_y, L_{ky}) satisfies a $H^{m+1}(\mathbb{H}_y)$ -regularity estimate, that is, there is $C_y > 0$ such that

$$\|v\|_{H^{m+1}(\mathbb{H}_y)} \leq C_y \left(\|P_y v\|_{H^{m-1}(\mathbb{H}_y)} + \|(L_{ky}u)v\|_{H^{m+1/2-m_k}(\partial\mathbb{H}_y)} + \|v\|_{H^m(\mathbb{H}_y)} \right), \quad (38)$$

for all $v \in H^1(\mathbb{H}_y)$.

Recall the diffeomorphism $\chi_y : B_y^2(r) \times [0, r) \rightarrow W_y \subset \Omega^\nu := \Omega \cup \partial^\nu \Omega$ of Corollary 1, $B_y^2(r) \times [0, r) \subset \mathbb{H}_y$, the closed half-space determined by the vector $\nu(y) = -y$. We define the Sobolev spaces on \mathbb{H}_y and $T_y \partial^\nu \Omega = \partial\mathbb{H}_y$ using the Euclidean structure (metric) on these spaces.

The following lemma is crucial for proving regularity in \mathcal{K}_a^m -spaces.

Lemma 15. *Assume that $\overline{\Omega}$ is compact and that the family $\{P; L_k\}$ satisfies the regularity assumption upon freezing the coefficients. Then there exists $C_P > 0$ independent of y such that*

$$\|u \circ \chi_y\|_{H^{m+1}(\mathbb{H}_y)} \leq C_P \left(r_\Omega(y)^2 \|(Pu) \circ \chi_y\|_{H^{m-1}(\mathbb{H}_y)} + r_\Omega(y)^{m_k} \|(L_k u) \circ \chi_y\|_{H^{m+1/2-m_k}(\partial\mathbb{H}_y)} + \|u \circ \chi_y\|_{H^m(\mathbb{H}_y)} \right),$$

for any $y \in D_k$ and any u with support in $W_y := \chi_y(B_y^2(r) \times [0, r))$.

Proof. The diffeomorphism $\chi_y : B_y^2(r) \times [0, r) \rightarrow W_y \subset \Omega^\nu$ of Corollary 1 allows us to define a second-order differential operator Q_y on $B_y^2(0; r) \times (0, r)$ and the boundary condition M_y by the formulas

$$Q_y(u \circ \chi_y) := r_\Omega(x)^2 (Pu) \circ \chi_y \quad \text{and} \quad M_y(u \circ \chi_y) := r_\Omega(y)^{m_k} (L_k u) \circ \chi_y,$$

where $x = \kappa(y)$.

The family Q_y depends continuously on $y \in \Sigma D_k$. Let $\xi^2 + \zeta^2 = 1$ be two real valued smooth functions with compact support and $\tilde{Q}_y := \xi Q_y \xi + \zeta \Delta \zeta$, which is defined if $\xi = 0$ on the complement of $B^2(0; r) \times (0, r)$. We can also assume that $\xi = 1$ on $B^2(0; r') \times (0, r')$, for some $0 < r' < r$. Changing coordinates preserves regularity, and hence

$$\|v\|_{H^{m+1}(\mathbb{H}_y)} \leq C_y \left(\|\tilde{Q}_y v\|_{H^{m+1}(\mathbb{H}_y)} + \|M_y v\|_{H^{m+1/2-m_k}(\partial\mathbb{H}_y)} + \|v\|_{H^m(\mathbb{H}_y)} \right).$$

The continuity of the family (\tilde{Q}_y, M_y) shows that C_y is lower semi-continuous. Since ΣD_k is compact, it follows that C_y is bounded from below by a constant $C > 0$. The result then follows since $\tilde{Q}_y(u \circ \chi_y) = r_\Omega(x)^2 (Pu) \circ \chi_y$ and $M_y(u \circ \chi_y) = r_\Omega(y)^{m_k} (L_k u) \circ \chi_y$ for u with support in $\chi_y(B_0^2(r') \times (0, r'))$. \square

6.5. Coercivity and positivity

We continue to assume that $\partial^\nu \Omega = \partial_N \Omega \cup \partial_D \Omega$, where $\partial_N \Omega$ and $\partial_D \Omega$ are unions of oriented open faces of Ω . (Recall that $\partial^\nu \Omega = \partial \Omega \setminus \partial_{\text{sing}} \Omega$ if Ω has no cracks.) Furthermore, we assume that $\partial_D \Omega \neq \emptyset$. We set $Lu = u = 0$ on $\partial_D \Omega$ and $Lu = D_\nu^P u = 0$ on $\partial_N \Omega$.

Definition 5. We say that the operator P is *coercive* if there exist positive constants C_1, C_2 such that

$$B_P(u, u) \geq C_1 \|u\|_{\mathcal{K}_1^1(\Omega)}^2 - C_2 \|r_\Omega^{-1} u\|_{L^2(\Omega)}^2, \quad u \in \mathcal{K}_1^1(\Omega)^\mu, \quad u = 0 \quad \text{on} \quad \partial_D \Omega,$$

where B_P is the bilinear form of equation (26).

Note that our definition of coercive operators is weaker than the usual definition: $B_P(u, u) \geq C_1 \|u\|_{H^1(\Omega)}^2 - C_2 \|u\|_{L^2(\Omega)}^2$, $u = 0$ on $\partial_D \Omega$, because of the factor r_Ω^{-1} .

We have the following basic result. Recall that for any oriented face D of Ω , ΣD denotes the desingularization of that face.

Proposition 4. *Assume P is coercive. Then P satisfies the assumption of regularity upon freezing coefficients with a constant C_y independent of $y \in \Sigma D$.*

Proof. Let us fix an arbitrary oriented face D_k of Ω and let $y \in \Sigma D_k$. Let also (P_y, L_{ky}) be as in the assumption of regularity upon freezing the coefficients (Definition 4). Assume first that $y \in D_k$ and let $\chi_y : B_x^2(r) \times [0, r) \rightarrow \Omega^\nu := \Omega \cup \partial^\nu \Omega$ be the diffeomorphism defined in Corollary 1. Let $\phi \in \mathcal{C}_c^\infty(\mathbb{H}_y)$ and $\phi_t(z) = t^{3/2} \phi(tz)$.

We use Proposition 3, to conclude that P satisfies regularity upon freezing coefficients at x for $m = 0$ first. Since we are on a half-space, the classical Nirenberg argument (see e.g. [37, 73, 77, 86]) shows that regularity upon freezing the coefficients can be bootstrapped so it holds for any $m > 0$. \square

See [80] for an application of Nirenberg's argument to non-linear problems.

Remark 6. If P is second order, strongly elliptic operator and the L_k are the restrictions to the boundary (*i. e.*, Dirichlet boundary conditions), then P satisfies a Gårding type inequality, and therefore regularity upon freezing the coefficients [77, 85, 92].

Other examples of coercive operators are the Laplace operator Δ and the elasticity operator $\text{div}(\mathbf{C} \cdot \nabla)$ with mixed boundary conditions (with \mathbf{C} positive definite).

A stronger condition than coercivity will be needed in general to establish well-posedness.

Definition 6. We say that the operator P is *strictly positive* if the bilinear form B_p is positive-definite on \mathcal{K}_1^1 , that is, there is a $\gamma > 0$ such that

$$B_P(u, u) \geq \gamma \|u\|_{\mathcal{K}_1^1(\Omega)}^2.$$

For example, a weighted form of Poincaré inequality, Theorem 6 below, will give that the Laplace operator is positive in the sense of the above definition with Dirichlet or Neumann boundary conditions provided that no two adjacent faces are endowed with Neumann boundary conditions and $\partial_D \Omega \neq \emptyset$. Moreover, if the elasticity tensor $\mathbf{C} = (C_{pq}^{ij})$ is positive definite on symmetric matrices, then the elasticity operator $\operatorname{div}(\mathbf{C} \cdot \nabla)$ is positive under the same conditions on the boundary. This result follows by an application of Korn's inequality, which will be discussed later (in Section 10.2).

Some authors use the term “strictly coercive” instead of “strictly positive.” Again, we feel that the term “coercive” is abused and often used with different meaning, so we preferred “strictly positive.”

7. Proof of the regularity theorem

We include in this section the proof of Theorem 1. Its proof is reduced to the Euclidean case using the partition of unity \mathcal{P}_Ω of Lemma 13 and the diffeomorphisms χ_k of Corollary 1.

We continue to assume that P is a second order differential operator as in Equation (25) and that L_k are either Dirichlet or Neumann boundary conditions. We also continue to assume that $\partial^\nu \Omega$ is written as a disjoint union $\partial^\nu \Omega = \partial_N \Omega \cup \partial_D \Omega$, with $\partial_N \Omega$ and $\partial_D \Omega$ union of oriented faces of Ω .

We begin with a preliminary result. Let $x \in \Omega$ and α_x be the dilation with center x and ration r_Ω , as defined in Equation (15).

Lemma 16. *Assume that $\overline{\Omega}$ is compact, and that the operator P is uniformly elliptic. Then there exists a constant $C > 0$ such that*

$$\|u \circ \alpha_x\|_{H^{m+1}(\mathbb{R}^3)} \leq C(r_\Omega^2 \|(Pu) \circ \alpha_x\|_{H^{m-1}(\mathbb{R}^3)} + \|u \circ \alpha_x\|_{H^m(\mathbb{R}^3)}),$$

for any $x \in \Omega$ and any $u \in C_c^\infty(B^3(x, \vartheta(x)) \cap \Omega)$, where $B^3(x, \vartheta(x)) \subset \mathbb{R}^3$ is the ball of radius $\vartheta(x)$ and center x .

