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Weak solutions, renormalized solutions and enstrophy defects in 2D turbulence

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Abstract

Enstrophy, half the integral of the square of vorticity, plays a role in 2D turbulence theory analogous to that played by kinetic energy in the Kolmogorov theory of 3D turbulence. It is therefore interesting to obtain a description of the way enstrophy is dissipated at high Reynolds number. In this article we explore the notions of viscous and transport enstrophy defect, which model the spatial structure of the dissipation of enstrophy. These notions were introduced by G. Eyink in an attempt to reconcile the Kraichnan-Batchelor theory of 2D turbulence with current knowledge of the properties of weak solutions of the equations of incompressible and ideal fluid motion. Three natural questions arise from Eyink's theory: (1) Existence of the enstrophy defects (2) Conditions for the equality of transport and viscous enstrophy defects (3) Conditions for the vanishing of the enstrophy defects. In [10], Eyink proved a number of results related to these questions and formulated a conjecture on how to answer these problems in a physically meaningful context. In the present article we improve and extend some of Eyink's results and present a counterexample to his conjecture.

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Contents

1. Introduction

This article is concerned with certain properties of weak solutions of the incompressible Euler equations in two space dimensions and with the corresponding vanishing viscosity limit in connection with the modeling of two-dimensional turbulence. To put our discussion in context it is useful to recall some of the basic features of the Kraichnan-Batchelor (KB) theory of two-dimensional turbulence, introduced in [12, 2]. This is a phenomenological theory, modeled after Kolmogorov's theory of 3D turbulence. The notion of *enstrophy* cascade plays a central role in KB theory, similar to the role of the energy cascade in Kolmogorov's theory. Enstrophy is half the integral of the square of vorticity, a conserved quantity for smooth ideal 2D flow, which is dissipated in viscous flow. In the cascade picture, the nonlinearity transports enstrophy from large to small scales, where it is dissipated by viscosity. A key issue in the KB theory is that such a picture must be sustained as viscosity vanishes, in a way that allows the rate at which enstrophy is dissipated to remain bounded away from zero as viscosity disappears. For details and the associated literature we refer the reader to [11], especially Section 9.7, and references there contained.

Let us consider a family of viscous flows, which we assume to have uniformly bounded enstrophy as viscosity vanishes. This sequence is compatible with the KB cascade if the enstrophy dissipation rate is bounded away from zero. Taking subsequences as needed, such a family leads to a weak solution of the 2D incompressible Euler equations, see [15], which must dissipate enstrophy. The difficulty one faces is that weak solutions of the incompressible 2D Euler equations with finite enstrophy conserve enstrophy exactly, a known fact which we will examine in detail later. We note that this difficulty does not occur in 3D, as energy dissipative solutions of the incompressible 3D Euler equations with finite initial energy have been shown to exist, see [8, 18].

Recently, G. Eyink proposed a way around the paradox outlined above, see [10], by considering flows with unbounded local enstrophy. Eyink's idea raises the mathematical problem of assigning meaning to enstrophy dissipation for flows with infinite enstrophy. In [10], Eyink introduced two notions of enstrophy defect in his attempt to describe the spatial structure of the enstrophy dissipation. These enstrophy defects are limits of enstrophy source terms in approximating enstrophy balance equations. When the relevant approximation is vanishing viscosity, this limit gives rise to a viscous enstrophy defect. The other defect introduced by Eyink was a purely inviscid enstrophy defect associated with mollifying a weak solution, which we call transport enstrophy defect. Eyink formulated a conjecture stating that both enstrophy defects are well-defined, that they give rise to the same distribution in the limit and that they do not always vanish. One of the main purposes of the present work is to present a counterexample to Eyink's conjecture.

Beyond the description of 2D turbulence, there are two other concerns that motivate this paper. The first is the problem of uniqueness of weak solutions for incompressible 2D Euler, a long-standing open problem. Existence of weak solutions is known for compactly supported initial vorticities in the space $(\mathcal{B}M_+ + L^1) \cap H^{-1}_{\text{loc}}$, where $\mathcal{B}M_+$ is the cone of nonnegative bounded Radon measures, see [5, 17, 23]. In contrast, uniqueness of weak solutions is only known for vorticities which are bounded or nearly so, see [22, 24, 25]. It is conceivable that the usual notion of weak solution is too weak to guarantee uniqueness, and that a criterion is required to select the 'correct' weak solution. Properties that distinguish those weak solutions which are inviscid limits are particularly interesting, and we will encounter some of these properties in this paper.

The second concern is connected with the general issue of inviscid dissipation. Transport by smooth volume-preserving flows merely rearranges the transported quantity. This property is maintained even when the flow is not smooth, as long as we restrict ourselves to renormalized solutions of the transport equations, in the sense of DiPerna and Lions, see [6]. Weak solutions (in the sense of distributions) of transport equations by divergence-free vector fields are always renormalized solutions if the transported quantity and the transporting velocity are sufficiently smooth. In the special case of weak solutions of the 2D Euler equations vorticity is always a renormalized solution of the vorticity equation, regarded as a linear transport equation, as long as enstrophy is finite. Consequently, for finite enstrophy flows the distribution function of vorticity is conserved in time. What happens with the distribution function of vorticity under less regular flows is a very interesting problem, closely related to the present work.

The remainder of this article is divided into six sections. In Section 2 we review the DiPerna-Lions transport theory and we apply it to ideal, incompressible, two-dimensional flow. In Section 3 we introduce the enstrophy defects, we prove that the viscous enstrophy defect vanishes for flows with finite enstrophy and we formulate a version of Eyink's conjecture. In Section 4 we prove that the enstrophy density associated to a viscosity solution is a weak solution of a transport equation as long as vorticity lies in the space $L^2(\log L)^{1/4}$, an Orlicz space slightly smaller than L^2. We also show that the transport enstrophy defect exists as a distribution for vorticities in $L^2(\log L)^{1/4}$ and vanishes if the weak solution in this space happens to be an inviscid limit. In Section 5 we present examples showing that the results obtained in the previous section are nearly sharp. In Section 6 we exhibit a counterexample to Eyink's conjecture. Finally, we draw some conclusions and highlight open problems in Section 7.

Technically speaking, we make use of the framework usually found in the study of nonlinear problems through weak convergence methods as well as harmonic analysis and function space theory. One distinction between our work and [10] is that we consider flows in the full plane with compactly supported initial vorticity, whereas Eyink dealt with periodic flows. Working in the plane is convenient because of the simpler expression for the Biot-Savart law and because it is easier to find the function space results we require. The trade-off is the need to work around problems arising from infinity, such as loss of tightness along vorticity sequences.

