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# Decomposition of Besov-Morrey Spaces

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Abstract. We establish a decomposition of Besov-Morrey spaces in terms of smooth "wavelets" obtained from a Littlewood-Paley partition of unity, or more generally molecules concentrated on dyadic cubes. We show that an expansion in atoms supported on dyadic cubes holds. We study atoms in Morrey spaces and prove a Littlewood-Paley theorem. Our results extend those of M. Frazier and B. Jawerth for Besov spaces, and are related to work of Uchiyama for BMO.

## 1. Introduction

In this paper, we consider several decompositions for Besov-Morrey spaces, which were introduced by H. Kozono and M. Yamazaki to study solutions of the Navier-Stokes equation with critical smoothness [**KY**]. Each element is expanded into either a fixed family of smooth oscillating functions, a "wavelet" decomposition, or into elementary units, "atoms", that satisfy certain size and cancellation conditions.

Besov-Morrey or BM spaces are modified Besov spaces where the base norm is of Morrey type, instead of $L^p$. We developed the theory of BM spaces and study further their applications to non-linear PDEs in [**Maz**].

Besov spaces can be described in terms of Littlewood-Paley components (see e.g. [**Trieb**]). Therefore, they behave well under the action of singular integrals and pseudo-differential operators. As a matter of fact, they form a microlocalizable scale [**Ru**],[**Ma**],[**Bour**]. Additionally, they measure the oscillatory properties of a distribution more accurately than Sobolev spaces, while possessing the same smoothness and scaling properties. For example, it is possible to construct functions $u$ with unit norm in $L^n(\mathbb{R}^n)$, while their norm in the homogeneous Besov space $\dot{B}^{3/q-1}_{q,\infty}$, $q > 3$, is arbitrarily small, by taking $u$ to be sufficiently smooth and oscillating. This observation has important consequences for the analysis of the Navier-Stokes equation, in particular existence of self-similar solutions [**Can**].

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The significance of oscillations is emphasized by the crucial role played by BMO in recent developments [**KoT**],[**KT**].

Similarly, Morrey spaces describe local regularity more precisely than $L^p$. For example, the following Sobolev-type embedding theorem holds for the Morrey space $M^p_1(\mathbb{R}^n)$:

$$M^p_1(\mathbb{R}^n) \subset C^{-n/p}_*(\mathbb{R}^n), \tag{1.1}$$

where $C^{-n/p}_*$ is a Hölder-Zygmund space. Note that it is stronger than the usual Sobolev embedding, as $M^p_1$ is strictly larger than $L^p$. From (1.1), *Morrey's Lemma* follows:

$$\nabla f \in M^p_1(\mathbb{R}^n), \quad p > n \Rightarrow f \in C^\alpha_{\text{loc}}(\mathbb{R}^n), \quad \alpha = 1 - n/p. \tag{1.2}$$

C.B. Morrey introduced $M^p_1$ and used this result to extend De Giorgi-Nash-Moser theory to quasi-linear inhomogeneous elliptic PDE [**Mo**].

In the context of fluid dynamics, Morrey spaces have been used to model flow when vorticity (the curl of the velocity field) is a singular measure supported on certain sets in $\mathbb{R}^n$ — for example, Jordan curves for $n = 3$, which correspond to the so-called *vortex rings* [**GM**]. Here $M^p_1(\mathbb{R}^n)$ is replaced by a corresponding space of measures $\tilde{M}^p(\mathbb{R}^n)$. They also provide a large class of examples of mild solutions to the Navier-Stokes system [**L-R**].

Besov-Morrey spaces combine several features of Besov and Morrey spaces. While they are strictly larger than Morrey spaces for the same scaling and smoothing — for instance p.v.$(1/x)$ belongs to the BM space $\mathcal{N}^0_{1,1,\infty}(\mathbb{R})$, but p.v.$(1/x) \neq \tilde{M}^1(\mathbb{R})$— they are better behaved under many respects, in particular under the action of pseudo-differential operators.

BM spaces "interpolate" (in a sense that we will not make precise) between Besov spaces and BMO. Indeed, while the techniques used here are essentially the same as those in [**FJ1**] and [**FJ2**], based on a discrete version of Calderon's reproducing formula, a careful analysis at each scale and location on the dyadic grid is necessary. Additional technical difficulties arise because BM and Morrey spaces lack good duality and interpolation properties [**BRV**].

As it is the case with classical spaces, our wavelet and atomic decompositions may provide concrete realizations on domains with boundary and trace theorems. They may also prove useful in analyzing PDE other than Navier-Stokes or semi-linear parabolic equations, where Littlewood-Paley theory is satisfactory. We reserve to address these questions in future work.

The paper is organized as follows. In Section 2, we introduce all the necessary definitions and state our main results, which we compare with those in [**FJ1**],[**FJ2**], and [**Uch**]. Detailed proofs are given in Section 3. Section 4 is devoted to investigate atoms and wavelets in Morrey spaces. In particular, we establish a theorem of Littlewood-Paley type. While it is known that *certain* Morrey spaces admit a wavelet expansion [**Fe**],[**Can**], the author is not aware of any result concerning atoms.

We conclude with some notational remarks.. Firstly, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote respectively the Fourier and inverse Fourier transforms, and we set:

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi), \ (\mathcal{F}^{-1}g)(x) = \check{g}(x).$$

Secondly, we consider functions or distributions on Euclidean space $\mathbb{R}^n$, where $n$ is fixed, and we usually omit the reference to the underlying space, i.e., we write

$L^p$, $\mathcal{M}^p_q$ for $L^p(\mathbb{R}^n)$, $\mathcal{M}^p_q(\mathbb{R}^n)$. Lastly, we employ the symbol $\approx$ to indicate norm equivalence.

## 2. Definitions and results

We start by recalling some definitions and basic properties of Morrey and Besov-Morrey spaces.

DEFINITION 2.1. For $p$ and $q$ satisfying $1 \le q \le p < \infty$, the *homogeneous Morrey space*, $\mathcal{M}^p_q$, and the *inhomogeneous* or *local Morrey space*, $M^p_q$, are defined as

$$\mathcal{M}^p_q \equiv \{f \in L^q_{\mathrm{loc}} \mid \|f\|_{\mathcal{M}^p_q} = \sup_{x_0\in\mathbb{R}^n} \sup_{0<R} R^{n/p-n/q} \|f\|_{L^q(B(x_0,R))} < \infty\}\,, \tag{2.1}$$

and

$$M^p_q \equiv \{f \in L^q_{\mathrm{loc}} \mid \|f\|_{M^p_q} = \sup_{x_0\in\mathbb{R}^n} \sup_{0<R\le 1} R^{n/p-n/q} \|f\|_{L^q(B(x_0,R))} < \infty\}\,, \tag{2.2}$$

where $B(x_0, R)$ is the closed ball of $\mathbb{R}^n$ with center $x_0$ and radius $R$.

Morrey spaces can be seen as a complement to $L^p$ spaces. In fact, $\mathcal{M}^p_p \equiv L^p$ and $L^p \subset M^p_q$. They are part of a larger class, that of Morrey-Campanato spaces $\mathcal{L}^k_{q,\lambda}$, which also include Lipschitz spaces and BMO, the space of functions of bounded mean oscillation.