Proof. We have that $(Pu) \circ \alpha_x = r_\Omega(x)^{-2} Q_x(u \circ \alpha_x)$, for

$$(Q_x)_{p,q} = - \sum_{i,k=1}^3 (a_{pq}^{ij} \circ \alpha_x) \partial_j \partial_i, \quad 1 \leq p, q, \leq \mu. \quad (39)$$

Therefore standard elliptic regularity applied to the operator Q_x shows that there exists $C_x > 0$ such that

$$\|u \circ \alpha_x\|_{H^{m+1}(\mathbb{R}^3)} \leq C_x (r_\Omega(x)^2 \|(Pu) \circ \alpha_x\|_{H^{m+1}(\mathbb{R}^3)} + \|u \circ \alpha_x\|_{H^m(\mathbb{R}_+^3)}).$$

Let us choose for each x the least C_x with this property. We only need to show that we can choose C_x independent of x .

Since $\partial_l(a_{pq}^{ik} \circ \alpha_x) = r_\Omega(x)(\partial_l a_{pq}^{ik}) \circ \alpha_x$, the family of functions $\{a_{pq}^{ik} \circ \alpha_x\}$ depends continuously on $x \in \overline{\Omega}$ in $\mathcal{C}^\infty(B^2)$. Hence the family Q_x of differential operators on $\mathcal{C}_c^\infty(B^2)^\mu$ depends continuously on $x \in \overline{\Omega}$. Since the function $x \rightarrow C_x$ is upper semi-continuous and $\overline{\Omega}$ is compact, it follows that the function C_x is bounded. This completes the proof. \square

We are now ready to state our main regularity result. Some applications of this result for the Laplace operator with mixed boundary conditions were given in [15]. By contrast, it is known that in the framework of the usual Sobolev spaces $H^m(\Omega)$, the smoothness of the solution of (1) is bounded [29, 41, 43, 47, 46, 70–72]. Our strategy for avoiding the loss of regularity is to “geometrically remove the singular points.” More concretely, we use a conformal change of metric that has the effect of “sending the singular points to infinity.” This procedure was used in other contexts by several authors [45, 64, 67, 66].

Theorem 4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a polyhedral structure. Assume that the family $\{P, L_k\}$ satisfies the regularity assumption upon freezing the coefficients. Let $m \in \mathbb{Z}$, $m \geq 1$, and $a \in \mathbb{R}$. Assume that $u \in \mathcal{K}_{a+1}^1(\Omega)^\mu$. Then there exists $C > 0$ such that*

$$\|u\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)} \leq C(\|Pu\|_{\mathcal{K}_{a-1}^{m-1}(\Omega)} + \sum_k \|L_k u\|_{\mathcal{K}_{a+1/2-m_k}^{m+1/2-m_k}(D_k)} + \|u\|_{\mathcal{K}_{a+1}^m(\Omega)}).$$

In particular, if $u \in \mathcal{K}_{a+1}^1(\Omega)^\mu$, $Pu \in \mathcal{K}_{a-1}^{m-1}(\Omega)^\mu$, and for all oriented faces D_k of Ω , $L_k u|_{\partial D \Omega} \in \mathcal{K}_{a+1/2-m_k}^{m+1/2-m_k}(D_k)^\mu$, then $u \in \mathcal{K}_{a+1}^{m+1}(\Omega)^\mu$.

Proof. By Remark 5, it is enough to assume that P is homogeneous of degree 2:

$$P = - \sum_{i,j=1}^3 a_{pq}^{ij} \partial_j \partial_i, \quad 1 \leq p, q \leq \mu.$$

(The boundary conditions are already homogeneous.) We shall extend the coefficients of P to the whole plane to satisfy the same estimates uniformly on \mathbb{R}^3 . This extension is possible because the coefficients of P are assumed defined and smooth on a neighborhood of Ω . Furthermore, since the constant C_y in Equation (38) is upper semi-continuous in y , we can replace C_y by C uniform if Ω is bounded.

Recall the partition of unity $\mathcal{P}_\Omega = \{\psi_j\}$ of Lemma 13. Also, recall that we have fixed x_j in the support of each ψ_j , with x_j in the boundary if possible. Let $\theta_j = \vartheta(x_j)$, with ϑ the distance to the set of edges. We denote again by α_j the dilation $\theta_j x + x_j$, and by $\chi_j = \chi_{y_j}$, $x_j = \kappa(y_j)$ the boundary diffeomorphism of Corollary 1, whenever $x_j \in \partial^\nu \Omega$. By decreasing the supports of the initial choices defining the partition of unity \mathcal{P}_Ω , we can assume that the functions $u_j = (\psi_j u) \circ \alpha_j$, if $x_j \in \Omega$, or $(\psi_j u) \circ \chi_j$, if

$x_j \in \partial^\nu \Omega$, are all supported in the fixed ball $B^3 = B(0; 1)$ (of radius 1 and center the origin). We implicitly assume that the half space \mathbb{H}_j has been rotated and translated to agree with \mathbb{R}_+^3 .

We next bound the commutator $[P, \psi_j]$ using Lemma 13. We have $[P, \psi_j] = Q_j$, where Q_j is a differential operator of order 1 with coefficients that depend on the first and second derivatives of ψ_j , so that $\theta_j^2 \alpha_j Q_j \alpha_j^{-1} : H^m(B^3) \rightarrow H^{m-1}(B^3)$ is uniformly bounded in j . (Here, $\alpha_j Q_j \alpha_j^{-1}(v) = [Q_j(v \circ \alpha_j^{-1})] \circ \alpha_j$.) Consequently, for $m \in \mathbb{Z}_+$,

$$\|([P, \psi_j] u) \circ \alpha_j\|_{H^{m-1}(B^3)} = \|(\alpha_j Q_j \alpha_j^{-1}) u_j\|_{H^{m-1}(B^3)} \leq C \theta_j^{-2} \|u_j\|_{H^m(B^3)}, \quad (40)$$

with C independent of j , since $\{\psi_j \circ \alpha_j\}$ is uniformly bounded in $C_c^\infty(B^2)$ again by Lemma 13. A similar estimate holds for $\chi_j Q_j \chi_j^{-1}$, using also Corollary 1.

If the support of ψ_j does not intersect the boundary of Ω , we conclude using also Lemma 16 that

$$\begin{aligned} \theta_j^{3/2-a-1} \|(\psi_j u) \circ \alpha_j\|_{H^{m+1}(B^3)} &\leq C \theta_j^{3/2-a-1} \left(\theta_j^2 \| (P \psi_j u) \circ \alpha_j \|_{H^{m-1}(B^3)} \right. \\ &\quad \left. + \theta_j^2 \|([\psi_j, P] u) \circ \alpha_j\|_{H^{m-1}(B^3)} + \|u \circ \alpha_j\|_{H^m(B^3)} \right) \\ &\leq C \theta_j^{3/2-a-1} \left(\theta_j^2 \|(\psi_j P u) \circ \alpha_j\|_{H^{m-1}(B^3)} + \|u \circ \alpha_j\|_{H^m(B^3)} \right) \\ &\leq C \theta_j^{3/2-a-1} \left(\theta_j^2 \|(\psi_j P u) \circ \alpha_j\|_{H^{m-1}(B^3)} + \|u \circ \alpha_j\|_{H^m(B^3)} \right) \\ &\leq C \left(\theta_j^{3/2-(a-1)} \|(\psi_j P u) \circ \alpha_j\|_{H^{m-1}(B^3)} + \theta_j^{3/2-a-1} \|u \circ \alpha_j\|_{H^m(B^3)} \right) \end{aligned}$$

with constants C independent of j . By squaring and using $(a+b)^2 \leq 2(a^2 + b^2)$ we obtain

$$\begin{aligned} \theta_j^{1-2a} \|(\psi_j u) \circ \alpha_j\|_{H^{m+1}(B^3)}^2 &\leq C \left(\theta_j^{3-2(a-1)} \|(\psi_j P u) \circ \alpha_j\|_{H^{m-1}(B^3)}^2 + \theta_j^{1-2a} \|u \circ \alpha_j\|_{H^m(B^3)}^2 \right). \quad (41) \end{aligned}$$

On the other hand, if the support of ψ_j does intersect the boundary of Ω , we can estimate the commutators of ψ_j with L_k , similarly to (40), to obtain for $m_k = 1$

$$\theta^{m_k} \|([L_k, \psi_j] u) \circ \chi_y\|_{H^{m+1/2-m_k}(\mathbb{R}^2)} \leq \|u \circ \chi_j\|_{H^m(\mathbb{R}_+^3)}.$$

(For $m_k = 0$ the commutator vanishes, so the analogous equation is trivially true.)

Lemma 15 gives

$$\begin{aligned}
\theta_j^{3/2-a-1} \|(\psi_j u) \circ \chi_j\|_{H^{m+1}(\mathbb{R}_+^3)} &\leq C \theta_j^{3/2-a-1} (\theta_j^2 \| (P \psi_j u) \circ \chi_j \|_{H^{m-1}(\mathbb{R}_+^3)} + \\
&\quad \theta^{m_k} \| (L_k \psi_j u) \circ \chi_j \|_{H^{m+1/2-m_k}(\mathbb{R}^2)} + \|u \circ \chi_j\|_{H^m(\mathbb{R}_+^3)}) \\
&\leq C \theta_j^{3/2-a-1} (\theta_j^2 \|(\psi_j P u) \circ \chi_j\|_{H^{m-1}(\mathbb{R}_+^3)} + \theta_j^2 \|([P, \psi_j] u) \circ \chi_j\|_{H^{m-1}(\mathbb{R}_+^3)} \\
&\quad + \theta^{m_k} \|(\psi_j L_k u) \circ \chi_j\|_{H^{m+1/2-m_k}(\mathbb{R}^2)} + \theta^{m_k} \|([L_k, \psi_j] u) \circ \chi_j\|_{H^{m+1/2-m_k}(\mathbb{R}^2)} \\
&\quad + \|u \circ \chi_j\|_{H^m(\mathbb{R}_+^3)}) \\
&\leq C \theta_j^{3/2-a-1} (\theta_j^2 \|(\psi_j P u) \circ \chi_j\|_{H^{m-1}(\mathbb{R}_+^3)} + \theta^{m_k} \|(\psi_j L_k u) \circ \chi_j\|_{H^{m+1/2-m_k}(\mathbb{R}^2)} \\
&\quad + \|u \circ \chi_j\|_{H^m(\mathbb{R}_+^3)}) \quad (42)
\end{aligned}$$

We now observe that $r_\Omega(x)$ is comparable to θ_j for x in the support of ψ_j , so that combining (41) with (42) (after we similarly square that relation), we conclude by Proposition 3 that

$$\|u\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)} \leq C \left(\|P u\|_{\mathcal{K}_{a-1}^{m-1}(\Omega)} + \sum_k \|L_k u\|_{\mathcal{K}_{a-1}^{m+1/2-m_k}(D_k)} + \|u\|_{\mathcal{K}_{a+1}^m(\Omega)} \right).$$

for any $u \in \mathcal{K}_{a+1}^1(\Omega)$ and a constant C that depends on m , a , and Ω . \square

Remark 7. The above regularity estimate holds for $m = 0$ provided the operator and boundary conditions are defined by (32). Moreover, the theorem could be improved in the following direction. Sometimes a vertex Q touches a smooth part of the boundary, like in the left picture of Figure 1. In that case, we could take the canonical weight (r_D or r_Ω) to be $O(1)$ on the smooth side of the boundary near Q . This additional generality, to be treated in a forthcoming paper [58], would make our presentation significantly more complicated with only minimum gain.