We conclude this introduction by fixing notation. We denote by $B(x;r)$ the disk centered at x with radius r in the plane. The characteristic function of a set E is denoted by χ_E. If X is a function space then X_c denotes the subspace of functions in X with compact support and X_{loc} denotes the space of functions which are locally in X. We use alternatively C_c^∞ or $\mathcal{D}$ to denote the space of smooth compactly-supported test functions. We use $W^{k,p}$ and H^s to denote the classical Sobolev spaces. We denote by $L^{p,q}$ the Lorentz spaces and $B^s_{p,q}$ the Besov spaces as defined respectively in [4] and [3].

2. Weak solutions and renormalized solutions

The purpose of this section is to discuss the relation between weak solutions of the incompressible 2D Euler equations and DiPerna-Lions renormalized solutions of linear transport equations.

We begin by recalling the vorticity formulation of the two dimensional Euler equations:

$$\partial_t \omega + u \cdot \nabla \omega = 0, \tag{2.1a}$$

$$u = K * \omega, \tag{2.1b}$$

with the Biot-Savart kernel K given by

$$K(x) \equiv \frac{x^\perp}{2\pi |x|^2},$$

$(x_1, x_2)^\perp = (-x_2, x_1)$, and where the convolution in (2.1b) occurs only in the spatial variable. Note that the specific form of the Biot-Savart kernel implies that $\operatorname{div} u = 0$.

Identity (2.1a) is a transport equation for the vorticity. Therefore, if u is sufficiently smooth so that ω is a classical solution, the vorticity itself and any function of it are transported along the flow induced by u. In particular, the enstrophy density function $\vartheta(x,t) = |\omega(x,t)|^2/2$ is conserved along particle trajectories, and, as the velocity u is divergence-free, the enstrophy $\Omega(t) \equiv \int \vartheta(x,t)\,dx$ is a globally conserved quantity in time.

There is a well-developed theory of weak solutions for (2.1). Well-posedness for weak solutions has been established for those initial vorticities which are bounded or nearly so, see [24, 25, 21, 22]. If vorticity belongs to L^p then, by Calderon-Zygmund theory and the Hardy-Littlewood-Sobolev inequality, $u \in W^{1,p}_{\text{loc}}$ so that, if $p \geq 4/3$ then $u \in L^{p'}$ with $p' = p/(p-1)$. Hence the relevant nonlinear term, $u\,\omega$, is locally integrable and the transport equation (2.1a) lends itself to a standard weak formulation. To be precise we recall the weak formulation of the initial-value problem for (2.1). Let $\omega_0 \in L^p(\mathbb{R}^2)$, $p \geq 4/3$.

Definition 1. Let $\omega = \omega(x,t) \in L^\infty([0,T);L^p(\mathbb{R}^2))$ for some $p \geq 4/3$ and let $u = K * \omega$. We say ω is a weak solution of the initial-value problem for (2.1) if, for any test function $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^2)$, we have:

$$\int_0^T \int_{\mathbb{R}^2} \varphi_t \omega + \nabla\varphi \cdot u\,\omega \; dxdt + \int_{\mathbb{R}^2} \varphi(x,0)\omega_0(x)\; dx = 0.$$

In addition, we require that the velocity field $u \in L^\infty([0,T);L^2(\mathbb{R}^2) + L^\infty(\mathbb{R}^2))$.

Existence of weak solutions has been established for initial vorticities $\omega_0 \in (\mathcal{B}M_{c,+} + L^1_c) \cap H^{-1}_{\text{loc}}$, see [7,5,23,17]; however, these results require a more elaborate weak formulation in order to accommodate the additional irregularity in vorticity. If the vorticity is in L^p for some $p \geq 4/3$ then all weak formulations reduce to the one in Definition 1. In this paper we are mostly concerned with flows whose vorticity is in L^2 or nearly so, and for these flows, Definition 1 is adequate. There is one situation of present interest for which Definition 1 cannot be used, namely, that of vorticities in the Besov space $B^0_{2,\infty}$. In this case a weak velocity formulation, see [7], should be used instead.

Given that, for vorticities in L^p, the velocities are only $W^{1,p}_{\text{loc}}$, it is natural to consider weak solutions of (2.1) in the context of the theory of renormalized solutions for linear transport equations, introduced by DiPerna and Lions [6]. We recall below the definition of renormalized solution for linear transport equations without lower-order term.

If $E \subseteq \mathbb{R}^n$ then $|E|$ denotes the Lebesgue measure of E. Let L^0 be the set of all measurable functions f on $\mathbb{R}^n$ such that $|\{|f(x)| > \alpha\}| < \infty$, for each $\alpha > 0$. Let $v \in L^1([0,T];W^{1,1}_{\text{loc}})$ such that

$$(1 + |x|)^{-1}\, v \in L^1([0,T];L^1) + L^1([0,T];L^\infty). \tag{2.2}$$

Definition 2. A function $\omega \in L^\infty([0,T];L^0)$ is called a *renormalized* solution to the linear transport equation

$$\omega_t + v \cdot \nabla\omega = 0$$

if, in the sense of distributions,

$$\partial_t \beta(\omega) + v \cdot \nabla\beta(\omega) = 0, \tag{2.3}$$

for all $\beta \in \mathcal{A} = \{\beta \in C^1, \beta \text{ bounded, vanishing near } 0\}$.

The most important property of renormalized solutions is that, in general, they are unique. The connection between weak solutions of the Euler equations and renormalized solutions of the vorticity equation (2.1a), regarded as a linear transport equation with given velocity, is known. However, this relation has not been clearly stated in the literature. We address this omission in the following result.

Proposition 1. *Let $p \geq 2$. If $\omega = \omega(x,t) \in L^\infty([0,T); L^p(\mathbb{R}^2))$ is a weak solution of the Euler equations then ω is a renormalized solution of transport equation* (2.1a) *with velocity $u = K * \omega$. Let $1 < p < 2$. If ω is a weak solution of the Euler equations obtained as a weak limit of a sequence of exact smooth solutions (generated, for example, by mollifying initial data and exactly solving the equations) then ω is a renormalized solution of* (2.1a).