Following Peetre [**Pe2**], we say that a locally integrable function $f$ belongs to $\mathcal{L}^k_{q,\lambda}$, $1 \le q < \infty$, $0 \le \lambda \le n + (k+1)q$, $k \in \mathbb{N}$, if

$$\sup_{x\in\mathbb{R}^n, R>0} R^{-\lambda} \inf_{P\in\mathcal{P}_k} \int_{B(x,R)} |f(y) - P(y)|^q \, dy < \infty, \tag{2.3}$$

where $\mathcal{P}_k$ is the space of all polynomials in $n$ variables of degree less than or equal to $k$.

$\mathcal{M}^p_q$ corresponds to the choice of parameter $\lambda = n(1 - q/p)$, so $0 \le \lambda < n$, for any $k$ ( Campanato [**Ca**]). Campanato's proof is actually valid only when $\mathbb{R}^n$ is replaced by a bounded, open, connected domain $\Omega$, but it can be easily adapted to the case $\Omega$ unbounded by means of a limiting argument.

BMO is obtained with $\lambda = n$, $k = 0$ (John-Nirenberg [**JN**]), which also corresponds to the limit $p \to \infty$, $q \in [1, \infty)$, in the Morrey class.

$\mathrm{Lip}^\alpha \equiv \mathcal{L}^0_{q,\lambda}$ for $n < \lambda < n + q$ with $\alpha = (\lambda - n)/q$, which was also shown by Campanato (see [**Pe2**]) and which is an extension of Morrey's Lemma.

We remark that (2.3) defines a seminorm that vanishes precisely when $f$ is a polynomial. Hence, the above identifications must be thought of modulo polynomials. This problem is discussed at some length in Section 4.

DEFINITION 2.2. For $1 \le q \le p < \infty$, $s \in \mathbb{R}$, and $r \in [1, \infty]$ we say that $f \in \mathcal{S}'$ belongs to the space $N^s_{p,q,r}$ if

$$\|f\|_{N^s_{p,q,r}} = \|\varphi_0(D)f\|_{M^p_q} + \left\{ \sum_{\nu\ge 0} \left( 2^{s\nu} \, \|\psi_\nu(D)f\|_{M^p_q} \right)^r \right\}^{1/r} < \infty\,, \tag{2.4}$$

and, similarly, we say that $f \in \mathcal{S}'/\mathcal{P}$[1] belongs to the space $\mathcal{N}^s_{p,q,r}$ if

$$\begin{aligned}\|f\|_{\mathcal{N}^s_{p,q,r}} &= \left\{ \sum_{\nu\in\mathbb{Z}} \left( 2^{s\nu}\, \|\psi_\nu(D) f\|_{\mathcal{M}^p_q} \right)^r \right\}^{1/r} \\ &= \left\| \{2^{s\nu}\, \|\psi_\nu(D) f\|_{\mathcal{M}^p_q}\}_{\nu=-\infty}^{\infty} \right\|_{\ell^r} < \infty.\end{aligned} \tag{2.5}$$

Here, $\{\phi_0, \psi_\nu\}$, $\nu \geq 0$, ($\{\psi_\nu\}$, $\nu \in \mathbb{Z}$) is a inhomogeneous (res. homogeneous) Littlewood-Paley resolution of unity, i.e.,

$$\psi_\nu(\xi) = \phi_\nu(\xi) - \phi_{\nu-1}(\xi), \qquad \phi_\nu(\xi) = \phi_0(\xi/2^\nu),$$

where $\phi_0 \in C^\infty_0(\mathbb{R}^n)$ real, such that $\phi_0 \equiv 1$ on the ball $B(0,1)$, $\operatorname{supp}\phi_0 \subset B(0,2)$. Then, it is easy to see that

$$(\phi_0 + \sum_{\nu\geq 0} \psi_\nu)\, u = u, \quad u \in \mathcal{S}',$$

while

$$\sum_{\nu\in\mathbb{Z}} \psi_\nu\, u = u,$$

if $u$ is a distribution supported away from the origin. Above, $\psi_\nu(D)$ is the operator associated to the Fourier multiplier $\psi_\nu$ in the usual way

$$\psi_\nu(D) f(x) = \mathcal{F}^{-1}(\psi_\nu \cdot \hat{f})(x). \tag{2.6}$$

For brevity, we will call $\mathcal{N}^s_{p,q,r}$ and $N^s_{p,q,r}$ respectively the *homogeneous BM* and *inhomogeneous BM space*. $N^s_{p,q,r}$ and $\mathcal{N}^s_{p,q,r}$ are complete Banach spaces under the norms (2.4) and (2.5) [**KY**].

BM spaces also generalize classical spaces, in particular:

$$N^s_{2,2,2} \equiv H^s, \quad N^s_{p,p,r} \equiv B^s_{p,r},$$

and similarly for the homogeneous counterparts.

While Littlewood-Paley decompositions localize in frequency, wavelet expansions localize in physical space. The type of wavelets used here are not compactly supported, although they are concentrated on compact sets (dyadic cubes); on the other hand, they have infinitely many vanishing moments.

For simplicity, we will state and prove theorems for homogeneous spaces with straightforward adaptation to the inhomogeneous case.

Definition 2.3. Let $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. The set

$$\mathrm{Q}_{\nu k} = \{x \in \mathbb{R}^n \mid 2^{-\nu} k_i \leq x_i \leq 2^{-\nu}(k_i + 1),\ i = 1, \ldots, n\}, \tag{2.7}$$

is called a *dyadic cube*. For fixed $\nu$, the collection $\{\mathrm{Q}_{\nu k}, k \in \mathbb{Z}^n\}$ tiles the whole space and the cubes are pairwise disjoint. Each cube $\mathrm{Q} = \mathrm{Q}_{\nu k}$ is uniquely identified by its length $\ell(\mathrm{Q}) = 2^{-\nu}$ and a preferred corner $x_\mathrm{Q} = 2^{-\nu} k$.

We use both Q and J to denote cubes, not necessarily dyadic. $|\mathrm{Q}|$ stands for the volume of the cube Q.

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[1] $\mathcal{S}'/\mathcal{P}$ is the space of equivalence classes of tempered distributions modulo polynomials with the quotient topology.

NOTATION 2.4. As we will be working in physical space $\mathbb{R}^n_x$, we set

$$\check{\psi}_\nu(x) = \sigma_\nu(x) = 2^{\nu n}\sigma(2^\nu x)\,, \quad \sigma = \check{\psi}_0,$$

and

$$\sigma_{\nu k}(x) = \sigma(2^\nu x - k).$$

Each $\sigma_{\nu k}$ is a smooth "wavelet" concentrated on the dyadic cube $\mathrm{Q}_{\nu k}$ with infinite vanishing moments, i.e.,

$$\int_{\mathbb{R}^n} x^M\, \sigma_{\nu k}(x)\, dx = 0\,, \tag{2.8a}$$

$$|\sigma_{\nu k}(x)| \leq C_M\,(1 + 2^\nu|x - 2^{-\nu}k|)^{-M}. \tag{2.8b}$$

for all $M \geq 0$. We put the word "wavelet" in quotes just to emphasize that $\sigma_{\nu k}$ are not $L^2$-orthonormal.

In [**Maz**] we proved the following result.