Proposition 4 and Theorem 4 give right away the following corollary.

Corollary 2. *Assume P is coercive and Ω is compact, then P satisfies $\mathcal{K}_a^m(\Omega)$ regularity for all $m \geq 1$ and $a \in \mathbb{R}$.*

8. Well-posedness for strictly positive operators

We continue to assume that $\Omega \subset \mathbb{R}^3$ is a fixed domain with a polyhedral structure, and that its oriented boundary $\partial^\nu \Omega$ is decomposed into a disjoint union $\partial^\nu \Omega = \partial_N \Omega \cup \partial_D \Omega$, with $\partial_N \Omega$ and $\partial_D \Omega$ unions of oriented faces of Ω such that $\partial_D \Omega \neq \emptyset$. In this section, we shall further assume that P is strictly positive (see Definition 6).

We will consider for each non-negative integer m the equation $[P; L_k]_m(u) = (f, g_N, g_D)$. For $m \geq 1$, this equation is simply the boundary value problem

$$\begin{cases} Pu = f \in \mathcal{K}_{a-1}^{m-1}(\Omega) & \text{in } \Omega \\ u = g_D \in \mathcal{K}_{a+1/2}^{m+1/2}(\partial_D \Omega) & \text{on } \partial_D \Omega \\ D_\nu^P u = g_N \in \mathcal{K}_{a-1/2}^{m-1/2}(\partial_D \Omega) & \text{on } \partial_N \Omega. \end{cases} \quad (43)$$

Recall that for $m = 0$, $[P; L_k]_m(u)$ is defined in a weak sense. We have the following well-posedness result.

Theorem 5. *Let $m \in \mathbb{Z}_+$ and let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a polyhedral structure. Assume P is strictly positive. Then there exists $\eta > 0$ such that the boundary value problem (43) has a unique solution $u \in \mathcal{K}_{a+1}^{m+1}(\Omega)^\mu$ for any $f \in \mathcal{K}_{a-1}^{m-1}(\Omega)^\mu$, any $g_D \in \mathcal{K}_{a+1/2}^{m+1/2}(\partial_D \Omega)^\mu$, any $g_N \in \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N \Omega)^\mu$, and any $|a| < \eta$. This solution depends continuously on f , g_D , and g_N .*

If $m = a = 0$ and $g_D = 0$, this solutions is the solution of the associated variational problem, which is obtained from equation (31).

Proof. First, let us notice that the surjectivity of the trace map (Theorem 3) allows us to reduce the proof to the case when $g_D = 0$.

Recall that $\mathcal{H}_a := \{u \in \mathcal{K}_{a+1}^1(\Omega), u = 0 \text{ on } \partial_D \Omega\}$. Thus we will look for solutions $u \in \mathcal{H}_a$. The maps $[P; L_k]_m$ of Equations (29) and (32) restrict to maps $\mathcal{K}_{a+1}^{m+1}(\Omega)^\mu \cap \mathcal{H}_a \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega)^\mu \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N \Omega)$, respectively $\mathcal{H}_a \rightarrow \mathcal{H}_{-a}^*$ for $m = 0$. We shall denote all these maps by $\tilde{P}_{m,a}$.

Our assumption that P is strictly positive implies that there exists $\gamma > 0$ such that

$$(Pu, u) = B_P(u, u) \geq \gamma \|u\|_{\mathcal{K}_1^1(\Omega)}^2$$

for any $u \in \mathcal{H}_0$. In particular, B_P satisfies the assumptions of the Lax-Milgram lemma, and hence $\tilde{P}_{0,0} = B_P^* : \mathcal{H}_0 \rightarrow \mathcal{H}_0^*$ is an isomorphism. The result for $m = 0$ and $a = 0$ is proved.

Recall the function r_Ω introduced in 14. The operators $r_\Omega^{-a} \tilde{P}_{m,a} r_\Omega^a$ will all act on the same space and by Proposition 2, these operators depend continuously on a . Let $m = a = 0$. Then $r_\Omega^{-a} \tilde{P}_{m,a} r_\Omega^a = P_{0,0}$ is an isomorphism, and hence we can conclude that there exists $\eta > 0$ such that $\tilde{P}_{0,a}$ is an isomorphism for any $|a| < \eta$.

We now prove the result for $|a| < \eta$ and $m \in \mathbb{Z}_+$ arbitrary. Corollary 2 then gives that

$$\tilde{P} : \mathcal{K}_{a+1}^{m+2}(\Omega)^\mu \cap \mathcal{H}_a \rightarrow \mathcal{K}_{a-1}^m(\Omega)^\mu$$

is surjective. Since this map is also continuous (Proposition 2) and injective (from the case $m = 0$), it is an isomorphism by the open mapping theorem. The proof is now complete. \square

9. Poincaré 's inequality and applications to the Laplace operator

In this section we will establish a weighted Poincaré inequality from which we will obtain solvability of some mixed boundary value problems for the Laplace operator Δ . The assumptions on Ω and its boundary of the previous section continue to hold.

Precisely, we will show that $P = \alpha^2/r_\Omega(x)^2 - \Delta$, $\alpha > 0$ large enough, with arbitrary mixed (Dirichlet or Neumann) boundary conditions, or $P = -\Delta$ in the case that $\partial_N\Omega$ contains no adjacent faces, satisfy the conditions of Theorem 5. The fact that P is strictly positive will be a consequence of a weighted form of Poincaré inequality.

9.1. Poincaré inequality

We begin with some preliminaries lemmas that settle the case of domains with polygonal structure.

With a little abuse of notation, we shall write $u(r, \theta) := u(r \cos \theta, r \sin \theta)$ for a function $u(x_1, x_2)$ expressed in polar coordinates. Below, $dx = dx_1 dx_2 \dots dx_n$, $n = 2$ or 3 . The following two lemmas are standard. See [15] or [74], for example.

Lemma 17. *Let $\mathcal{C} = \mathcal{C}_{R,\alpha} := \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2, 0 < r < R, 0 < \theta < \alpha\}$, $0 < \alpha \geq 2\pi$. Then*

$$\int_{\mathcal{C}} \frac{|u|^2}{r^2} dx \leq \left(\frac{2\alpha}{\pi}\right)^2 \int_{\mathcal{C}} \frac{|\partial_\theta u|^2}{r^2} dx \leq \left(\frac{2\alpha}{\pi}\right)^2 \int_{\mathcal{C}} |\nabla u|^2 dx$$

for any u , $\nabla u \in L^2(\mathcal{C})$ satisfying $u(r, \theta) = 0$ if $\theta = 0$, in the trace sense.

Observe that no condition is imposed on $u(r, \alpha)$ in the above lemma and that we allow $\alpha = 2\pi$, but we distinguish the limit $\theta \rightarrow 0^+$ from the limit $\theta \rightarrow 2\pi^-$ in order to consider domains with cracks.

In [15], a weighted Poincaré inequality for a curvilinear polygon was derived from Lemma 17. In fact, that proof applies to a general domain D with polygonal structure provided the oriented boundary $\partial^\nu D$ is used in place of the usual boundary ∂D .

Lemma 18. *Let $D \subset \mathbb{R}^2$ be a connected domain with a polygonal structure. Let $r_D(z)$ be the canonical weight function on D and let $\partial_D D$ be a non-empty closed subset of $\partial^\nu D$ such that $\partial_N D := \partial^\nu D \setminus \partial_D D$ is a union of oriented open sides of D , no two of which are adjacent. Then there exists a constant $C_D > 0$ such that*

$$\|u\|_{K^0_1(D)}^2 := \int_D \frac{|u(z)|^2}{r_D(x)^2} dz \leq C_D \int_D |\nabla u(z)|^2 dz$$

for any $u \in H^1(D)$ satisfying $u = 0$ on $\partial_D D$.

We will clarify further the need for the oriented boundary in the proof of Poincaré inequality for domains in \mathbb{R}^3 in Theorem 6.

We will also need a version of Lemma 17 in spherical coordinates. The proof exploits the usual, unweighted Poincaré's inequality on domains on the unit sphere. We include a proof, since domains with polygonal structure are not extension domains for the standard Sobolev spaces when cracks are

present. This observation holds also for domains with polyhedral structure. Recall that a domain $\Omega \in \mathbb{R}^n$ is called an *extension domain* for $H^s(\Omega)$ if there exists a bounded extension operator from $H^s(\Omega)$ to $H^s(\mathbb{R}^n)$ (see e.g. [97]).

For simplicity, we state and prove Poincaré's inequality for planar domains in \mathbb{R}^2 . The same proof applies for domains on the sphere, where ∇ is to be intended as the covariant derivative.