Proof. If $p \geq 2$, then the velocity u belongs to $L^\infty([0,T); W^{1,p}_{\text{loc}})$ and hence to $L^\infty([0,T); W^{1,p'}_{\text{loc}})$, as $p \geq p'$. The velocity u satisfies the mild growth condition (2.2) because $L^2 + L^\infty$ is contained in $L^1 + L^\infty$ and an $L^2 + L^\infty$ estimate on velocity was required in the definition of weak solution. Hence, we are under the conditions of the consistency result, Theorem II.3 in [6], so we may conclude that ω is a renormalized solution. The statement regarding weak solutions that are limits of exact smooth solutions is a consequence of the stability result contained in Theorem II.4 in [6].

It is an interesting question whether the vanishing viscosity limit gives rise to a renormalized solution as well, if the initial vorticity is in L^p, $1 < p < 2$.

Let $\omega \in L^\infty([0,T); L^2_c(\mathbb{R}^2))$ be a weak solution of (2.1). By Proposition 1 ω is also a renormalized solution. Since the velocity is divergence-free, we may conclude, using the full strength of the DiPerna-Lions theory of renormalized solutions, that the distribution function of ω is time-independent, i.e.:

$$\lambda_\omega(s,t) \equiv |\{x \in \mathbb{R}^2 \,|\, |\omega(x,t)| > s\}| = \lambda_\omega(s,0) \equiv \lambda_{\omega_0}(s), \tag{2.4}$$

see the second Theorem III.2 of [6]. Therefore, all rearrangement-invariant norms of vorticity are conserved in time. In particular, the enstrophy $\Omega(t)$ is preserved for any weak solution of the 2D Euler equations with finite initial enstrophy.

3. Two notions of enstrophy defect and Eyink's conjecture

In this section we will introduce two notions of enstrophy defect, one associated with enstrophy dissipation due to viscosity and another associated with enstrophy disappearance due to irregular transport. We will also state precisely a version of Eyink's conjecture in the setting of full-plane flow.

Let $\omega \in L^\infty([0,T); L^{4/3}(\mathbb{R}^2))$ be a weak solution of (2.1). Set $j_\epsilon(x) = \epsilon^{-2} j(\epsilon^{-1}x)$ to be a Friedrichs mollifier and write

$$\omega_\epsilon = j_\epsilon * \omega,$$

$$u_\epsilon = j_\epsilon * u,$$

$$(u\,\omega)_\epsilon = j_\epsilon * (u\,\omega).$$

Then ω_ϵ solves

$$\begin{aligned} &\partial_t \omega_\epsilon + \operatorname{div}\left[u_\epsilon \omega_\epsilon + ((u\omega)_\epsilon - u_\epsilon \omega_\epsilon)\right] = 0, \\ &\omega_\epsilon(0) = j_\epsilon * \omega_0. \end{aligned} \tag{3.1}$$

The associated enstrophy density $\vartheta_\epsilon(x,t) = |\omega_\epsilon(x,t)|^2/2$ satisfies

$$\partial_t \vartheta_\epsilon + \operatorname{div}\left[u_\epsilon \vartheta_\epsilon + \omega_\epsilon\left((u\omega)_\epsilon - u_\epsilon \omega_\epsilon\right)\right] = -Z_\epsilon(\omega), \tag{3.2}$$

where

$$Z_\epsilon(\omega) = -\nabla \omega_\epsilon \cdot \left((u\omega)_\epsilon - u_\epsilon \omega_\epsilon\right).$$

The behavior of Z_ϵ as $\epsilon \to 0$ is a description of the space-time distribution of enstrophy dissipation of the weak solution ω due to irregular transport. We use this notion to define the enstrophy defect.

Definition 3. The transport enstrophy defect associated to ω is:

$$Z^T(\omega) \equiv \lim_{\epsilon \to 0} Z_\epsilon(\omega),$$

whenever the limit exists in the sense of distributions. The weak solution ω is said to be *dissipative* if $Z^T(\omega)$ exists and $Z^T(\omega) \geq 0$.

Given that the transport enstrophy defect is intended to describe the space-time structure of enstrophy dissipation and taking into account that finite-enstrophy weak solutions conserve enstrophy, one would hope that $Z^T(\omega) \equiv 0$ if $\omega_0 \in L^2_c(\mathbb{R}^2)$. Actually, this seems to be a difficult problem, to which we will return later on in this work. Recall that, in the 3D case, it is known that finite energy solutions may dissipate energy, see [8,18].

From a physical point of view it is natural to consider weak solutions arising through the vanishing viscosity limit. We denote by ω_ν the solution to the two-dimensional Navier-Stokes equations in velocity-vorticity form:

$$\partial_t \omega_\nu + u_\nu \cdot \nabla \omega_\nu - \nu \Delta \omega_\nu = 0, \tag{3.3a}$$

$$u_\nu = K * \omega_\nu, \tag{3.3b}$$

with initial data ω_0. Note that $\operatorname{div} u_\nu = 0$.

The Navier-Stokes evolution naturally dissipates enstrophy, though only through diffusion. The viscous enstrophy density ϑ_ν satisfies the following parabolic equation:

$$\partial_t \vartheta_\nu + u_\nu \cdot \nabla \vartheta_\nu - \nu \Delta \vartheta_\nu = -Z^\nu(\omega_\nu), \tag{3.4}$$

where

$$Z^\nu(\omega_\nu) = \nu |\nabla \omega_\nu|^2.$$

Note that $Z^\nu(\omega_\nu) \geq 0$ always. We use Z^ν to define a viscous enstrophy defect. Let $\omega = \omega(x,t) \in L^\infty([0,T); L^p(\mathbb{R}^2))$, $p \geq 4/3$, be a weak solution of the 2D Euler equations which was obtained as a vanishing viscosity limit. More precisely, we assume that ω is a limit of a sequence of solutions to the 2D Navier-Stokes equations (3.3) with fixed initial data ω_0 and with viscosity $\nu_k \to 0$. In what follows we will refer to such a weak solution as a *viscosity solution.* Let ω_{ν_k} be such an approximating sequence of solutions, with $\omega_{\nu_k} \rightharpoonup \omega$, weak-$*$ in $L^\infty([0,T);L^p(\mathbb{R}^2))$. Henceforth we will abuse terminology and identify the sequence $\{\omega_{\nu_k}\}$ with its weak (inviscid) limit ω.

Definition 4. The viscous enstrophy defect associated to ω is defined as:

$$Z^V(\omega) \equiv \lim_{\nu_k \to 0} Z^{\nu_k}(\omega_{\nu_k}),$$

whenever the limit exists in the sense of distributions.