THEOREM 1. *Let $f \in \mathcal{N}^s_{p,q,r}$, then*

$$f = \sum_{\mathrm{Q}\ dyadic} s_{\mathrm{Q}}\, \sigma_{\mathrm{Q}}$$

*in $\mathcal{S}'/\mathcal{P}$, and*

$$\left[\sum_{\nu\in\mathbb{Z}}\left(\sup_{\substack{\mathrm{J}\ dyadic\\ \ell(\mathrm{J})\geq 2^{-\nu}}}\left(\frac{1}{|\mathrm{J}|}\right)^{1-q/p}\sum_{\mathrm{Q}_{\nu k}\subset \mathrm{J}}|\mathrm{Q}|^{1-q/p}|s_{\mathrm{Q}}|^q\right)^{r/q}\right]^{1/r} \leq C\,\|f\|_{\mathcal{N}^s_{p,q,r}}, \tag{2.9}$$

*for some $C = C(n,p,q,r)$.*

Dependence on the index $p$ and $s$ is hidden in $\sigma_{\mathrm{Q}}$ and $s_{\mathrm{Q}}$, which are defined as follows:

$$\begin{aligned} \sigma_{\mathrm{Q}} &= |\mathrm{Q}|^{s/n-1/p}\,\sigma_{\nu k}, \\ s_{\mathrm{Q}} &= 2^{\nu(s-n/p)}\,\sigma_\nu * f(2^{-\nu}k), \end{aligned} \tag{2.10}$$

if $\mathrm{Q} = \mathrm{Q}_{\nu k}$.

Here we establish the converse to Theorem 1. Furthermore, as for Besov spaces, $\sigma_{\mathrm{Q}}$ can be replaced by an $(s,p)$-molecule.

DEFINITION 2.5. A function $m$ is called an $(s,p)$-*molecule*, $s \in \mathbb{R}$, $1 \leq p \leq \infty$, if the following oscillation and decay conditions hold:

$$\int_{\mathbb{R}^n} x^\alpha\, m(x)\, dx = 0, \quad |\alpha| \leq L, \tag{2.11a}$$

$$|\partial^\alpha m(x)| \leq C_M\, 2^{\nu(n/p-s+|\alpha|)}(1 + 2^\nu|x - x_0|)^{-(M+|\alpha|)}, \quad |\alpha| \leq K, \tag{2.11b}$$

for some point $x_0 \in \mathbb{R}^n$ and $\nu \in \mathbb{Z}$. Here, $K \geq ([s]+1)$, $M$ is large, fixed ($M > L + 2n$, where $L \geq \max(-s + n/p, -1)$).

We say that a molecule is *concentrated* on the cube Q if $x_0 = x_{\mathrm{Q}}$. In this case, we denote the molecule by $m_{\mathrm{Q}}$. In particular, for every $s$ $\sigma_{\mathrm{Q}}$ is an $(s,p)$ molecule concentrated on Q.

THEOREM 2. *If* $\{s_{\mathrm{Q}}\}$ *is any sequence of scalars indexed by dyadic cubes,* $f = \sum_{\mathrm{Q}} s_{\mathrm{Q}} m_{\mathrm{Q}}$ *in the sense of distributions (modulo polynomials), and*

$$\|f\|_* = \left[ \sum_{\nu\in\mathbb{Z}} \left( \sup_{\substack{\mathrm{J}\ dyadic \\ \ell(\mathrm{J})\geq 2^{-\nu}}} \left(\frac{1}{|\mathrm{J}|}\right)^{1-q/p} \sum_{\mathrm{Q}_{\nu k}\subset \mathrm{J}} |\mathrm{Q}|^{1-q/p} |s_{\mathrm{Q}}|^q \right)^{r/q} \right]^{1/r} < \infty, \tag{2.12}$$

*then* $f \in \mathcal{N}^s_{p,q,r}$ *with* $\|f\|_{\mathcal{N}^s_{p,q,r}} \approx \|f\|_*$.

When $p = q$ we correctly recover Frazier and Jawerth's decomposition of Besov spaces [**FJ1**]. However, our proof require more vanishing moments for $m_{\mathrm{Q}}$. Indeed, the optimal choice for $M$ and $L$ is:

$$M \sim n/p + 2n + 2 + L, \qquad L \sim -s + n/p,$$

while $L \geq \max(-s, -1)$ for Besov spaces, which correspond to setting $p = q$.

On the other hand, it is consistent with the decomposition of BMO obtained by Uchiyama [**Uch**]. There, $K = 1$, $L = 0$, $M = 2n + 2$. Indeed, we have already pointed out that BMO corresponds to the limit $p \to \infty$ in the Morrey class (cf. (2.3)), so by Littlewood-Paley theory to the limit $p \to \infty$, $s = 0$, $r = 2$ in the BM class. In fact, Uchiyama's work yields:

$$\|f\|_{\mathrm{BMO}} \approx \sup_{\mathrm{J\ dyadic}} \left( \frac{1}{|\mathrm{J}|} \sum_{\nu=-\log_2(\ell(\mathrm{J}))}^{+\infty} \sum_{\mathrm{Q}_{\nu k}\subset\mathrm{J}} |s_{\mathrm{Q}}|^2 |\mathrm{Q}| \right)^{1/2}. \tag{2.13}$$

REMARK 2.6. Note that, in the indicated limit, the corresponding expression for $\|f\|_*$ is stronger. It is easy to see that for any collection $\{\alpha_{\mathrm{Q}}\}$ indexed by dyadic cubes,

$$\sup_{\mathrm{J\ dyadic}} \left( \sum_{\nu\geq -\log_2(\ell(\mathrm{J}))} \sum_{\mathrm{Q}_{\nu k}\subset\mathrm{J}} \alpha_{\mathrm{Q}_{\nu k}} \right) \leq \sum_{\nu\in\mathbb{Z}} \left( \sup_{\ell(\mathrm{J})\geq 2^{-\nu}} \sum_{\mathrm{Q}_{\nu k}\subset\mathrm{J}} \alpha_{\mathrm{Q}_{\nu k}} \right). \tag{2.14}$$

Indeed, let $\mathrm{J}_0$ be any fixed dyadic cube and set $\nu_0 = -\log_2(\ell(J_0))$. Split the sum over $\nu$ in the right hand side of (2.14) into $\sum_{\nu\leq\nu_0} + \sum_{\nu\geq\nu_0}$. Then, the second sum is a majorant for the left hand side of (2.14), as clearly $\ell(\mathrm{J}_0) \geq 2^{-\nu}$ for all such $\nu$. This observation should be compared with the well-know fact that the homogeneous Besov space $\dot{B}^0_{\infty,2}$ is strictly contained in BMO [**Trieb**].

Finally, we consider an atomic decomposition for BM spaces. We will use the $(s,p)$-atoms in [**FJ1**].

DEFINITION 2.7. A function $a$ is called an $(s,p)$-*atom*, if the following support, smoothness, and cancellation conditions are satisfied:

$$\operatorname{supp} a \subseteq 3\mathrm{Q} \tag{2.15a}$$

$$\int_{\mathbb{R}^n} x^\alpha\, a(x)\, dx = 0, \quad |\alpha| \leq L, \tag{2.15b}$$

$$|\partial^\alpha a(x)| \leq C\, |\mathrm{Q}|^{(s-n/p-|\alpha|)/n}, \quad |\alpha| \leq K, \tag{2.15c}$$

for some cube Q (not necessarily dyadic). 3Q is the cube concentric with Q of side-length $3\ell(\mathrm{Q})$. The numbers $K$ and $L$ are as in Definition 2.5.