Lemma 19. *Let $D \subset \mathbb{R}^2$ be a connected, compact domain with polygonal structure. Let $u \in H^1(D)$ be such that $u \equiv 0$ in trace sense on a set A of positive measure in ∂D . Then, there exists a positive constant $c = c_D$ independent of u , such that*

$$\|u\|_{L^2(D)} \leq c_D \|\nabla u\|_{L^2(D)}. \quad (44)$$

Proof. We begin by establishing a version of Rellich's Theorem for domains with a polygonal structure that will be used in the course of the proof, that is, we show that the embedding $H^1(D) \hookrightarrow L^2(D)$ is compact. To this effect, we write D as a finite union of Lipschitz domains D_1, \dots, D_N , following Lemma 1. Let $\{u_n\} \subset H^1(D)$, $\|u_n\|_{H^1(D)} \leq 1$, $\forall n$. On each subdomain D_j , Rellich's Theorem applies. Therefore, by a diagonal argument, we can construct a subsequence u_{n_k} such that its restriction to D_j converges to a function u_j strongly in $L^2(D_j)$, $j = 1, \dots, N$. By the uniqueness of the limit, $u_j = u_k$ on $D_k \cap D_j$, so that the u_j extend to a function u on D such that $u_{n_k} \rightarrow u$ strongly in $L^2(D)$ as $k \rightarrow \infty$.

The proof then follows exactly as in the standard case of Lipschitz domains using the continuity of the trace. (Recall that we define Sobolev spaces on the boundary always using the topology of the oriented boundary, so that for example non-tangential limits on each side of a crack can have different values.) \square

We can now state and prove our last preliminary lemmas. We will write $x = (\rho, x')$, $x' \in S^2 = \partial B^3$, in spherical coordinates in \mathbb{R}^3 , and again with abuse of notation set $u(\rho, \omega) = u(x_1, x_2, x_3)$. Recall that the canonical weight is comparable to ρ near a vertex away from any edge, and it is comparable to the distance to the edge close to it. We address these cases in two separate lemmas.

Lemma 20. *Let ω be a connected domain with polygonal structure on the unit sphere. Let $\mathcal{C} = \mathcal{C}_{R,\omega} = \{(\rho, x'), 0 < \rho < R, x' \in \omega\}$. Then, if $u, \nabla u \in L^2(\mathcal{C})$, and $u(R, \omega) = 0$ on a subset of positive measure on $\partial\omega$ in trace sense, then*

$$\int_{\mathcal{C}} \frac{|u|^2}{\rho^2} dV \leq C \int_{\mathcal{C}} |\nabla_{x'} u(\rho, x')|^2 dS d\rho \leq C \int_{\mathcal{C}} |\nabla u(x)|^2 dV,$$

where $\nabla_{x'}$ is the covariant derivative on S^2 .

Proof. Let (φ, θ) be spherical coordinates on ω . Then, the standard Poincaré inequality on the sphere gives:

$$\int_{\omega} |u|^2 dS \leq C \int_{\omega} \left(u_{\varphi}^2 + \frac{1}{\sin^2 \varphi} u_{\theta}^2 \right) \sin \varphi d\varphi d\theta := C \int_{\omega} |\nabla_{x'} u|^2 dS.$$

Then, we have

$$\begin{aligned} \int_{\mathcal{C}} \frac{|u|^2}{\rho^2} dV &= \int_0^R \int_{\omega} |u|^2 dS d\rho \leq C \int_0^R \int_{\omega} \left(u_{\varphi}^2 + \frac{1}{\sin^2 \varphi} u_{\theta}^2 \right) dS d\rho \\ &\leq C \int_0^R \int_{\omega} \left(u_{\rho}^2 + \frac{1}{\rho^2} u_{\varphi}^2 + \frac{1}{\rho^2 \sin^2 \varphi} u_{\theta}^2 \right) \rho^2 dS d\rho = C \int_{\mathcal{C}} |\nabla u(x)|^2 dV, \end{aligned} \quad (45)$$

since $|\nabla u|^2 = u_{\rho}^2 + \frac{1}{\rho^2} u_{\varphi}^2 + \frac{1}{\rho^2 \sin^2 \varphi} u_{\theta}^2$. \square

We observe that in the most extreme case, ω is the sphere S^2 with a segment removed. Its oriented boundary consists of two copies of this segment, and it is enough for u to vanish in trace sense on one copy. More precisely, it is enough that u vanishes on a one-sided, non-tangential approach to the boundary.

We last replace ρ in the above lemma with the canonical weight, which is comparable to the distance from the singular set, as it is this function that appears in the definition of the weighted Sobolev spaces.

Lemma 21. *Assume that the hypotheses of Lemma 20 hold, and assume further that $0 < R < \epsilon$, where ϵ is sufficiently small. Then, if $r_{\mathcal{C}}$ is the canonical weight associated to $\mathcal{C} = \mathcal{C}_{R,\omega}$,*

$$\|u\|_{\mathcal{K}_1^0(\mathcal{C})} := \int_{\mathcal{C}} \frac{|u|^2}{r_{\mathcal{C}}^2} dV \leq C \int_{\mathcal{C}} |\nabla u(x)|^2 dV. \quad (46)$$

Proof. We continue to denote by (ρ, φ, θ) spherical coordinates on \mathcal{C} . For each vertex A of ω , we construct an open set $U_A \subset \mathcal{C}$ as follows. By the definition of a domain with polygonal structure, there is an open set $V_A \subset S^2$ and a diffeomorphism ϕ_A such that $\phi_A(\omega \cap V_A) = \{(r, \theta), r \in (0, \delta), \theta \in \varpi_A\}$, where ϖ_A is a disjoint, finite union of intervals in $[-1, 1]$, and δ is independent of A . We let then $U_A = \{(\rho, x'), 0 < \rho < R, x' \in \omega \cap V_A\}$, and write $U = \mathcal{C} \setminus \bigcup_A U_A$. On U , we observe that $C^{-1} \rho \leq r_{\mathcal{C}} \leq C \rho$ for some positive constant C , since ϵ is small, and use Lemma 20:

$$\int_U \frac{|u|^2}{r_{\mathcal{C}}^2} dV \leq C \int_{\mathcal{C}} \frac{|u|^2}{\rho^2} dV \leq C \int_{\mathcal{C}} |\nabla u(x)|^2 dV.$$

Next we parameterize each U_A by the coordinates (ρ, r, θ) , $0 < \rho < R$, $0 < r < \delta$, $\theta \in \varpi_A$. If $\epsilon > \delta$ are chosen small enough, then on U_A we have

$C^{-1}r\rho \leq r_C \leq Cr\rho$, with C uniform in A , so that applying Lemma 18 to the domain $\phi_A(\omega \cap V_A)$

$$\begin{aligned}
\int_{U_A} \frac{|u(x)|^2}{r_\Omega^2} dV &\leq C \int_0^R \int_{\phi_A(\omega \cap V_A)} \frac{|u(\rho, r, \theta)|^2}{\rho^2 r^2} dr d\theta d\rho \\
&\leq C \int_0^R \int_{\phi_A(\omega \cap V_A)} \frac{|\nabla_{(r, \theta)} u|^2}{\rho^2} dr d\theta d\rho \\
&\leq C' \int_0^R \int_{\omega \cap V_A} \frac{1}{\rho^2} \left(u_\phi^2 + \frac{1}{\sin^2 \phi} u_\theta^2 \right) d\phi d\theta d\rho \\
&\leq C' \int_0^R \int_{\omega \cap V_A} \left[u_\rho^2 + \frac{1}{\rho^2} \left(u_\phi^2 + \frac{1}{\sin^2 \phi} u_\theta^2 \right) \right] d\phi d\theta d\rho \\
&\leq C' \int_0^R \int_{\omega \cap V_A} |\nabla u|^2 dV.
\end{aligned}$$

By summing all the inequality for U_A and for U , we finally obtain (46). \square

We now turn to the proof of the desired weighted Poincaré inequality in three dimensions. The proof reduces to the case covered by Lemmas 17, 20 and 21, by applying the diffeomorphism ϕ_p of Definition 2.

Again, we denote with $\partial_D \Omega$ the part of the oriented boundary $\partial^\nu \Omega$ with Dirichlet conditions, and recall that a dihedral angle is a cone in our notation.

Theorem 6. *Assume that $\bar{\Omega}$ is compact or a cone. Assume also that $\partial_D \Omega \neq \emptyset$ and that $\partial_N \Omega = \partial^\nu \Omega \setminus \partial_D \Omega$ is a union of open oriented faces of Ω , no two of which have an edge in common. Then there exists C_Ω , depending only on Ω , such that*

$$\|u\|_{\mathcal{K}_1^0(\Omega)}^2 := \int_\Omega \frac{|u(x)|^2}{r_\Omega^2} dx \leq C_\Omega \int_\Omega |\nabla u(x)|^2 dx,$$

for any $u \in H^1(\Omega)$ that satisfies $u = 0$ on $\partial_D \Omega$.

Proof. The idea of the proof is to cover the domain Ω with a finite number of open sets $\tilde{\Omega}$ on which the integration simplifies and we can reduce our proof to the usual Poincaré inequality. The result will follow by adding the corresponding inequalities. Also, by Theorem 3 and standard density arguments, we can assume that u is a smooth function.

We shall write $dx = dV = dx_1 dx_2 dx_3$ for the volume element. Also, recall that r_Ω is the canonical weight function, which is comparable to $\vartheta(x)$, the distance from $x \in \bar{\Omega}$ to the edges of Ω .