Before we formulate Eyink's conjecture we show that, if the initial vorticity has finite enstrophy, then the viscous enstrophy defect vanishes identically.

Proposition 2. *Let $\omega_0 \in L^2_c(\mathbb{R}^2)$. Let $\omega \in L^\infty([0,T);L^2(\mathbb{R}^2))$ be a viscosity solution with initial vorticity ω_0. Then $Z^V(\omega)$ exists and it is identically zero.*

Proof. Suppose that the viscosity solution $\omega = \omega(x,t)$ is the limit of the approximating sequence ω_{ν_k} of solutions to the Navier-Stokes equations. We may assume that $\omega_{\nu_k} \to \omega$ in $C([0,T), w-L^2)$, where $w-L^2$ is L^2 endowed with the weak topology, see [13], Appendix C. Multiplying (3.3) by ω_{ν_k}, integrating by parts, and using the divergence-free condition on u_{ν_k}, gives for each fixed ν_k and $t > 0$

$$\frac{d}{dt}\int_{\mathbb{R}^2} \omega_{\nu_k}^2(t)\,dx + 2\nu_k \int_{\mathbb{R}^2} |\nabla \omega_{\nu_k}|^2(t)\,dx = 0.$$

By integrating in time, we then obtain the same energy estimate as for the heat equation, namely

$$\|\omega_{\nu_k}(t)\|_{L^2}^2 - \|\omega_0\|_{L^2}^2 = -2\nu_k \int_0^t \int_{\mathbb{R}^2} |\nabla \omega_{\nu_k}|^2\,dx\,ds, \quad \forall\, 0 < t < T. \tag{3.5}$$

From Proposition 1 it follows that ω is a renormalized solution to (2.1a) and hence $\|\omega(t)\|_{L^2}^2 = \|\omega_0\|_{L^2}^2$. Therefore, if $\omega_{\nu_k}(t)$ converges *strongly* in L^2 to $\omega(t)$, for each $0 < t < T$, then we have that

$$\lim_{\nu_k \to 0} \int_0^t \int_{\mathbb{R}^2} \nu_k |\nabla \omega_{\nu_k}|^2\,dx\,ds = 0. \tag{3.6}$$

This means in particular that $\lim_{\nu_k \to 0} Z^{\nu_k}(\omega_{\nu_k}) = Z^V(\omega) \equiv 0$ in the sense of distributions.

To establish strong convergence of the approximating sequence, we notice that, from (3.5), $\|\omega_{\nu_k}(t)\|_{L^2} \le \|\omega_0\|_{L^2}$ for each $t > 0$, so that

$$\limsup_{\nu_k \to 0} \|\omega_{\nu_k}(t)\|_{L^2} \le \|\omega_0\|_{L^2} = \|\omega(t)\|_{L^2}.$$

On the other hand, it follows from the weak lower semicontinuity of the norm that

$$\liminf_{\nu_k \to 0} \|\omega_{\nu_k}(t)\|_{L^2} \geq \|\omega(t)\|_{L^2},$$

as $\omega_{\nu_k} \to \omega$ in $C([0,T); w - L^2)$. Thus $\|\omega_{\nu_k}(t)\|_{L^2} \to \|\omega(t)\|_{L^2}$ for each $0 < t < T$, from which the desired strong convergence follows.

Remark 1. In the case of periodic flow it is possible to show that, if ω is a dissipative weak solution in $L^\infty([0,T); L^2)$, then the transport enstrophy defect $Z^T(\omega)$ vanishes identically. The proof is an easy adaptation of what was presented above. For the full plane, there are serious technical difficulties with controlling the behavior of Z_ϵ near infinity, which are connected with understanding the possibility of enstrophy leaving the compact parts of the plane. The main concern of the present article is with local enstrophy dissipation so we will avoid this issue of escape to infinity.

Turbulence theory requires flows that dissipate enstrophy at a rate which does not vanish as viscosity goes to zero. A vanishing viscous enstrophy defect excludes precisely such flows. From Proposition 2, we see that in order to model two-dimensional turbulence, one should consider flows with infinite enstrophy. Is it possible for flows with infinite enstrophy to dissipate enstrophy in a meaningful way? This is the main point in Eyink's work and it is precisely what we wish to explore.

A natural choice of space which allows for infinite enstrophy is the L^2-based Besov space $B^0_{2,\infty}$. The choice of the space $B^0_{2,\infty}$ is motivated by the Kraichnan-Batchelor theory of two-dimensional turbulence, which predicts, in the limit of vanishing viscosity, an energy spectrum of the form

$$E(\kappa, t) \sim \eta(t)^{2/3}\, \kappa^{-3}. \tag{3.7}$$

Above, $\eta(t)$ is the average rate of enstrophy dissipation per unit volume, and $E(\kappa, t)$ is the density of the measure μ given by

$$\mu(A) = \int_A E(\kappa, t)\, d\kappa = \int_{A \times S^1} |\widehat{u}(k,t)|^2\, dk,$$

with $\kappa = |k|$, for any measurable subset A of the real line.

By Calderon-Zygmund, $\omega \in L^2([0,T]; B^0_{2,\infty})$ implies that the velocity $u \in$
$L^2([0,T]; B^1_{2,\infty})$ locally, and [20]

$$\|u\|^2_{B^1_{2,\infty}} \approx \sup_{0 < s \leq 1} s^2 \|\psi(s\cdot)\widehat{u}\|^2_{L^2}, \tag{3.8}$$

for ψ a smooth cut-off function supported in the dyadic shell $\{k \mid 1/2 < |k| < 2\}$. By rescaling (here $s = \kappa^{-1}$), a finite $B^1_{2,\infty}$ norm gives a decay rate like (3.7) for the energy spectrum.

In this situation, Eyink's conjecture embodies the expectation that the transport enstrophy defect accounts for the residual rate of viscous enstrophy dissipation in the limit of vanishing viscosity. One of the main results in the present work is an example showing that this is not necessarily the case.