Since atoms are a particular kind of molecules with compact support, it is sufficient to establish the analog of Theorem 1.

THEOREM 3. *Let* $f \in \mathcal{N}^s_{p,q,r}$*, then*

$$f = \sum_{\mathrm{Q}\ dyadic} s_{\mathrm{Q}}\, a_{\mathrm{Q}}$$

*in* $\mathcal{S}'/\mathcal{P}$*, where* $a_{\mathrm{Q}}$ *are* $(s,p)$*-atoms. Moreover,*

$$\left[ \sum_{\nu \in \mathbb{Z}} \left( \sup_{\substack{\mathrm{J}\ dyadic \\ \ell(\mathrm{J}) \geq 2^{-\nu}}} \left( \frac{1}{|\mathrm{J}|} \right)^{1-q/p} \sum_{\mathrm{Q}_{\nu k} \subset \mathrm{J}} |\mathrm{Q}|^{1-q/p} |s_{\mathrm{Q}}|^q \right)^{r/q} \right]^{1/r} \leq C\, \|f\|_{\mathcal{N}^s_{p,q,r}}, \tag{2.16}$$

*for some* $C = C(n,p,q,r)$*.*

Atoms give an exact localization and are useful, for examples, in proving trace theorems. However, the $a_{\mathrm{Q}}$ are not taken from a fixed family of functions and depend on $f$ non-linearly. $a_{\mathrm{Q}}$ and $s_{\mathrm{Q}}$ will be defined later.

NOTATION 2.8. In the rest of the paper, we indicate dyadic cubes by Q, while J may or may not be dyadic. In addition, $C$ stands for any immaterial constant, which will in general depend on $n$, $p$, $q$, $r$ and possibly $K$, $M$ or $L$, but not on the scale $\nu$ or location $k$ on the dyadic grid.

## 3. Proofs

The main ingredient is the following discrete Calderon reproducing formula (see [**FJ1**] for details).

LEMMA 3.1. *Let* $f \in \mathcal{S}'/\mathcal{P}$ *and let* $\{\psi_\nu\}$ *be a (homogeneous) Littlewood-Paley partition of unity. Then*

$$f(\cdot) = \sum_{\nu \in \mathbb{Z}} 2^{-n\nu} \sum_{k \in \mathbb{Z}^n} \check{\psi}_\nu * f(2^{-\nu}k)\, \check{\psi}_\nu(\cdot - 2^{-\nu}k)\,, \tag{3.1}$$

*where the convergence is in* $\mathcal{S}'/\mathcal{P}$*.*

The main idea behind the proof is to exploit the compact support of $\psi_\nu$ to extend $\psi_\nu\, \hat{f}$ to a periodic function of period $2^{\nu+1}\pi$ in $\mathbb{R}^n_\xi$, and then use Fourier series to represent it as a discrete sum of "wavenumbers" $2^{-\nu}k$ coupled to the frequency $\xi$. $f$ is now a superposition of elements of a given family of test functions with coefficients equal to the value of $f_\nu = \check{\psi}_\nu * f$ at the *sampling points* $2^{-\nu}k$.

The sampling functions can be different than $\psi_\nu$, as long as they satisfy some compatibility conditions [**FJ1**]. This more general version (given by the so-called $\phi$-transform) of Calderon's formula will be used later when dealing with atoms.

Using the notation introduced in the previous section and after some simple manipulations, (3.1) becomes

$$f = \sum_\nu 2^{-\nu n} \sum_k (\sigma_{\nu k}, f)\, \sigma_{\nu k}, \tag{3.2}$$

which is an almost orthogonal decomposition, as $(\sigma_{\nu k}, \sigma_{\mu l}) = 0$ if $|\nu - \mu| \geq 2$, or anyway negligible if $|k - l|$ is large enough.

Theorem 1 then follows from a Plancherel-Polya type estimate for Morrey norms [**Maz**].

Lemma 3.2. *Let $1 \le q \le p < \infty$, $\nu \in \mathbb{Z}$, $f \in \mathcal{S}'$. Suppose* $\operatorname{supp} \hat{f} \subset B(0, 2^{\nu+1})$. *Then,*

$$\left( \sup_{\ell(\mathrm{J}) \ge 2^{-\nu}} \left( \frac{1}{|\mathrm{J}|} \right)^{1-q/p} \sum_{\mathrm{Q}_{\nu k} \subset \mathrm{J}} |\mathrm{Q}|^{1-q/p} \sup_{\mathrm{Q}_{\nu k}} |f(x)|^q \right)^{1/q} \le C\, 2^{\nu(n/p)} \, \|f\|_{\mathcal{M}^p_q}, \tag{3.3}$$

*where* J *are dyadic cubes.*

We split the proof of Theorem 2 in two parts. We first establish a "wavelet" decomposition and, then, we indicate how to modify it for molecules.

Proposition 3.3. *If $s_{\mathrm{Q}}$ are numbers indexed by dyadic cubes, $f = \sum_{\mathrm{Q}} s_{\mathrm{Q}} \sigma_{\mathrm{Q}}$ in $\mathcal{S}'/\mathcal{P}$, and :*

$$\|f\|_* = \left[ \sum_{\nu \in \mathbb{Z}} \left( \sup_{\substack{\mathrm{J}\, dyadic \\ \ell(\mathrm{J}) \ge 2^{-\nu}}} \left( \frac{1}{|\mathrm{J}|} \right)^{1-q/p} \sum_{\mathrm{Q}_{\nu k} \subset \mathrm{J}} |\mathrm{Q}|^{1-q/p} |s_{\mathrm{Q}}|^q \right)^{r/q} \right]^{1/r} < \infty, \tag{3.4}$$

*then $f \in \mathcal{N}^s_{p,q,r}$ with $\|f\|_{\mathcal{N}^s_{p,q,r}} \approx \|f\|_*$.*

Proof. First, notice that:

$$\|f\|_{\mathcal{M}^p_q} \approx \sup_{\mathrm{J}\ \text{dyadic}} |\mathrm{J}|^{1/p - 1/q} \, \|f\|_{L^q(\mathrm{J})}, \tag{3.5}$$

since any cube J of length $2^{-\mu}$ ($\mu \in \mathbb{Z}$) can be covered by a fixed number $N$ of dyadic cubes Q of comparable length, $N$ depending only on the dimension $n$.