Let us consider the inequality

$$\|u\|_{\mathcal{K}_1^0(\tilde{\Omega})}^2 := \int_{\tilde{\Omega}} \frac{|u(x)|^2}{r_\Omega(x)^2} dV \leq C \int_{\tilde{\Omega}} |\nabla u(x)|^2 dV, \quad u = 0 \text{ on } \partial_D \Omega \quad (47)$$

for subdomains $\tilde{\Omega} \subset \Omega$. The statement of the theorem is exactly the inequality (47) for $\tilde{\Omega} = \Omega$. The proof of our inequality for $\tilde{\Omega}$ will be obtained by adding the inequality (47) for suitable subdomains $\tilde{\Omega}$. These domains will be either of the form $V_p \cap \Omega$ or a single interior domains that is at a positive distance from the edges. On the domain that is away from the edges, we can use the usual Poincaré inequality. If $\tilde{\Omega} = V_p \cap \Omega$, we notice that the inequality (47) is preserved under the change of coordinates ϕ_p with constants uniform in p . V_p is the open neighborhood of a point $p \in \partial^\nu \Omega$ of Definition 2

Depending whether p is a vertex Q or belongs to an edge e , we denote

$$\Omega_Q = \phi_Q(V_Q \cap \Omega), \quad \Omega_e = \phi_p(V_p \cap \Omega).$$

With abuse of notation, we still denote a point in the transformed space by $x = (x_1, x_2, x_3)$. By modifying slightly the definition of ϕ_p , we can take Ω_Q to satisfy all the hypotheses of Lemma 21 uniformly in Q , while Ω_e can be characterized in suitable cylindrical coordinates, by

$$\Omega_e = \{(r, \theta, z), 0 < r < \delta, 0 < \theta < \theta_e, 0 < z < z_e := |e| - 2\epsilon\},$$

where $|e|$ is the length of the curve $\phi_p(e)$, and ϵ, δ are chosen small enough uniformly in e . Moreover, r is the distance to the singular set on Ω_e . In these cylindrical coordinates, the image of the edge e corresponds to the the z -axis (in particular, $r = 0$ on e).

For $\tilde{\Omega}_Q = \phi_Q^{-1}(\Omega_Q)$ we use Lemma 21. In fact, by hypothesis $u = 0$ on at least one side of the oriented boundary $\partial^\nu \tilde{\Omega}_Q$, so that $u \circ \phi_Q^{-1} = 0$ on on at least one side of $\partial^\nu \Omega_Q$. Therefore, (46) holds for Ω_Q , and hence for $\tilde{\Omega}_Q$ with uniform constants, by making a change of variables.

We next prove the inequality (47) for $\tilde{\Omega}_e = \phi_p^{-1}(\Omega_e)$. We write $\Omega_e = W_\delta \times (0, z_e)$, where $W_\delta = \{0 < r < \delta, 0 < \theta < \theta_e\}$ in suitable cylindrical coordinates (r, θ, z) . The hypothesis on $\partial\Omega \setminus \partial_D \Omega$ imply that in trace sense $u(r, 0, z) = 0$ or $u(r, \theta_e, z) = 0$, and we can always arrange the coordinate system locally in Ω_e so that $u(r, 0, z) = 0$. It is also crucial that on $\tilde{\Omega}_e$, $c^{-1}r \leq r_\Omega \leq Cr$, with C uniform in e , since r is exactly the distance to the

singular set on Ω_e . By Fubini's Theorem and Lemma 17, we have:

$$\begin{aligned}
& \int_{\tilde{\Omega}_e} \frac{|u|^2}{r\Omega} dV \leq C_\Omega \int_{\Omega_e} \frac{|u \circ \phi_p^{-1}|^2}{r^2} dx_1 dx_2 dx_3 \\
& = C_\Omega \int_0^{z_e} \int_{W_\delta} \frac{|u \circ \phi_p^{-1}|^2}{r} dr d\theta dz \leq C_\Omega \int_0^{z_e} \int_{W_\delta} \left(\frac{|\partial_\theta(u \circ \phi_p^{-1})|^2}{r} \right) dr d\theta dz \\
& \leq C'_\Omega \int_0^{z_e} \int_{W_\delta} |\nabla_{x_1, x_2}(u \circ \phi_p^{-1})|^2 dx_1 dx_2 \\
& \leq C'_\Omega \int_0^{z_e} \int_{W_\delta} (|\nabla_{x_1, x_2}(u \circ \phi_p^{-1})|^2 + |\partial_z(u \circ \phi_p^{-1})|^2) dx_1 dx_2 dz \\
& = C'_\Omega \int_{\Omega_e} |\nabla(u \circ \phi_p^{-1})|^2 dx_1 dx_2 dx_3 \leq C''_\Omega \int_{\tilde{\Omega}_e} |\nabla u|^2 dV,
\end{aligned} \tag{48}$$

which is exactly (47) with $\tilde{\Omega}_e = \tilde{\Omega}$.

Finally, we add the inequalities (48) for all $\tilde{\Omega} = \tilde{\Omega}_e$ and the inequalities (46) for all $\tilde{\Omega} = \tilde{\Omega}_Q$, and the usual Poincaré inequality for

$$U = \Omega \setminus \bigcup_{e, Q} (\Omega_Q \cup \Omega_e),$$

with ϵ and δ replaced by $\epsilon/2, \delta/2$, where Q ranges over all the vertices and e ranges over all edges of Ω . This concludes the proof for u smooth, and hence by density for $u \in H^1(\Omega)$. \square

From the above Poincaré type inequalities we obtain the following corollary from [15].

Corollary 3. *Let Ω be a domain with a polyhedral structure and assume that $\partial_D \Omega \neq \emptyset$, that $\partial_N \Omega \subset \partial^\nu \Omega$ is a union of open oriented faces of Ω no two of which have an edge in common. Then the norms $\|\cdot\|_{H^1(\Omega)}$, $\|\cdot\|_{\mathcal{K}_1^1(\Omega)}$, and the seminorm $|\cdot|_{H^1(\Omega)}$ are equivalent on $H_D^1(\Omega) = \{u \in \mathcal{K}_1^1(\Omega), u = 0 \text{ on } \partial_D \Omega\}$. In particular, $H_0^1(\Omega) = \mathring{\mathcal{K}}_1^1(\Omega)$.*

Proof. The weighted Poincaré inequality of Theorem 6 immediately gives that for $u \in H_D^1(\Omega)$, $\|u\|_{\mathcal{K}_1^1} \leq C|u|_{H^1} \leq C\|u\|_{H^1}$. The reverse inequality follows observing that $1/r_\Omega(x) \geq 1/\alpha$, for some large enough constant $\alpha > 0$, so that the \mathcal{K}_1^1 norm is stronger than the H^1 norm. \square

9.2. Well-posedness and regularity for the Laplace operator

Let Δ be the Laplace operator on \mathbb{R}^3 . Here, we illustrate an application of the results of the previous sections to the solvability of the mixed boundary value problem (43) ($Pu = f$ in Ω , $u = g_D$ on $\partial_D \Omega$ and $u = g_N$ on $\partial_N \Omega$) for $P = -\Delta$ and $P = \alpha^2/r_\Omega(x)^2 - \Delta$. Recall that $r_\Omega(x)$ is comparable to $\vartheta(x)$, but $r_\Omega \in C^\infty(\Sigma_\Omega)$ while ϑ does not.

We first observe that the Laplace operator with Dirichlet or Neumann boundary conditions satisfies the assumption of regularity upon freezing the coefficients of Definition 4 (see, e.g. [85] Proposition 11.12), and therefore full regularity in weighted Sobolev spaces follows from Theorem 4.

We now set $P = \alpha^2/r_\Omega(x)^2 - \Delta$, with α a positive constant such that

$$\alpha/r_\Omega(x) \geq 1, \quad \forall x \in \overline{\Omega}, \quad (49)$$

(in fact, we can always arrange that $r_\Omega \leq 1$) and impose arbitrary mixed (Dirichlet or Neumann) boundary conditions on the faces D_k of Ω . Then,

$$B_P(u, u) = \|u/r_\Omega\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \equiv \|u\|_{\mathcal{K}_1^1(\Omega)},$$

so that P is positive on $\mathcal{K}_1^1(\Omega)$, which correspond in our notation to $a = 0$.

Next, we set $P = -\Delta$ and impose again Dirichlet boundary conditions on $\partial_D\Omega$ and Neumann boundary conditions on $\partial_N\Omega$, but we assume in addition that $\partial_N\Omega$ contains no adjacent faces. Then, from the weighted Poincaré inequality 47 we obtain that

$$B_P(u, u) = \|\nabla u\|_{L^2(\Omega)}^2 \geq C(\Omega) \|u/r_\Omega\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \equiv \|u\|_{\mathcal{K}_1^1(\Omega)},$$

that is, P is positive on $\mathcal{K}_1^1(\Omega)$.

Therefore, by applying Theorem 5, we obtain solvability of the boundary value problem (43) on \mathcal{K}_{a+1}^{m+1} for any $m \geq 0$ and $|a|$ small enough. We summarize this result in the following theorem.

Theorem 7. *Let $m \in \mathbb{Z}_+$ and let $\Omega \subset \mathbb{R}^3$ be a domain with a polyhedral structure. Let $P = \alpha^2/r_\Omega(x)^2 - \Delta$ with arbitrary mixed (Dirichlet or Neumann) boundary conditions or $P = -\Delta$, but such that $\partial_D\Omega \neq \emptyset$ and that $\partial_N\Omega$ contains no adjacent faces. Then there exists $\eta > 0$ such that the boundary value problem (3) has a unique solution $u \in \mathcal{K}_{a+1}^{m+1}(\Omega)$ for any $f \in \mathcal{K}_{a-1}^{m-1}(\Omega)$, any $g_D \in \mathcal{K}_{a+1/2}^{m+1/2}(\partial_D\Omega)$, any $g_N \in \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N\Omega)$, and any $|a| < \eta$. This solution depends continuously on f , g_D , and g_N .*

This result is in contrast to the case of (unweighted) Sobolev spaces $H^m(\Omega)$, for which solvability of the boundary value problem holds only for a finite range of values of m . For example, if Ω is a polygon in \mathbb{R}^2 , the Poisson's equation with homogeneous Dirichlet boundary conditions and L^2 data admits a solution in $H^2(\Omega)$ if Ω is convex, but the solution may fail to be H^2 otherwise. For a proof we refer in the vast literature on the subject to [29, 41, 48].