Conjecture 1 (Eyink). Let ω be a weak solution of the incompressible 2D Euler equations, obtained by the vanishing viscosity method, such that $\omega \in L^2((0,T); B^0_{2,\infty}(\mathbb{R}^2))$. We assume that there exists ω_{ν_k}, solutions of the incompressible Navier-Stokes equations (3.3), such that

$$\omega_{\nu_k} \rightharpoonup \omega \text{ in weak-} * \ L^2((0,T); B^0_{2,\infty}(\mathbb{R}^2))$$

Then both limits, $\lim_{\nu\to 0^+} Z^\nu(\omega_\nu)$ and $\lim_{\epsilon\to 0^+} Z_\epsilon(\omega)$, exist and are equal, so that we may write $Z(\omega) = Z^V(\omega) = Z^T(\omega)$ in this case. Furthermore, ω is a dissipative solution. Lastly, there exist one such ω with $Z(\omega) > 0$.

The space $B^0_{2,\infty}$ has the disadvantage of not being rearrangement-invariant, which means that it provides no natural estimate for vorticity. In addition, $B^0_{2,\infty}$ is not contained in $L^{4/3}$, so that a weak solution in this Besov space has to be defined in a different way than what we did in Definition 1, namely using the weak velocity formulation as in [7].

From an analytical standpoint, it is natural to reformulate Eyink's conjecture replacing $B^0_{2,\infty}$ by a rearrangement invariant space containing L^2. In that case, the existence of a viscosity weak solution follows from appropriate hypotheses on initial data, so that the statement of the conjecture would become simpler. One straighforward choice is the Marcinkiewicz space $L^{2,\infty}$, which is rearrangement invariant. Additionally, vorticities in $L^{2,\infty}$ which are supported in sets of finite measure also belong to $L^{4/3}$, so that Definition 1 can be used. Although $L^{2,\infty}$ and $B^0_{2,\infty}$ are both endpoints of secondary scales of spaces based on L^2, the precise relation between them has not been clearly stated in the literature.

The conjecture stated above differs from Eyink's original formulation in that it refers to full-plane instead of periodic flow, a distinction which is more technical than substantive. One of the purposes of the present article is to produce an example of a weak solution, under the constraints of the conjecture, for which both Z^T and Z^V exist, $Z^T \equiv 0$ but Z^V does not vanish. The example we will present belongs to $L^{2,\infty} \cap B^0_{2,\infty}$. Before we present the construction of this example, we will examine in more detail the behavior of the enstrophy defects in the case of finite enstrophy. This is the subject of the next two sections.

4. Transport enstrophy defect and local balance of enstrophy

We have established that, if the initial vorticity has finite enstrophy, then the (renormalized) weak solution conserves enstrophy exactly (Proposition

1 and subsequent observation) and that for viscosity solutions, the viscous enstrophy defect Z^V vanishes. This result implies that, for modeling 2D turbulence, flows with bounded enstrophy are not useful, since they cannot support a cascade. However, independently from its physical relevance, the idea of transport enstrophy defect is very intriguing from the point of view of nonlinear PDE. One of the most interesting problems is whether transport enstrophy dissipation occurs at all, a nontrivial open question . In [10], Eyink proved that if the vorticity is in L^p, $p > 2$ then Z^T exists and vanishes identically. Our main purpose in this section is to examine transport enstrophy dissipation in more detail, looking for criticality in spaces which are logarithmic perturbations of L^2.

We begin by considering local balance of enstrophy. One of the ways in which this balance can be expressed is by showing that the enstrophy density ϑ satisfies the transport equation $\vartheta_t + u \cdot \nabla \vartheta = 0$. We first note that, if the initial vorticity ω_0 belongs to L^2_c and if ω is any weak solution with initial vorticity ω_0, then the corresponding enstrophy density $\vartheta = |\omega|^2/2$ is a *renormalized* solution of the above transport equation. The proof of this fact follows from the knowledge that ω itself is a renormalized solution (in this case) and that, if $\beta(s)$ is an admissible renormalization, then so is $\beta(s^2)$. This observation is a Lagrangian perspective on local enstrophy balance, but it cannot be immediately translated into Eulerian information. We cannot prove that ϑ is a weak (distributional) solution of the same transport equation because of the difficulty in making sense of the term $u\vartheta$ for arbitrary L^2 vorticity. This difficulty arises since, if the vorticity is in L^2, then the associated velocity is only H^1_{loc} and hence not necessarily bounded. We will explore this issue further in the following section through examples. Our next result is an attempt to determine the critical space in which viscosity solutions have enstrophy densities that solve the transport equation in the sense of distributions. The key idea is to identify a critical space where we can make sense of the nonlinear term $u\vartheta$. Let us begin by recalling some basic facts regarding Orlicz and Lorentz spaces.

Let $f \in L^1_c(\mathbb{R}^2)$ and denote by $\lambda_f = \lambda_f(s) \equiv |\{x \mid |f(x)| > s\}|$ its distribution function. Let f^* denote the standard nonincreasing rearrangement of f, see [4] for details. We consider the Lorentz spaces $L^{(1,q)}_{\text{loc}}$, based on the maximal function of f^*, $f^{**}(s) = \frac{1}{s}\int_0^s f^*(r)dr$, $1 \le q < \infty$:

$$L^{(1,q)}_{\text{loc}}(\mathbb{R}^2) \equiv \{f \in L^1_c(\mathbb{R}^2) \mid \|s\, f^{**}(s)\|_{L^q(ds/s)} < \infty\}. \tag{4.1}$$

There are two ways of defining Lorentz spaces, one based on f^{**} and the other based on f^*. The two definitions are equivalent if $p > 1$, but they lead to two slightly different spaces if $p = 1$, which are usually denoted $L^{1,q}$ and $L^{(1,q)}$. The spaces $L^{(1,q)}_{\text{loc}}$ play a distinguished role in the study of incompressible 2D Euler: if $1 \le q < 2$ they can be compactly imbedded in H^{-1}_{loc}. If $q = 2$ then the imbedding is merely continuous, see [14]. In fact, it was observed by P.-L. Lions in [13] that $L^{(1,2)}_{\text{loc}}(\mathbb{R}^2)$ is the *largest* rearrangement invariant Banach space that can be continuously imbedded in $H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$.

Let $1 \leq p < \infty$ and $a \in \mathbb{R}$. Define $A_{p,a} = A_{p,a}(s) \equiv [s \log^a(2+s)]^p$, for $s > 0$. Then this is a Δ-regular N-function (see [1] for the basic definitions). In particular $A_{p,a}$ is nondecreasing and convex. The associated Orlicz space is the Zygmund space $L^p(\log L)^a$ defined by:

$$L^p(\log L)^a(\mathbb{R}^2) \equiv \left\{ f \in L^1_{\mathrm{loc}} \;\middle|\; \int_{\mathbb{R}^2} A_{p,a}(|f(x)|)\, dx \, < \infty. \right\} \tag{4.2}$$

The Orlicz spaces are Banach spaces when equipped with the Luxemburg norm:

$$\|f\|_{p,a} = \inf \left\{ k > 0 \;\middle|\; \int A_{p,a}\left(\frac{|f(x)|}{k}\right) dx \leq 1 \right\}. \tag{4.3}$$

If f does not vanish identically then the infimum is attained.