Next, we write:

$$f = \sum_{\nu \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}^n} s_{\mathrm{Q}_{\nu k}} \sigma_{\mathrm{Q}_{\nu k}} \right) = \sum_{\nu \in \mathbb{Z}} f_\nu,$$

and we observe that $\hat{f}_\nu$ is supported on the dyadic shell $D_\nu = \{\xi \mid 2^{\nu-1} < |\xi| < 2^{\nu+1}\}$. Therefore, it is enough to bound the Morrey norm of $f_\nu$ appropriately, in view of the following simple lemma, which was proved in [**Maz**]:

Lemma 3.4. *Let $\{f_\nu\}$ , $\nu \in \mathbb{Z}$, be a sequence of tempered distributions such that for some $A > 0$, $\operatorname{supp} \hat{f}_\nu \subset \{\xi \in \mathbb{R}^n \mid A\, 2^{\nu-1} < |\xi| < A\, 2^{\nu+1}\}$. Then*

$$\| \sum_\nu f_\nu \|_{\mathcal{N}^s_{p,q,r}} \le C(A) \, \left[ \sum_{\nu \in \mathbb{Z}} 2^{rs\nu} \|f_\nu\|^r_{\mathcal{M}^p_q} \right]^{1/r}, \tag{3.6}$$

*for all $s \in \mathbb{R}$, $1 \le q \le p < \infty$, $r \in [1, \infty]$.*

We fix $\nu$ for the moment and consider $f_\nu$ only. To simplify formulas, we put:

$$A_\nu = \left( \sup_{\ell(\mathrm{J}) \ge 2^{-\nu}} \left( \frac{1}{|\mathrm{J}|} \right)^{1-q/p} \sum_{\mathrm{Q}_{\nu k} \subset \mathrm{J}} |\mathrm{Q}|^{1-q/p} |s_{\mathrm{Q}}|^q \right)^{1/q}.$$

$A_\nu$ is finite because of (3.4). Then, by rescaling we can always assume that

$$A_\nu = 1. \tag{3.7}$$

Consequently, it is enough to show that

$$\|f_\nu\|_{\mathcal{M}^p_q} \le C\, 2^{-\nu s}, \tag{3.8}$$

with $C$ independent of $\nu$.

Let J be a dyadic cube and suppose $\ell(\mathrm{J}) = 2^{-\mu} \geq 2^{-\nu}$. The case $\ell(\mathrm{J}) < 2^{-\nu}$ will be consider later. We need to evaluate $\|f_\nu\|_{L^q(\mathrm{J})}$ in terms of $\ell(\mathrm{J})$.

Following [**Uch**] (see also [**St**]), we decompose $f_\nu$ into $f_\nu^{(1)} + f_\nu^{(2)}$, where:

$$f_\nu^{(1)} = \sum_{\mathrm{Q}_{\nu k} \subset \tilde{\mathrm{J}}} s_{\mathrm{Q}} \sigma_{\mathrm{Q}},$$
$$f_\nu^{(2)} = \sum_{\mathrm{Q}_{\nu k} \cap \tilde{\mathrm{J}} = \emptyset} s_{\mathrm{Q}} \sigma_{\mathrm{Q}},$$

and $\tilde{\mathrm{J}} = 3\mathrm{J}$. The idea is that $\|f_\nu^{(1)}\|_{L^q(\mathrm{J})} \approx \|f_\nu^{(1)}\|_{L^q(\mathbb{R}^n)}$, while the contribution of $f_\nu^{(2)}$ is small because the cubes $\mathrm{Q}_{\nu k}$ are now well separated from J.

The first estimate follows from the lemma below, which is a simple modification of Lemma 3.4 in [**FJ1**]. We postpone its proof.

Lemma 3.5. *Let $\mathcal{I}$ be any sub-collection of dyadic cubes of size $2^{-\nu}$. Let*

$$f = \sum_{\mathrm{Q} \in \mathcal{I}} s_{\mathrm{Q}} \sigma_{\mathrm{Q}}.$$

*Then,*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C\, 2^{-\nu s} \left( \sum_{\mathrm{Q} \in \mathcal{I}} |s_{\mathrm{Q}}|^q |\mathrm{Q}|^{1-q/p} \right)^{1/q}. \tag{3.9}$$

We apply the Lemma to $f_\nu^{(1)}$ with $\mathcal{I} = \{\mathrm{Q}_{\nu k} \subset \tilde{J}\}$. There is a small point to analyze, namely $\tilde{\mathrm{J}}$ is not a dyadic cube in general. However, as we observed before, it can be covered by a finite number of dyadic cubes $\mathrm{Q}^{(i)}$ of length $3\ell(\mathrm{J})$, $i = 1, \dots, N$. Furthermore, (3.9) is monotonic in $\mathcal{I}$. In particular, by the $1/q$-triangle inequality $(1/q \leq 1)$:

$$\left( \sum_{\mathrm{Q} \subset \tilde{J}} |s_{\mathrm{Q}}|^q |\mathrm{Q}|^{1-q/p} \right)^{1/q} \leq \sum_{i=1}^{N} \left( \sum_{\mathrm{Q} \subset \mathrm{Q}^{(i)}} |s_{\mathrm{Q}}|^q |\mathrm{Q}|^{1-q/p} \right)^{1/q}.$$

Consequently,

$$\begin{aligned} |\mathrm{J}|^{1/p-1/q} \|f_\nu^{(1)}\|_{L^q(\mathrm{J})} &\leq C\, 3^{n(1/q-1/p)} |\mathrm{Q}^{(i)}|^{1/p-1/q} \|f_\nu^{(1)}\|_{L^q(\mathbb{R}^n)} \\ &\leq C\, 2^{-\nu s}\, |\mathrm{Q}^{(i)}|^{1/p-1/q} \sum_{i=1}^{N} \left( \sum_{\mathrm{Q} \subset \mathrm{Q}^{(i)}} |s_{\mathrm{Q}}|^q |\mathrm{Q}|^{1-q/p} \right)^{1/q} \\ &\leq C\, 2^{-\nu s}, \end{aligned} \tag{3.10}$$

because of (3.7).

We now concentrate on $f_\nu^{(2)}$ and proceed similarly. From (3.7), $|s_{\mathrm{Q}}| \leq 1$, so that

$$
\begin{aligned}
(3.11)\qquad |\mathrm{J}|^{1/p-1/q}\|f_\nu^{(2)}\|_{L^q(\mathrm{J})} &\leq 2^{-\nu s}|\mathrm{J}|^{1/p-1/q} \sum_{\mathrm{Q}_{\nu k}\cap\tilde{\mathrm{J}}=\emptyset} |\mathrm{Q}|^{-1/p}\|\sigma_{\mathrm{Q}_{\nu k}}\|_{L^q(\mathrm{J})} \\
&\leq C\, 2^{-\nu s}\left(\frac{|\mathrm{J}|}{|\mathrm{Q}|}\right)^{1/p} \sum_{\mathrm{Q}_{\nu k}\cap\tilde{\mathrm{J}}=\emptyset} \frac{1}{(1+2^\nu|x_{\mathrm{Q}}-x_{\mathrm{J}}|)^M}.
\end{aligned}
$$

The factor $(|\mathrm{J}|/|\mathrm{Q}|)$ can be large, but $|x_{\mathrm{Q}}-x_{\mathrm{J}}| > \ell(\mathrm{J}) = 2^{-\mu}$ for the collection of cubes that appear in $f_\nu^{(2)}$. Moreover, as $\mu \leq \nu$, $x_{\mathrm{J}} = 2^{-\nu}\tilde{k}$, for some $\tilde{k} \in \mathbb{Z}^n$. Consequently,

$$1 + 2^\nu|x_{\mathrm{Q}} - x_{\mathrm{J}}| \sim 2^{\nu-\mu}|k - \tilde{k}|,$$

with $k \neq \tilde{k}$. Hence, by choosing $M$ appropriately, say $M = n/p + \alpha$, $\alpha > n$, we obtain:

$$
(3.12)\qquad |\mathrm{J}|^{1/p-1/q}\|f_\nu^{(2)}\|_{L^q(\mathrm{J})} \leq C\, 2^{-\nu s} \sum_{k|\mathrm{Q}_{\nu k}\cap\tilde{J}=\emptyset} \frac{1}{|k-\tilde{k}|^\alpha} \leq C\, 2^{-\nu s}.
$$

If $\ell(\mathrm{J}) < \ell(\mathrm{Q})$, i.e., $\mu > \nu$, the situation is actually simpler, because there are relatively few cubes that are not well separated from J.