10. A weighted Korn's inequality and applications to linear elasticity

The purpose of this section is to highlight the applicability of our main result, Theorem 5, to systems of uniformly elliptic operators, specifically

those appearing in linear elasticity. Singular domains, and especially domains with cracks, arise naturally in elasticity, as a result of large applied stresses.

We first derive a weighted form of Korn's inequality, which will imply that under certain conditions on the elasticity parameters, the matrix operator $P = \operatorname{div}(\mathbf{C} \cdot \nabla)$ of linear elasticity is positive on $\mathcal{K}_1^1(\Omega)$. The assumptions on Ω and its oriented boundary $\partial^\nu \Omega$ of the previous sections continue to hold here. We then study the regularity and solvability of the boundary value problem (43) on $\mathcal{K}_{m+1}^{a+1}(\Omega)$.

We begin by introducing the equations of linear elasticity and the elasticity tensor.

10.1. Linear elasticity

Linear elastostatics is modeled by the following 3×3 system of equations:

$$(P_{\mathbf{C}} u)_i := \sum_{j,k,l=1}^3 \partial_{x_j} (\mathbf{C}_{ijkl} \partial_{x_l} u_k) := \operatorname{div}(\mathbf{C} \cdot \nabla u)_i = f_i, \quad i = 1, 2, 3, \quad (50)$$

for an unknown vector field $u = (u_1, u_2, u_3)$ on Ω , where f are volume forces. For convenience, throughout this section, we use the convention of summation over repeated indices.

Above, Ω represents a bounded elastic body in \mathbb{R}^3 and u is the displacement at the point x under the elastic deformation of Ω by the given volume forces f . $\mathbf{C} = [\mathbf{C}_{ijkl}]$ is a fourth-order tensor field, called the *elasticity tensor*, with coefficients $\mathbf{C}_{ijkl} \in C^\infty(\bar{\Omega})$ and symmetries:

$$\mathbf{C}_{ijkl} = \mathbf{C}_{klij} = \mathbf{C}_{jikl} = \mathbf{C}_{ijlk}.$$

The components of \mathbf{C} , sometimes referred to as *elastic moduli*, encode the elastic response of the medium to deformations. For example, if a material is isotropic, then \mathbf{C} has the simple form:

$$\mathbf{C}_{ijkl}^{iso} = \lambda(x) \delta_{ij} \delta_{kl} + \mu(x) [\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}]$$

with λ, μ the so-called *Lamé parameters*, and δ is the Kronecker symbol ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise). In general, there can be up to 21 independent elastic moduli.

The system (50) is obtained from the laws of balance of energy and momentum by linearizing around an unperturbed or natural state of the elastic body, and therefore holds in the regime of small deformations. (For a derivation, we refer for example to [60]). The principal part of $P_{\mathbf{C}}$ is the matrix operator

$$P = \mathbf{C}[\partial, \partial] = [\mathbf{C}_{ijkl} \partial_{x_j} \partial_{x_k}],$$

We will consider two types of boundary conditions: Dirichlet or *displacement* boundary conditions $u = g$ on $\partial_D \Omega$, and natural or *traction* boundary conditions $(\partial_\nu^P u)_i = (\nu \cdot (\mathbf{C} \cdot \nabla u))_i := \nu_j \mathbf{C}^{ijkl} \partial_l u_k = (g_N)_i$, $i = 1, 2, 3$, on $\partial_N^\nu(\Omega)$. The traction $\nu \cdot (\mathbf{C} \cdot \nabla u)$ is the normal vector component of the stress at the boundary.

Remark 8. In domains with cracks, linear elastostatics can model small perturbations around cracks in equilibrium. If the crack is a material surface, and not just a surface of discontinuity for the displacement and the stress, then the laws of balance of momentum implies that the traction has to be continuous across the crack. Therefore, the most physically motivated boundary problems in this case are transmission problems, which we considered for planar domains in [58].

Definition 7. The elasticity tensor \mathbf{C} is called (uniformly) strongly elliptic if there is a constant $c > 0$ such that, for any $x \in \Omega$,

$$\mathbf{C}_{ijkl}(x)V^iW^jV^kW^l \geq c|\mathbf{V}|^2|\mathbf{W}|^2$$

for all vectors \mathbf{V}, \mathbf{W} in the tangent space $T_x\Omega$. Similarly, \mathbf{C} is called positive-definite if, for any $x \in \Omega$, $\mathbf{C}_{ijkl}(x)E^{ij}E^{kl} \geq 0$ for any *symmetric* two-tensor $E = [E^{ij}]$ in $T_x^2\Omega$.

In the isotropic case, \mathbf{C} is strongly elliptic if, and only if, $\mu > 0$ and $\lambda + 2\mu > 0$, while \mathbf{C} is positive definite if, and only if, $\mu > 0$ and $2\mu + 3\lambda > 0$. In practice, it is not too restrictive to assume that \mathbf{C} is positive definite, since this assumption implies that the elastic stored energy function is positive and convex in the strain (see again [60]). If \mathbf{C} is a positive-definite elasticity tensor, then the elasticity operator $\mathbf{C}[\partial, \partial]$ with mixed displacement-traction boundary conditions is positive in the sense of Definition 6, provided $\partial_N(\Omega)$ does not contains two adjacent faces. This result is a consequence of Korn's inequality and will be proved in Lemma 22 below.

10.2. A weighted Korn's inequality

For $u \in H_{\text{loc}}^1(\Omega)^3$, we shall denote by $\epsilon(u)$ the *deformation tensor*

$$\epsilon_{ij}(u) := (\partial_i u_j + \partial_j u_i)/2. \quad (51)$$

Then, $\epsilon(u) := (\epsilon_{ij}(u)) \in L_{\text{loc}}^2(\Omega)^9$ with $\|\epsilon(u)\|_{L^2(\Omega)}^2 := \sum_{ij} \int_{\Omega} |\epsilon_{ij}(u)|^2 dx$.

The usual Korn's inequality states that as a system $\epsilon(u)$ is H^1 -regular (that is, coercive) or even positive, depending on the boundary conditions. It is in fact a special case of Gårding inequality for systems.

If $\partial\Omega$ is smooth, this result is classical (see e.g., [39], [33], and again [60] with references therein). We will use here Korn's inequality for Lipschitz domains [78], [17].

Proposition 5. *Let Ω be a connected, bounded, Lipschitz domain and $u \in H_{\text{loc}}^1(\Omega)^3$. There exists $C > 0$, depending only on Ω , such that*

$$\|u\|_{H^1(\Omega)}^2 \leq C(\|\epsilon(u)\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \quad (52)$$

for all $u \in H^1(\Omega)^3$. Assume $\partial_D\Omega \neq \emptyset$. Then there exists $C > 0$, again depending only on Ω , such that

$$\|u\|_{H^1(\Omega)}^2 := \sum_{ij} \int_{\Omega} |\partial_i u_j|^2 dx \leq C\|\epsilon(u)\|_{L^2(\Omega)}^2 \quad (53)$$

for all $u \in H_{\text{loc}}^1(\Omega)^3$ that satisfy $u = 0$ on $\partial_D \Omega$.

The inequality (53) is referred to as “first Korn’s inequality,” while (52) is “second Korn’s inequality.” Second Korn’s inequality can be equivalently formulated as

$$\|u\|_{H^1(\Omega)} \leq C \|\epsilon(u)\|_{L^2(\Omega)},$$

for all $u \in H^1(\Omega)^3$ that satisfy $\int_{\Omega} (\partial_i u_j - \partial_j u_i) dx = 0$ for all i, j . See [21] for a Korn type inequality on Riemannian manifolds.

Combining Proposition 5 with the weighted Poincaré inequality of Theorem 6, we obtain the following corollary.

Corollary 4. *Let Ω be a domain with polyhedral structure. Assume $\overline{\Omega}$ is compact, $\partial_D \Omega \neq \emptyset$, and $\partial_N \Omega$ is a union of open faces, no two of which are adjacent. Then, there exists $C = C(\Omega) > 0$ such that*

$$\begin{aligned} \|u\|_{\mathcal{K}_1^1(\Omega)}^2 &:= \int_{\Omega} \frac{|u(x)|^2}{r_{\Omega}(x)^2} dx + \int_{\Omega} |\nabla u(x)|^2 dx + \\ &\leq C \|\epsilon(u)\|_{L^2(\Omega)}^2 \end{aligned} \quad (54)$$

for all $u \in \mathcal{K}_1^1(\Omega)^3$ that satisfy $u = 0$ on $\partial_D \Omega$.

Proof. We begin by writing Ω as a finite disjoint union of connected, bounded Lipschitz domains Ω_j , $j = 1, \dots, N$, as in Lemma 1. In each domain Ω_j , inequality (53) holds. Then, from the weighted Poincaré inequality we have:

$$\begin{aligned} \|u\|_{\mathcal{K}_1^1(\Omega)}^2 &\leq C \|\nabla u\|_{L^2(\Omega)}^2 \\ &\leq C \|u\|_{H^1(\Omega)}^2 = \sum_j \|u\|_{H^1(\Omega_j)}^2 \\ &\leq \tilde{C} \sum_j \|\epsilon(u)\|_{L^2(\Omega_j)}^2 = \tilde{C} \|\epsilon(u)\|_{L^2(\Omega)}^2, \end{aligned}$$

since the domains Ω_j are disjoint. \square

10.3. Solvability and regularity for the elasticity system

In this section, we apply the weighted Korn’s inequality to study the existence, uniqueness, and regularity of solutions to the elasticity system (50) with mixed traction-displacement boundary conditions.