If $p = 1$, these spaces are well-known logarithmic refinements of L^1 commonly denoted by $L(\log L)^a$; for arbitrary p they are logarithmic refinements of L^p. It was observed in [14] that $L(\log L)^{1/q} \subset L^{(1,q)} \subset L(\log L)^a$ for any $a < 1/q \leq 1$. The relevant case at present is $q = 2$.

We begin with a technical lemma.

Lemma 1. *Let α and β be functions in $L^2(\log L)^{1/4}(\mathbb{R}^2)$. Then the product $\alpha\beta$ belongs to $L(\log L)^{1/2}$ and*

$$\|\alpha\beta\|_{1,1/2} \leq 4\left(\max\{\|\alpha\|_{2,1/4}; \|\beta\|_{2,1/4}\}\right)^2.$$

Proof. We may assume without loss of generality that neither α nor β vanish identically, otherwise the result is immediate. Thus the infimum in the Luxemburg norm (4.3) is attained for both α and β, i.e.,

$$\int A_{2,1/4}\left(\frac{|\alpha(x)|}{\|\alpha\|_{2,1/4}}\right) dx = 1 \text{ and } \int A_{2,1/4}\left(\frac{|\beta(x)|}{\|\beta\|_{2,1/4}}\right) dx = 1.$$

It is an easy exercise to show that $A_{2,1/4}(2s) \geq 4A_{2,1/4}(s)$, for any $s > 0$. Thus it follows that

$$\int A_{2,1/4}\left(\frac{|\alpha(x)|}{2\|\alpha\|_{2,1/4}}\right) dx \leq \frac{1}{4} \text{ and } \int A_{2,1/4}\left(\frac{|\beta(x)|}{2\|\beta\|_{2,1/4}}\right) dx \leq \frac{1}{4}.$$

Let $k = \max\{\|\alpha\|_{2,1/4}; \|\beta\|_{2,1/4}\}$. Then:

$$\int A_{1,1/2}\left(\frac{\alpha(x)\beta(x)}{4k^2}\right) dx =$$

$$= \int_{\alpha \geq \beta} A_{1,1/2}\left(\frac{\alpha(x)\beta(x)}{4k^2}\right) dx + \int_{\beta > \alpha} A_{1,1/2}\left(\frac{\alpha(x)\beta(x)}{4k^2}\right) dx$$

$$\leq \int \frac{|\alpha|^2}{(2k)^2} \log^{1/2}\left(2 + \frac{|\alpha|^2}{(2k)^2}\right) dx + \int \frac{|\beta|^2}{(2k)^2} \log^{1/2}\left(2 + \frac{|\beta|^2}{(2k)^2}\right) dx$$
$$\leq \sqrt{2} \int A_{2,1/4}\left(\frac{\alpha}{2k}\right) dx + \sqrt{2} \int A_{2,1/4}\left(\frac{\beta}{2k}\right) dx$$
$$\leq \sqrt{2}(1/4 + 1/4) < 1,$$

where the last estimate holds in view of the fact that $A_{2,1/4}$ is nondecreasing.

It follows that

$$\|\alpha\beta\|_{1,1/2} \leq 4k^2,$$

as we wished.

We are now ready to prove that the enstrophy density is a weak solution of the appropriate transport equation, if the vorticity is an inviscid limit and belongs to $L^2(\log L)^{1/4}$.

Theorem 1. *Let $\omega_0 \in (L^2(\log L)^{1/4})_c(\mathbb{R}^2)$. Consider a viscosity solution $\omega \in L^\infty([0,T); L^2(\log L)^{1/4}(\mathbb{R}^2))$ with initial data ω_0. Then the following equation holds in the sense of distributions:*

$$\partial_t(|\omega|^2) + \operatorname{div}(u|\omega|^2) = 0, \tag{4.4}$$

where $u = K * \omega$.

Proof. Let ω_{ν_k}, $\nu_k \to 0$, be a sequence of solutions of the 2D Navier-Stokes equations (3.3), with initial vorticity ω_0, such that $\omega_{\nu_k} \rightharpoonup \omega$ weak-$*$ in $L^\infty([0,T); L^2(\mathbb{R}^2))$. The existence of such a sequence is guaranteed by the fact that ω is a viscosity solution with initial vorticity $\omega_0 \in L^2(\log L)^{1/4} \subset L^2$.

We will begin by showing an *a priori* bound, uniform in viscosity, in the space $L^\infty([0,T); L^2(\log L)^{1/4}(\mathbb{R}^2))$ for ω_{ν_k}. To this end we multiply (3.3a) by

$$\frac{1}{m} A'_{2,1/4}\left(\frac{\omega_{\nu_k}}{m}\right),$$

for arbitrary $m > 0$. Here, $A'_{2,1/4}$ is the derivative of $A_{2,1/4}$ with respect to its argument. Then $\frac{1}{m}\omega_{\nu_k}$ satisfies the following equation:

$$\begin{aligned}\partial_t\left(A_{2,1/4}\left(\frac{\omega_{\nu_k}}{m}\right)\right) + u_{\nu_k} \cdot \nabla A_{2,1/4}\left(\frac{\omega_{\nu_k}}{m}\right) = \\ \nu_k \Delta A_{2,1/4}\left(\frac{\omega_{\nu_k}}{m}\right) - \frac{\nu_k}{m^2} A''_{2,1/4}\left(\frac{\omega_{\nu_k}}{m}\right)|\nabla\omega_{\nu_k}|^2.\end{aligned} \tag{4.5}$$

We integrate (4.5) in all of $\mathbb{R}^2$, use the divergence-free condition on velocity and the convexity of $A_{2,1/4}$ to conclude that, for any $m > 0$,

$$\frac{d}{dt} \int_{\mathbb{R}^2} A_{2,1/4}\left(\frac{\omega_{\nu_k}(x,t)}{m}\right) dx \leq 0.$$

Thus, since the norm in $L^2(\log L)^{1/4}$ is the Luxemburg norm (4.3), it follows that

$$\|\omega_{\nu_k}(\cdot,t)\|_{2,1/4} \le \|\omega_0\|_{2,1/4}, \tag{4.6}$$

for any $0 \le t < T$.