Given J, there is a unique dyadic cube $\mathrm{Q}_{\mathrm{J}}$ at scale $2^{-\nu}$ that contains J. Again, only cubes $\mathrm{Q}_{\nu k}$ inside $3\mathrm{Q}_{\mathrm{J}}$ are not well separated from J, but now there are just $N = 3^n - 1$ of such cubes. On the “bad” cubes, $\sigma_{\nu k}$ is of order one, while on the remaining cubes we have the estimate:

$$
\begin{aligned}
(3.13)\qquad \sup_{x\in\mathrm{J}} |\sigma_{\nu k}(x)| &\leq C\,\frac{1}{(1+2^\nu|x_{\mathrm{Q}_J} - x_{\mathrm{Q}_{\nu k}}|)^M} \\
&\leq C\,\frac{1}{(1+|\tilde{k}-k|)^M},
\end{aligned}
$$

with $x_{\mathrm{Q}_{\mathrm{J}}} = 2^{-\nu}\tilde{k}$.
Therefore, similarly to (3.10) and (3.11):

$$
\begin{aligned}
(3.14)\qquad |\mathrm{J}|^{1/p-1/q}\,\|f_\nu\|_{L^q(\mathrm{J})} &\leq C\,|\mathrm{J}|^{1/p-1/q}\,2^{-\nu s}|\mathrm{Q}|^{-1/p}\cdot \\
&\cdot\left(\sum_{\mathrm{Q}\subset 3\mathrm{Q}_{\mathrm{J}}} |\mathrm{J}|^{1/q} + \sum_{\mathrm{Q}\cap 3\mathrm{Q}_{\mathrm{J}}=\emptyset} \frac{1}{(1+2^\nu|x_{\mathrm{Q}_J}-x_{\mathrm{Q}_{\nu k}}|)^M}|\mathrm{J}|^{1/q}\right) \\
&\leq C\,2^{-\nu s}\,|\mathrm{J}|^{1/p}|\mathrm{Q}|^{-1/p}\left(N + \sum_{k\in\mathbb{Z}^n}\frac{1}{(1+|\tilde{k}-k|)^M}\right) \leq C\,2^{-\nu s},
\end{aligned}
$$

since by hypothesis $|\mathrm{J}|/|\mathrm{Q}| < 1$. □

Proof of Lemma 3.5. Decompose $\mathbb{R}^n$ into cubes $\mathrm{Q}_{\nu l}$, $l \in \mathbb{Z}^n$, and use (2.8b), along with the definition of $\sigma_{\mathrm{Q}}$:

$$\begin{aligned}
(3.15)\quad \int_{\mathbb{R}^n} |f|^q\,dx &\le 2^{-\nu s q} \sum_{l\in\mathbb{Z}^n} \int_{\mathrm{Q}_{\nu l}} \left( \sum_{\mathrm{Q}_{\nu k}\in\mathcal{I}} |s_{\mathrm{Q}_{\nu k}}||\mathrm{Q}|^{-1/p} |\sigma_{\mathrm{Q}_{\nu l}}|\,dx \right)^q \\
&\le C\,2^{-\nu s q} \sum_{l\in\mathbb{Z}^n} |\mathrm{Q}|^{1-q/p} \left( \sum_{\mathrm{Q}_{\nu k}\in\mathcal{I}} |s_{\mathrm{Q}}| \frac{1}{(1+2^\nu |x_{\mathrm{Q}_{\nu k}} - x_{\mathrm{Q}_{\nu l}}|)^M} \right)^q \\
&\le C\,2^{-\nu s q} \sum_{l\in\mathbb{Z}^n} |\mathrm{Q}|^{1-q/p} \left( \sum_{\mathrm{Q}_{\nu k}\in\mathcal{I}} |s_{\mathrm{Q}}| \frac{1}{(1+|k-l|)^M} \right)^q .
\end{aligned}$$

Pick $M$ large and conclude using Young-Hausdorff inequality. □

As we indicated above, smooth $\sigma_{\mathrm{Q}}$ can be replaced by molecules $m_{\mathrm{Q}}$ of limited smoothness and vanishing moments. It is crucial that $m_{\mathrm{Q}}$ be concentrated on *dyadic* cubes, i.e., $x_0$ in Definition 2.5 must be of the form $x_{\mathrm{Q}} = 2^{-\nu}k$.

We rely on the following result of Frazier and Jawerth ([**FJ1**], Lemma 3.3) in order to bound the norm of $\psi_\mu(D)(\sum_{\ell(\mathrm{Q})=2^{-\nu}} m_{\mathrm{Q}})$, as Lemma 3.4 does not apply in this case:

$$(3.16)\qquad |\psi_\mu(D) m_{\mathrm{Q}}(x)| \le C\,2^{\nu(n/p-s)}\,2^{-(\nu-\mu)(L+n+1)} \frac{1}{(1+2^\mu|x-x_{\mathrm{Q}}|)^{M-L-n-1}},$$

if $\nu \ge \mu$, and

$$(3.17)\qquad |\psi_\mu(D) m_{\mathrm{Q}}(x)| \le C\,2^{\nu(n/p-s)}\,2^{-(\mu-\nu)K} \frac{1}{(1+2^\nu|x-x_{\mathrm{Q}}|)^{M-L-n-1}},$$

if $\nu \le \mu$. The proof uses the moment condition (2.11a).

Therefore, if $\nu \le \mu$, $\psi_\mu(D)m_{\mathrm{Q}}$ satisfies an estimate of the form (2.11b) and we can proceed as before to obtain:

$$(3.18)\qquad \|\psi_\mu(D)(\sum_{\ell(\mathrm{Q})=2^{-\nu}} s_{\mathrm{Q}} m_{\mathrm{Q}})\|_{\mathcal{M}^p_q} \le C\,2^{-\nu s} 2^{-(\mu-\nu)K}\,A_\nu,$$

which is the equivalent of (3.8), provided:

$$(3.19)\qquad M > L + n/p + 2n + 1.$$

When $\nu > \mu$, we simply rewrite (3.16) as

$$|\psi_\mu(D) m_{\mathrm{Q}}(x)| \le C\,2^{\nu(n/p-s)}\,2^{-(\nu-\mu)[2(L+n+1)-M]} \frac{1}{(1+2^\nu|x-x_{\mathrm{Q}}|)^{M-L-n-1}},$$

to conclude:

$$(3.20)\qquad \|\psi_\mu(D)(\sum_{\ell(\mathrm{Q})=2^{-\nu}} s_{\mathrm{Q}} m_{\mathrm{Q}})\|_{\mathcal{M}^p_q} \le C\,2^{-\nu s} 2^{-(\nu-\mu)[2(L+n+1)-M]}\,A_\nu,$$

with the same condition on $M$.