Lemma 22. *If \mathbf{C} is a positive-definite elasticity tensor, the bilinear form B_P of equation (26)*

$$B_P(u, v) = \int_{\Omega} \nabla u \cdot \mathbf{C} \cdot \nabla v \, \text{dvol} = \int_{\Omega} C^{ijkl}(x) \partial_j u_i(x) \partial_l v_k(x) \, dx,$$

is coercive on $H^1(\Omega)$, more precisely:

$$B_P(u, u) \geq C_{\Omega} \|\epsilon(u)\|_{L^2(\Omega)}^2. \quad (55)$$

Proof. We fix $x \in \Omega$, and let $F = (\nabla u)(x)$, a 3×3 matrix. We denote by E its symmetric part: $E = (\nabla u(x) + \nabla u(x)^T)/2$, a symmetric matrix. From the symmetry properties of \mathbf{C} and the assumption that \mathbf{C} is positive definite, we have

$$C^{ijkl}(x) F_{ij} F_{lk} = C^{ijkl}(x) E_{ij} E_{lk} \geq c |E|^2.$$

But, by hypothesis c is uniform in $x \in \Omega$, so that (55) follows by integrating over Ω . \square

By combining the above lemma with the inequality (54), we immediately have that then the elasticity operator $-\mathbf{C}[\partial, \partial]$ with mixed-boundary conditions is positive in the sense of Definition 6, provided no two adjacent faces of Ω have traction boundary conditions:

$$B_P(u, u) \geq C_\Omega \|\epsilon(u)\|_{L^2(\Omega)}^2 \geq C'_\Omega \|u\|_{\mathcal{K}_1^1(\Omega)}^2, \quad u = 0 \text{ on } \partial_D \Omega \neq \emptyset.$$

Similarly, by combining the above lemma with the inequality (52) instead, we can conclude that $\alpha^2/r_\Omega(x)^2 - \text{div}(\mathbf{C} \cdot \nabla)$, where α satisfies again (49), is positive in the sense of Definition 6, with arbitrary displacement-traction boundary conditions:

$$\begin{aligned} \alpha^2 \|u/r_\Omega(x)\|_{L^2(\Omega)}^2 + B_P(u, u) &\geq C_\Omega \|\epsilon(u)\|_{L^2(\Omega)}^2 + \|u/r_\Omega(x)\|_{L^2(\Omega)}^2 \\ &\geq C_\Omega \|\nabla(u)\|_{L^2(\Omega)}^2 + \|u/r_\Omega(x)\|_{L^2(\Omega)}^2 \geq C'_\Omega \|u\|_{\mathcal{K}_1^1(\Omega)}^2. \end{aligned}$$

Above we have used that $\alpha^2 \|u/r_\Omega(x)\|_{L^2(\Omega)}^2 \geq \|u\|_{L^2(\Omega)}^2$ and that the bilinear form of equation (26) for the operator $1/r_\Omega(x)^2 - \text{div}(\mathbf{C} \cdot \nabla)$ is exactly $\|u/r_\Omega(x)\|_{L^2(\Omega)}^2 + B_P(u, u)$, provided $\partial_D \Omega \neq \emptyset$.

Finally, solvability and regularity for the boundary value problem follows by applying Theorem 5. We summarize these results in the following theorem.

Theorem 8. *Let $m \in \mathbb{Z}_+$ and let $\Omega \subset \mathbb{R}^3$ be a domain with a polyhedral structure. Let \mathbf{C} be a positive-definite elasticity tensors. Set $P = 1/r_\Omega(x)^2 - \text{div}(\mathbf{C} \cdot \nabla)$ with arbitrary mixed (displacement or traction) boundary conditions or $P = -\text{div}(\mathbf{C} \cdot \nabla)$, but such that $\partial_D \Omega \neq \emptyset$ and that $\partial_N \Omega$ contains no adjacent faces. Then there exists $\eta > 0$ such that the boundary value problem (3) has a unique solution $u \in \mathcal{K}_{a+1}^{m+1}(\Omega)^\mu$ for any $f \in \mathcal{K}_{a-1}^{m-1}(\Omega)^q$, any $g_D \in \mathcal{K}_{a+1/2}^{m+1/2}(\partial_D \Omega)^q$, any $g_N \in \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N \Omega)^q$, and any $|a| < \eta$. This solution depends continuously on f , g_D , and g_N .*

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References

1. S. Agmon. *Lectures on elliptic boundary value problems*. Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. Van Nostrand Mathematical Studies, No. 2. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965.
2. S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. *Comm. Pure Appl. Math.*, 17:35–92, 1964.
3. B. Ammann, A. Ionescu, and V. Nistor. Sobolev spaces on Lie manifolds and regularity for polyhedral domains. *Doc. Math.*, 11:161–206 (electronic), 2006.
4. B. Ammann and V. Nistor. Weighted sobolev spaces and regularity for polyhedral domains. *Computer Methods in Applied Mechanics and Engineering*, 196:3650–3659, 2007.
5. T. Apel and S. Nicaise. The finite element method with anisotropic mesh grading for elliptic problems in domains with corners and edges. *Math. Methods Appl. Sci.*, 21(6):519–549, 1998.
6. T. Apel, A.-M. Sändig, and S. Solov'ev. Computation of 3D vertex singularities for linear elasticity: error estimates for a finite element method on graded meshes. *M2AN Math. Model. Numer. Anal.*, 36(6):1043–1070 (2003), 2002.
7. D. Arnold, R. Falk, and R. Winther. Multigrid preconditioning in $H(\text{div})$ on non-convex polygons. *Comput. Appl. Math.*, 17(3):303–315, 1998.
8. T. Aubin. Espaces de Sobolev sur les variétés riemanniennes. *Bull. Sci. Math. (2)*, 100(2):149–173, 1976.
9. T. Aubin and Y. Y. Li. On the best Sobolev inequality. *J. Math. Pures Appl. (9)*, 78(4):353–387, 1999.
10. I. Babuška. The rate of convergence for the finite element method. *SIAM J. Numer. Anal.*, 8:304–315, 1971.
11. I. Babuška and A. K. Aziz. Survey lectures on the mathematical foundations of the finite element method. In *The mathematical foundations of the finite element method with applications to partial differential equations (Proc. Sympos., Univ. Maryland, Baltimore, Md., 1972)*, pages 1–359. Academic Press, New York, 1972. With the collaboration of G. Fix and R. Kellogg.
12. I. Babuška and S. L. Sobolev. Optimization of numerical processes. *Apl. Mat.*, 10:96–129, 1965.
13. I. Babuška and V. Nistor. Boundary value problems in spaces of distributions and their numerical investigation. IMA Preprint.
14. C. Bacuta, A. L. Mazzucato, V. Nistor, and L. Zikatanov. Interface and mixed boundary value problems on n -dimensional polyhedral domains. In preparation. Preliminary version as IMA preprint #1984, August 2004.
15. C. Bacuta, V. Nistor, and L. T. Zikatanov. Improving the rate of convergence of high-order finite elements on polyhedra. I. A priori estimates. *Numer. Funct. Anal. Optim.*, 26(6):613–639, 2005.
16. J. Bramble. *Multigrid methods*, volume 294 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1993.
17. J. H. Bramble. A proof of the inf-sup condition for the Stokes equations on Lipschitz domains. *Math. Models Methods Appl. Sci.*, 13(3):361–371, 2003. Dedicated to Jim Douglas, Jr. on the occasion of his 75th birthday.
18. S. Brenner and R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 2002.
19. C. Băcuță, V. Nistor, and L. Zikatanov. Improving the rate of convergence of ‘high order finite elements’ on polygons and domains with cusps. *Numerische Mathematik*, 100:165–184, 2005.

20. M. Cheeger, J. Gromov and M. Taylor. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.*, 17(1):15–53, 1982.
21. W. Chen and J. Jost. A Riemannian version of Korn’s inequality. *Calc. Var. Partial Differential Equations*, 14(4):517–530, 2002.
22. O. Chkadua and R. Duduchava. Pseudodifferential equations on manifolds with boundary: Fredholm property and asymptotic. *Math. Nachr.*, 222:79–139, 2001.
23. M. Costabel. Boundary integral operators on curved polygons. *Ann. Mat. Pura Appl. (4)*, 133:305–326, 1983.
24. M. Costabel. Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal.*, 19(3):613–626, 1988.
25. M. Costabel and M. Dauge. Computation of corner singularities in linear elasticity. In *Boundary value problems and integral equations in nonsmooth domains (Luminy, 1993)*, volume 167 of *Lecture Notes in Pure and Appl. Math.*, pages 59–68. Dekker, New York, 1995.
26. M. Costabel and M. Dauge. Crack singularities for general elliptic systems. *Math. Nachr.*, 235:29–49, 2002.
27. M. Costabel, M. Dauge, and S. Nicaise. Corner singularities of Maxwell interface and eddy current problems. In *Operator theoretical methods and applications to mathematical physics*, volume 147 of *Oper. Theory Adv. Appl.*, pages 241–256. Birkhäuser, Basel, 2004.
28. F. Costanzo and J. Walton. Steady growth of a crack with a rate and temperature sensitive cohesive zone. *Journal of the Mechanics and Physics of Solids*, 50:1649–1679, 2002.
29. M. Dauge. *Elliptic boundary value problems on corner domains*, volume 1341 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988. Smoothness and asymptotics of solutions.
30. A. Duarte, O. Hamzeh, T. Liszka, and W. Tworzydło. A generalized finite element method for the simulation of three-dimensional dynamic crack propagation. *Computer Methods in Applied Mechanics and Engineering*, 190:2227–2262, 2001.
31. R. Duduchava and D. Natroshvili. Mixed crack type problem in anisotropic elasticity. *Math. Nachr.*, 191:83–107, 1998.
32. R. Duduchava and F.-O. Speck. Singular integral equations in special weighted spaces. *Georgian Math. J.*, 7(4):633–642, 2000.
33. G. Duvaut and J.-L. Lions. *Inequalities in mechanics and physics*. Springer-Verlag, Berlin, 1976. Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.
34. L.C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.
35. R. Falk. Nonconforming finite element methods for the equations of linear elasticity. *Math. Comp.*, 57(196):529–550, 1991.
36. R. Falk and J. Osborn. Remarks on mixed finite element methods for problems with rough coefficients. *Math. Comp.*, 62(205):1–19, 1994.
37. G. Fichera. *Linear elliptic differential systems and eigenvalue problems*, volume 8 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1965.
38. G. Folland. *Introduction to partial differential equations*. Princeton University Press, Princeton, NJ, second edition, 1995.
39. K.O. Friedrichs. On the boundary-value problems of the theory of elasticity and Korn’s inequality. *Ann. of Math. (2)*, 48:441–471, 1947.
40. G. Friesecke, R. James, and S. Müller. A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Comm. Pure Appl. Math.*, 55(11):1461–1506, 2002.