We have obtained that ω_{ν_k} is bounded in $L^\infty([0,T);L^2(\log L)^{1/4}(\mathbb{R}^2))$ and, as this is a Banach space, we may assume, passing to a subsequence if necessary, that $\omega_{\nu_k} \rightharpoonup \omega$ weak-$*$ in this space as $\nu_k \to 0$.

Next recall that $\vartheta_{\nu_k} = |\omega_{\nu_k}|^2/2$ satisfies the viscous enstrophy balance equation (3.4). Therefore, for any test function $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^2)$ we have:

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}^2} \varphi_t \vartheta_{\nu_k}\,dxdt + \int_0^T\int_{\mathbb{R}^2} \nabla\varphi\cdot u_{\nu_k}\vartheta_{\nu_k}\,dxdt = \\ &\qquad = \int_0^T\int_{\mathbb{R}^2} \nu_k \Delta\varphi\,\vartheta_{\nu_k}\,dxdt - \int_0^T\int_{\mathbb{R}^2} \varphi Z^{\nu_k}(\omega_{\nu_k})\,dxdt. \end{aligned} \tag{4.7}$$

We need to pass to the limit $\nu_k \to 0$ in each of the terms above. First recall, from the proof of Proposition 2, that $\vartheta_{\nu_k}(\cdot,t) \to \vartheta(\cdot,t)$ strongly in $L^1(\mathbb{R}^2)$ for each $0 < t < T$. Indeed, we used this fact to show that $Z^{\nu_k}(\omega_{\nu_k}) \to 0$ in $L^1([0,T)\times\mathbb{R}^2)$, see (3.6). Furthermore, as $\int \vartheta_{\nu_k}(\cdot,t)\,dx \le \int \vartheta_0\,dx$ it follows, by the Dominated Convergence Theorem, that $\vartheta_{\nu_k} \to \vartheta$ strongly in $L^1([0,T)\times\mathbb{R}^2)$. Therefore, the first term in (4.7) converges to

$$\int_0^T\int_{\mathbb{R}^2} \varphi_t\vartheta\,dxdt,$$

and the third term converges to zero due to the vanishing factor ν_k. The fourth term in (4.7) converges to zero, as was shown in (3.6) in the proof of Proposition 2. It remains to determine the limit behavior of the nonlinear term.

We start with the observation that $\omega_0 \in (L^2(\log L)^{1/4})_c \subset L^1_c$. Using the maximum principle it is easy to show that the L^1-norm of the solution ω_{ν_k} decreases in time:

$$\|\omega_{\nu_k}(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le \|\omega_0\|_{L^1(\mathbb{R}^2)}. \tag{4.8}$$

Thus, as the Biot-Savart kernel K is locally integrable and bounded near infinity, the convolution $K*\omega_{\nu_k}$ is well-defined. We may therefore use the Biot-Savart law $u_{\nu_k} = K*\omega_{\nu_k}$ to find:

$$\int_0^T\int_{\mathbb{R}^2} \nabla\varphi\cdot u_{\nu_k}\vartheta_{\nu_k}\,dxdt = \tag{4.9}$$

$$\begin{aligned} &= \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla\varphi(x,t)\cdot K(x-y)\omega_{\nu_k}(y,t)\vartheta_{\nu_k}(x,t)\,dydxdt \\ &= -\int_0^T\int_{\mathbb{R}^2} \omega_{\nu_k}(y,t)\int_{\mathbb{R}^2} K(y-x)\cdot\nabla\varphi(x,t)\vartheta_{\nu_k}(x,t)\,dx\,dydt, \end{aligned}$$

as K is antisymmetric. Thus we may write

$$(4.9) = -\int_0^T \int_{\mathbb{R}^2} \omega_{\nu_k}(y,t)\,\mathcal{I}_k(y,t)\,dydt,$$

with

$$\mathcal{I}_k \equiv \int_{\mathbb{R}^2} K(y-x)\cdot\nabla\varphi(x,t)\vartheta_{\nu_k}(x,t)\,dx.$$

Let $\vartheta = |\omega|^2/2$. Denote by $\mathcal{I}$ the function

$$\mathcal{I} \equiv \int_{\mathbb{R}^2} K(y-x)\cdot\nabla\varphi(x,t)\vartheta(x,t)\,dx,$$

which is well defined, as we will see later.

We deduce, from the *a priori* estimate (4.6) in $L^2(\log L)^{1/4}$, from Lemma 1, and from the fact that each component of $\nabla\varphi$ is a smooth test function, that $\{\nabla\phi\vartheta_{\nu_k}\}$ is bounded in $L^\infty((0,T);L(\log L)^{1/2}(\mathbb{R}^2))$ and, therefore, in $L^\infty((0,T);L^{(1,2)}_{\text{loc}}(\mathbb{R}^2))$ (see [14]). As already observed above, $L^{(1,2)}_{\text{loc}}$ can be continuously imbedded in H^{-1}_{loc}, so that

$$\{\mathcal{I}_k\} \text{ is bounded in } L^\infty((0,T);L^2_{\text{loc}}(\mathbb{R}^2)). \tag{4.10}$$

Thus it follows that, passing to a subsequence if necessary, $\mathcal{I}_k$ converges weak-$*$ in $L^\infty((0,T);L^2_{\text{loc}}(\mathbb{R}^2))$ to a weak limit. We will show that this weak limit is $\mathcal{I}$. We know that $\nabla\varphi\vartheta_{\nu_k} \to \nabla\varphi\vartheta$ strongly in $L^1((0,T)\times\mathbb{R}^2)$, because $\omega_{\nu_k} \to \omega$ strongly in $L^2((0,T)\times\mathbb{R}^2)$. Let $\eta \in \mathcal{D}((0,T)\times\mathbb{R}^2)$. Then we may write:

$$\langle \mathcal{I}_k, \eta\rangle = -\langle \nabla\varphi\vartheta_{\nu_k}, K*\eta\rangle,$$

using the antisymmetry of K. Since $K*\eta \in L^\infty((0,T)\times\mathbb{R}^2)$ we therefore obtain that

$$\langle \mathcal{I}_k, \eta\rangle \to -\langle \nabla\varphi\vartheta, K*\eta\rangle = \langle \mathcal{I}, \eta\rangle.$$

We have shown that $\mathcal{I}_k \to \mathcal{I}$ in the sense of distributions, so that, by uniqueness of limits, the weak limit of $\mathcal{I}_k$ is necessarily equal to $\mathcal{I}$. Hence, the whole sequence $\mathcal{I}_k$ converges weakly to $\mathcal{I}$, without the need to pass to a subsequence. In particular, we have established that the integral in the definition of $\mathcal{I}$ is well defined.