Then, Minkowski inequality ($p, q, r \geq 1$) gives:

$$\|\sum_{\mathrm{Q}} s_{\mathrm{Q}} m_{\mathrm{Q}}\|_{\mathcal{N}^s_{p,q,r}} \leq C \left[\sum_{\mu\in\mathbb{Z}} \left(\sum_{\nu=-\infty}^{\mu} 2^{-(\mu-\nu)(K-s)}\, A_\nu\right)^r\right]^{1/r} +$$

$$+ \left[\sum_{\mu\in\mathbb{Z}} \left(\sum_{\nu=\mu+1}^{+\infty} 2^{-(\nu-\mu)(2L+2n+2+s-M)}\, A_\nu\right)^r\right]^{1/r}$$

$$\leq C \left[\sum_{\nu\in\mathbb{Z}} \left(\sup_{\ell(\mathrm{J})\geq 2^{-\nu}} \left(\frac{1}{|\mathrm{J}|}\right)^{1-q/p} \sum_{\mathrm{Q}_{\nu k}\subset \mathrm{J}} |\mathrm{Q}|^{1-q/p} |s_{\mathrm{Q}}|^q\right)^{r/q}\right]^{1/r}.$$

The last line follows from Young-Hausdorff inequality: $\|a * b\|_{\ell^r} \leq \|a\|_{\ell^1} \|b\|_{\ell^r}$, as $K-s>0$ by construction, if in addition $M$ is chosen so that $2L+2n+2+s-M>0$. If we select $M \sim n/p + 2n + 2 + L$ to ensure (3.19), then it will be automatically satisfied, since $L \geq -s + n/p$.

We, now, turn to the proof of Theorem 3, namely the atomic decomposition of BM spaces.

Proposition 3.6. *If $f \in \mathcal{N}^s_{p,q,r}$, then there are $(s,p)$-atoms $a_{\mathrm{Q}}$ and numbers $s_{\mathrm{Q}}$ such that*

$$f = \sum_{\mathrm{Q}} s_{\mathrm{Q}}\, a_{\mathrm{Q}} \ \ \text{in } \mathcal{S}'/\mathcal{P},$$

*and*

$$\left[\sum_{\nu\in\mathbb{Z}} \left(\sup_{\ell(\mathrm{J})\geq 2^{-\nu}} \left(\frac{1}{|\mathrm{J}|}\right)^{1-q/p} \sum_{\mathrm{Q}_{\nu k}\subset \mathrm{J}} |\mathrm{Q}|^{1-q/p} |s_{\mathrm{Q}}|^q\right)^{r/q}\right]^{1/r} \leq C\, \|f\|_{\mathcal{N}^s_{p,q,r}}. \tag{3.21}$$

Again, $a_{\mathrm{Q}}$ must be in correspondence with dyadic cubes, since $p, q, r \geq 1$ (cf. [**Trieb**] and the discussion there).

$a_{\mathrm{Q}}$ and $s_{\mathrm{Q}}$ are constructed as follows. Pick a smooth, radial function $\theta$, supported on the unit ball with $L$ vanishing moments, and such that $\hat{\theta} \geq \epsilon > 0$ on the dyadic shell $\{1/2 < |\xi| < 2\}$. Then, there exist sampling functions $\psi_\nu$ so that (3.1) holds [**FJ1**].

For a fixed dyadic cube $\mathrm{Q} = \mathrm{Q}_{\nu k}$, define

$$a_{\mathrm{Q}}(x) = \frac{1}{s_{\mathrm{Q}}} \int_{\mathrm{Q}} \theta_\nu(x-y)(\sigma_\nu * f)(y)\, dy, \tag{3.22}$$

and

$$s_{\mathrm{Q}} = C\, 2^{\nu s} 2^{-\nu n/p} \sup_{x\in \mathrm{Q}} |\sigma_\nu * f(x)|, \tag{3.23}$$

with $C$ a large enough constant. $\theta_\nu$ is a rescaled version of $\theta$. By Lemma 3.2, (3.23) can actually be replaced by (2.10).

Then, the proposition is a straightforward consequence of Calderon's reproducing formula, in its more general form, and the Plancherel-Polya estimate (3.3), exactly as Theorem 1. We refer to [**FJ1**], [**Maz**] for details.

## 4. Atoms and wavelets in Morrey spaces

In [**FJ2**] using the $\phi$-transform, Frazier and Jawerth exhibit decompositions of $L^p$ and, more generally, Triebel-Lizorkin spaces into atoms and wavelet-like functions.

Since on one hand $\mathcal{M}^p_p = L^p$ and on the other $\mathcal{M}^p_q = BMO$ for $p \to \infty$, we expect a similar result to hold for Morrey spaces. The $L^p$ theory relies on interpolation for vector-valued singular integrals and duality. As pointed out earlier, Morrey spaces do not interpolate well [**BRV**] and are not dual of each others, so the extension is not straightforward. Here we hint at possible results, while complete proofs will be the focus of later work.

We start by obtaining a Littlewood-Paley theorem for $\mathcal{M}^p_q$ (and $M^p_q$). Without loss of generality, we can assume that the Littlewood-Paley components $\phi_0,\ \psi_\nu,\ j \geq 0$, are such that

$$\phi_0^2 + \sum_{\nu\geq 0} \psi_\nu^2 = 1.$$

PROPOSITION 4.1. *Let $f \in \mathcal{M}^p_q$, $1 < q \leq p < \infty$. Then,*

$$\|f\|_{\mathcal{M}^p_q} \approx \left\| \left( |\phi_0(D)f|^2 + \sum_{\nu\geq 0} |\psi_\nu(D)f|^2 \right)^{1/2} \right\|_{\mathcal{M}^p_q}. \tag{4.1}$$

PROOF. Consider the operator $\Psi : L^2(\mathbb{R}^n, \mathbb{R}) \to L^2(\mathbb{R}^n, \ell^2)$ defined by:

$$\Psi(f) = (\phi_0(D)f, \psi_1(D)f, \ldots, \psi_\nu(D)f, \ldots),$$

and its adjoint, $\Psi^* : L^2(\mathbb{R}^n, \ell^2) \to L^2(\mathbb{R}^n, \mathbb{R})$, given by:

$$\Psi^*(g) = \phi_0(D)g_0 + \sum_{\nu\geq 0} \psi_\nu(D)g_\nu,$$

if $g = \{g_\nu\}$. Clearly, $\Psi^* \circ \Psi = \mathrm{Id}$ on $L^2(\mathbb{R}^n)$. It is a standard fact (see, e.g. [**St**]) that $\Psi$ and $\Psi^*$ then extend as bounded operators on $L^p$ and $L^p(\ell^2)$ respectively. In fact, they are pseudo-differential operators with symbol in the class $\Sigma^0_1$. That is, the symbol of $\Psi$ satisfies:

$$\|D^\alpha_\xi p(\xi)\|_{\mathcal{L}(\mathbb{R},\ell^2)} \leq C_\alpha\, |\xi|^{-\alpha}, \quad \forall \alpha,$$

and similarly for $\Psi^*$, where the estimates are in the $\mathcal{L}(\ell^2, \mathbb{R})$ norm instead.

We need to prove that these operators act continuously on $\mathcal{M}^p_q$ and $\mathcal{M}^p_q(\mathbb{R}^n, \ell^2)$. $\mathcal{M}^p_q(\mathbb{R}^n, \ell^2)$ is defined in the obvious way:

$$\|f\|_{\mathcal{M}^p_q(\mathbb{R}^n,\ell^2)} = \sup_{x_0\in\mathbb{R}^n} \sup_{0<R} R^{n/p-n/q} \left( \int_{B(x_0,R)} \|f\|^q_{\ell^2}(y)\, dy \right)^{1/q}.$$

Peetre [**Pe1**] (see also [**Tay**]) showed that if $p$ is a scalar function in $\Sigma^0_1$, then

$$p(D) : \mathcal{M}^p_q \to \mathcal{M}^p_q, \quad \forall 1 < q \leq p < \infty, \tag{4.2}$$

with the same bound on the operator norm as in the $L^p$ case. The proof is based on corresponding mapping properties for $L^p$ spaces and a rescaling argument, so it extends to vector-valued operators like $\Psi$. $\square$

Note that interpolation is used only in proving $L^p$ estimates.