41. P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.
42. P. Grisvard. Singularités en élasticité. *Arch. Rational Mech. Anal.*, 107(2):157–180, 1989.
43. P. Grisvard. *Singularities in boundary value problems*, volume 22 of *Research in Applied Mathematics*. Masson, Paris, 1992.
44. W. Hackbusch. *Multigrid methods and applications*, volume 4 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1985.
45. E. Hunsicker and R. Mazzeo. Harmonic forms on manifolds with edges. *Int. Math. Res. Not.*, 2005(52):3229–3272.
46. D. Jerison and C. Kenig. The Neumann problem on Lipschitz domains. *Bull. Amer. Math. Soc. (N.S.)*, 4(2):203–207, 1981.
47. D. Jerison and C. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.*, 130(1):161–219, 1995.
48. D. S. Jerison and C. E. Kenig. Boundary value problems on Lipschitz domains. In *Studies in partial differential equations*, volume 23 of *MAA Stud. Math.*, pages 1–68. Math. Assoc. America, Washington, DC, 1982.
49. M. Kanninen and C. Popelar. *Advanced Fracture Mechanics*. Oxford University Press, 1985.
50. D. Kapanadze and B.-W. Schulze. *Crack theory and edge singularities*, volume 561 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 2003.
51. C. Kenig. Recent progress on boundary value problems on Lipschitz domains. In *Pseudodifferential operators and applications (Notre Dame, Ind., 1984)*, volume 43 of *Proc. Sympos. Pure Math.*, pages 175–205. Amer. Math. Soc., Providence, RI, 1985.
52. V. A. Kondrat'ev. Boundary value problems for elliptic equations in domains with conical or angular points. *Transl. Moscow Math. Soc.*, 16:227–313, 1967.
53. V. Kozlov, V. Maz'ya, and J. Rossmann. *Spectral problems associated with corner singularities of solutions to elliptic equations*, volume 85 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001.
54. V. A. Kozlov and V. G. Maz'ya. On stress singularities near the boundary of a polygonal crack. *Proc. Roy. Soc. Edinburgh Sect. A*, 117(1-2):31–37, 1991.
55. P. Laasonen. On the degree of convergence of discrete approximations for the solutions of the Dirichlet problem. *Ann. Acad. Sci. Fenn. Ser. A. I.*, 1957(246):19, 1957.
56. J. Lewis and C. Parenti. Pseudodifferential operators of Mellin type. *Comm. Partial Differential Equations*, 8(5):477–544, 1983.
57. J.E. Lewis. Layer potentials for elastostatics and hydrostatics in curvilinear polygonal domains. *Trans. Amer. Math. Soc.*, 320(1):53–76, 1990.
58. H. Li, A. L. Mazzucato, and V. Nistor. Analysis of the finite element method for transmission/mixed boundary value problems on general polygonal domains. Submitted.
59. J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 1*. Travaux et Recherches Mathématiques, No. 17. Dunod, Paris, 1968.
60. J. E. Marsden and T. Hughes. *Mathematical foundations of elasticity*. Dover Publications Inc., New York, 1994. Corrected reprint of the 1983 original.
61. V. G. Maz'ja and B. A. Plamenevskii. Elliptic boundary value problems on manifolds with singularities. In *Problems in mathematical analysis, No. 6: Spectral theory, boundary value problems (Russian)*, pages 85–142, 203. Izdat. Leningrad. Univ., Leningrad, 1977.

62. W. G. Mazja, S. A. Nazarov, and B. A. Plamenevski. *Asymptotische Theorie elliptischer Randwertaufgaben in singular gestörten Gebieten. I*, volume 82 of *Mathematische Lehrbücher und Monographien, II. Abteilung: Mathematische Monographien* [Mathematical Textbooks and Monographs, Part II: Mathematical Monographs]. Akademie-Verlag, Berlin, 1991. Störungen isolierter Randsingularitäten. [Perturbations of isolated boundary singularities].
63. V. Maz'ya and J. Roßmann. Weighted L_p estimates of solutions to boundary value problems for second order elliptic systems in polyhedral domains. *ZAMM Z. Angew. Math. Mech.*, 83(7):435–467, 2003.
64. R. Mazzeo. Elliptic theory of differential edge operators. I. *Commun. Partial Differ. Equations*, 16(10):1615–1664, 1991.
65. W. McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.
66. R. Melrose. Analysis on manifolds with corners. book: in preparation.
67. R. Melrose. *Geometric scattering theory*. Stanford Lectures. Cambridge University Press, Cambridge, 1995.
68. D. Mitrea, M. Mitrea, and M. Taylor. Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds. *Mem. Amer. Math. Soc.*, 150(713):x+120, 2001.
69. I. Mitrea. On the traction problem for the Lamé system on curvilinear polygons. *J. Integral Equations Appl.*, 16(2):175–219, 2004.
70. Irina Mitrea and Marius Mitrea. The Poisson problem with mixed boundary conditions in Sobolev and Besov spaces in non-smooth domains. *Trans. Amer. Math. Soc.*, 359(9):4143–4182 (electronic), 2007.
71. M. Mitrea and M. Taylor. Boundary layer methods for Lipschitz domains in Riemannian manifolds. *J. Funct. Anal.*, 163(2):181–251, 1999.
72. M. Mitrea and M. Taylor. Potential theory on Lipschitz domains in Riemannian manifolds: L^p Hardy, and Hölder space results. *Comm. Anal. Geom.*, 9(2):369–421, 2001.
73. C. B. Morrey, Jr. Second-order elliptic systems of differential equations. In *Contributions to the theory of partial differential equations*, Annals of Mathematics Studies, no. 33, pages 101–159. Princeton University Press, Princeton, N. J., 1954.
74. S. Nazarov and B. Plamenevsky. *Elliptic problems in domains with piecewise smooth boundaries*, volume 13 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.
75. S. Nicaise. Regularity of the solutions of elliptic systems in polyhedral domains. *Bull. Belg. Math. Soc. Simon Stevin*, 4(3):411–429, 1997.
76. Serge Nicaise and Anna-Margarete Sändig. Transmission problems for the Laplace and elasticity operators: regularity and boundary integral formulation. *Math. Models Methods Appl. Sci.*, 9(6):855–898, 1999.
77. L. Nirenberg. Estimates and existence of solutions of elliptic equations. *Comm. Pure Appl. Math.*, 9:509–529, 1956.
78. J. A. Nitsche. On Korn's second inequality. *RAIRO Anal. Numér.*, 15(3):237–248, 1981.
79. A.-M. Sändig, U. Richter, and R. Sändig. The regularity of boundary value problems for the Lamé equations in a polygonal domain. *Rostock. Math. Kolloq.*, (36):21–50, 1989.
80. G. Savaré. Regularity results for elliptic equations in Lipschitz domains. *J. Funct. Anal.*, 152(1):176–201, 1998.
81. C. Schwab. *P- And Hp- Finite Element Methods: Theory and Applications in Solid and Fluid Mechanics*. Oxford University Press, 1999.
82. M. Shubin. Spectral theory of elliptic operators on noncompact manifolds. *Astérisque*, 207:5, 35–108, 1992. Méthodes semi-classiques, Vol. 1 (Nantes, 1991).

83. L. Skrzypczak. Mapping properties of pseudodifferential operators on manifolds with bounded geometry. *J. London Math. Soc. (2)*, 57(3):721–738, 1998.
84. E. Sternberg and W. Koiter. The wedge under a concentrated couple: a paradox in the two-dimensional theory of elasticity. *J. Appl. Mech.*, 25:575–581, 1958.
85. M. Taylor. *Pseudodifferential operators*, volume 34 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1981 xi+452.
86. M. Taylor. *Partial differential equations I, Basic theory*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1995.
87. M. Taylor. *Partial differential equations II, Qualitative studies of linear equations*, volume 116 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996.
88. M. Taylor. *Tools for PDE*, volume 81 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000. Pseudodifferential operators, paradifferential operators, and layer potentials.
89. H. Triebel. Characterizations of function spaces on a complete Riemannian manifold with bounded geometry. *Math. Nachr.*, 130:321–346, 1987.
90. G. Verchota. Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. *J. Funct. Anal.*, 59(3):572–611, 1984.
91. G. Verchota and A. Vogel. The multidirectional Neumann problem in \mathbb{R}^4 . Preprint, 2003.
92. M. I. Vishik. Les opérateurs différentiels et pseudo-différentiels à une infinité de variables, les problèmes elliptiques et paraboliques. In *Colloque International CNRS sur les Équations aux Dérivées Partielles Linéaires (Univ. Paris-Sud, Orsay, 1972)*, pages 342–362. Astérisque, 2 et 3. Soc. Math. France, Paris, 1973.
93. L. Wahlbin. On the sharpness of certain local estimates for \mathring{H}^1 projections into finite element spaces: influence of a re-entrant corner. *Math. Comp.*, 42(165):1–8, 1984.
94. L. Wahlbin. Local behavior in finite element methods. In *Handbook of numerical analysis, Vol. II*, Handb. Numer. Anal., II, pages 353–522. North-Holland, Amsterdam, 1991.
95. W.L. Wendland. *Elliptic systems in the plane*. Pitman, London, 1979.
96. H. Whitney. Tangents to an analytic variety. *Ann. of Math. (2)*, 81:496–549, 1965.
97. William P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.

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