Remark 4.2. In (4.1) the Littlewood-Paley expansion can be replaced by a homogeneous sum:

$$\|f\|_{\mathcal{M}^p_q} \approx \left( \sup_{\text{J dyadic}} \frac{1}{|\mathrm{J}|^{1-q/p}} \int_{\mathrm{J}} \left( \sum_{\nu=-\infty}^{+\infty} |\psi_\nu(D) f|^2 \right)^{q/2} dx \right)^{1/q}. \tag{4.3}$$

It is enough to show that if $f \in \mathcal{M}^p_q$, $\sum_{\nu \le \mu} \psi_\nu(D)^2 f \to 0$ for $\mu \to -\infty$ so that $\sum_{\nu\in\mathbb{Z}} \psi_\nu(D)^2 f$ converges in $\mathcal{S}'$. The proof is identical to that for $L^p$ spaces. We refer to [**FJ2**] for details.

In fact, $\mathcal{M}^p_q \equiv \mathcal{L}^k_{q,n(1-q/p)}$, which is a homogeneous space, and the equality is intended in the following sense: for each equivalence class (modulo polynomials) $\tilde{f}$ in $\mathcal{L}^k_{q,n(1-q/p)}$ there is a unique canonical representative $f$ in $\mathcal{M}^p_q$, where $f$ is $\sum_{\nu\in\mathbb{Z}} \psi_\nu(D)^2 \tilde{f}$. For $p = q$, we recover the well-known identification $L^p \equiv F^0_{p,2} \equiv \dot{F}^0_{p,2}$ [**Trieb**]. Below we will not distinguish between $f$ and $\tilde{f}$.

We compare (4.3) with the definition of the homogeneous Triebel-Lizorkin space $\dot{F}^0_{\infty,r}$, $r \in [1,\infty)$ [**FJ2**]:

$$\|f\|_{\dot{F}^0_{\infty,r}} = \sup_{\text{J dyadic}} \left( \frac{1}{|\mathrm{J}|} \int_{\mathrm{J}} \sum_{\nu=-\log_2 \ell(\mathrm{J})}^{+\infty} |\psi_\nu(D) f|^r \, dx \right)^{1/r}. \tag{4.4}$$

Recall that $\dot{F}^0_{\infty,2} \equiv \mathrm{BMO}$.

Frazier and Javerth obtained an atomic decomposition for this space using the $\phi$-transform, so we expect that a similar result holds for Morrey spaces too.

Conjecture. *If $f \in \mathcal{M}^p_q$, then*

(1) $f = \sum_{\mathrm{Q}\ dyadic} s_{\mathrm{Q}} m_{\mathrm{Q}}$, *where $m_{\mathrm{Q}}$ are $(0,p)$ molecules (in particular, smooth "wavelets" $\sigma_{\mathrm{Q}}$),*

(2) $f = \sum_{\mathrm{Q}\ dyadic} s_{\mathrm{Q}} a_{\mathrm{Q}}$, *where $a_{\mathrm{Q}}$ are $(0,p)$ atoms,*

*and*

$$\|f\|_{\mathcal{M}^p_q} \approx \|f\|_* = \left\| \left( \sum_{\mathrm{Q}} |s_{\mathrm{Q}}|^2 \tilde{\chi}^2_{\mathrm{Q}} \right)^{1/2} \right\|_{\mathcal{M}^p_q}. \tag{4.5}$$

*with $\tilde{\chi}_{\mathrm{Q}} = |\mathrm{Q}|^{-1/p} \chi_{\mathrm{Q}}$ the $L^p$-normalized characteristic function of the cube* Q. $s_{\mathrm{Q}}$ *is defined as in* (3.23) *with* $s = 0$.

In particular, if $q = 2$, (4.5) becomes:

$$\|f\|_{\mathcal{M}^p_2} \approx \sup_{\text{J dyadic}} \left( \frac{1}{|\mathrm{J}|^{1-2/p}} \int_{\mathrm{J}} \sum_{\mathrm{Q}\subset\mathrm{J}} |s_{\mathrm{Q}}|^2 \tilde{\chi}_{\mathrm{Q}}(x)^2 \, dx \right)^{1/2} \tag{4.6}$$

$$= \sup_{\text{J dyadic}} \left( \frac{1}{|\mathrm{J}|^{1-2/p}} \sum_{\mathrm{Q}\subset\mathrm{J}} |s_{\mathrm{Q}}|^2 |\mathrm{Q}|^{1-2/p} \right)^{1/2}$$

So, we can think of $\|f\|^2_{\mathcal{M}_2^p}$ as equivalent to a modified Carleson norm, $\|\cdot\|_{C,p}$, of the measure

$$\mu = \sum_{\mathrm{Q}} |s_{\mathrm{Q}}|^2 |\mathrm{Q}|^{1-2/p}\, \delta_{(x_{\mathrm{Q}}, \ell(\mathrm{Q}))},$$

where

$$\|\mu\|_{C,p} = \sup_{\mathrm{J}\ \text{dyadic}} \left( \frac{1}{|\mathrm{J}|^{1-2/p}} \mu(\mathrm{T}(\mathrm{J})) \right),$$

and

$$\mathrm{T}(\mathrm{J}) = \{(x,t) \in \mathbb{R}^{n+1}_+ \mid 0 \le t \le \ell(\mathrm{J}), x \in \mathrm{J}\}.$$

Again, the limit $p \to \infty$ gives the usual characterization of BMO in terms of Carleson norms. (4.6) is also consistent with the wavelet expansion in [**Fe**] and [**Can**], if we take into account that the wavelets used there are $L^2$-normalized.

Unfortunately, the methods in [**FJ1**] do not seem to apply here. For a fixed dyadic cube J, only scales smaller than $\ell(\mathrm{J})$ appear in (4.4), so that essentially

$$\int_{\mathrm{J}} \sum_{\nu=-\log_2 \ell(\mathrm{J})}^{+\infty} |\psi_\nu(D) f|^r \, dx \approx \int_{\mathrm{J}} \sum_{\mathrm{Q}_{\nu k} \subset \mathrm{J}} (\sup_{\mathrm{Q}_{\nu k}} |\psi_\nu(D) f|)^r \chi_{\mathrm{Q}_{\nu k}}(x) \, dx.$$

Rigorous estimates are obtained through a reduction to the $L^p$ case, $1 < p < \infty$. On the other hand, there is a contribution at every scale in (4.3).

For $L^p$, (4.5) follows from the generalized version of Calderon's formula, the Littlewood-Paley theorem, and the Fefferman-Stein vector-valued maximal inequality. Though the Hardy-Littlewood maximal function $M$ is bounded on Morrey spaces [**CF**],[**Na**], the vector case does not follow from the scalar case, because $M$ is non-linear. For the same reason, we cannot reduce to the $L^p$ case by rescaling arguments.

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