The next step is to deal with the behavior of $\mathcal{I}_k$ at infinity. Note that each component of $\nabla\varphi\vartheta_{\nu_k}$ is compactly supported, uniformly in t and ν_k, in a ball, say, $B(0;R)$. As the viscous enstrophy decreases in time, we find that

$$\|\nabla\varphi\vartheta_{\nu_k}\|_{L^\infty((0,T);L^1(\mathbb{R}^2))} \le C(\varphi)\int_{\mathbb{R}^2}\vartheta_0\,dx \equiv C(\varphi)\,\Omega_0.$$

From this observation and the explicit expression for the kernel K, a direct estimate yields that

$$|\mathcal{I}_k(y,t)| \le \frac{\widetilde{C}(\varphi)\,\Omega_0}{|y|}$$

for $|y| \geq 2R$. Hence

$$\{\mathcal{I}_k\} \text{ is bounded in } L^\infty((0,T) \times (\mathbb{R}^2 \setminus B(0;2R))). \tag{4.11}$$

Using the same argument as was used above to establish that $\mathcal{I}_k \rightharpoonup \mathcal{I}$ weak-$*$ $L^\infty((0,T); L^2_{\text{loc}}(\mathbb{R}^2))$, we may conclude, from estimate (4.11), that $\mathcal{I}_k \rightharpoonup \mathcal{I}$ weak-$*$ in $L^\infty((0,T) \times (\mathbb{R}^2 \setminus B(0;2R)))$ as well, without the need to pass to a subsequence.

We claim that $\omega_{\nu_k} \to \omega$ strongly in $L^1((0,T) \times \mathbb{R}^2)$, as $\nu_k \to 0$, as well as in $L^2((0,T) \times \mathbb{R}^2)$. Assuming the claim, we can pass to the limit in the nonlinear term. Indeed, we write

$$(4.9) = -\left(\int_0^T \int_{B(0;2R)} \omega_{\nu_k}(y,t)\,\mathcal{I}_k(y,t)\,dydt + \right.$$
$$\left. + \int_0^T \int_{\mathbb{R}^2 \setminus B(0;2R)} \omega_{\nu_k}(y,t)\,\mathcal{I}_k(y,t)\,dydt\right),$$

which converges to

$$-\int_0^T \int_{B(0;2R)} \omega(y,t)\,\mathcal{I}(y,t)\,dydt - \int_0^T \int_{\mathbb{R}^2 \setminus B(0;2R)} \omega(y,t)\,\mathcal{I}(y,t)\,dydt,$$

as each integral forms a "weak-strong pair", by virtue of the convergence $\mathcal{I}_k \to \mathcal{I}$ established above, and noting that $L^\infty((0,T); L^2_{\text{loc}}) \subset L^2((0,T); L^2_{\text{loc}})$.

All that remains is to prove the claim. We begin by noting that the strong convergence in $L^2((0,T) \times \mathbb{R}^2)$ was observed in the proof of Proposition 2: it follows from the convergence of the norms together with weak convergence. To address strong convergence in L^1 we make use of the following fact (for $p = 1$), due to H. Brézis and E. Lieb, (see Theorem 8 of [9] for a proof): a sequence that converges weakly and almost everywhere and such that the L^p-norms also converge will converge strongly in L^p. We obtain weak convergence in $L^1((0,T) \times \mathbb{R}^2)$, passing to a subsequence if necessary, directly from the *a priori* estimate (4.8) on the L^1-norm of ω_{ν_k} together with strong convergence in L^2. We also have almost everywhere convergence passing to a further subsequence if needed. Finally, we can establish strong convergence of the L^1-norm by repeating the argument used in the proof of Proposition 2 to show that the L^2-norms converge. Consequently, strong convergence in L^1 holds for this particular subsequence. However, since we have identified the limit, we find that the whole sequence ω_{ν_k} converges to ω strongly in $L^1((0,T) \times \mathbb{R}^2)$ as $\nu_k \to 0$, as we wished.

Remark 2. The natural condition under which the argument above remains valid is $|\omega_0|^2 \in L^{(1,2)}$. We chose to present the result under the slightly stronger assumption $\omega_0 \in L^2(\log L)^{1/4}$ because it does not seem immediate to provide an *a priori* estimate on the square of ω in $L^{(1,2)}$ that is uniform in viscosity.

Remark 3. If we do not assume that the weak solution is a viscosity solution then the best result available on enstrophy density satisfying the transport equation (4.4) in the sense of distributions requires $\omega_0 \in L^p_c$, $p > 2$, see [10].

One is naturally led to ask what knowledge has been gained with Theorem 1. Where we previously knew that the enstrophy density satisfied the transport equation in the renormalized sense, we now know that the equation is satisfied in the sense of distributions. We apply this additional information in the proof of our next result.

In the remainder of this section, we are concerned with the conditions under which Z^T exists and vanishes for finite enstrophy flows. The key point in the proof of Theorem 1 is that we provided meaning to the term $u\vartheta$ for $\omega \in L^2(\log L)^{1/4}$, through the computation of (4.9). Assigning meaning to the nonlinearity $u\vartheta$ will also play a central role in the proof of the next result. We formalize the meaning we wish to adopt in a definition.

Definition 5. Let $\omega \in L^2(\log L)^{1/4}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Let $u = K * \omega$. Then we define $u\vartheta \in \mathcal{D}'(\mathbb{R}^2)$ by:

$$\begin{aligned}\langle u\vartheta, \Phi\rangle &= -\int_{\mathbb{R}^2} \omega(y) \int_{\mathbb{R}^2} K(y-x)\cdot \Phi(x)\vartheta(x)\,dx\,dy \\ &\equiv -\int_{\mathbb{R}^2} \omega(y)[K \cdot *(\Phi\vartheta)](y)\,dy,\end{aligned}$$

for any test vector field $\Phi \in \mathcal{D}(\mathbb{R}^2)$.

The integral above is well-defined as $\Phi\vartheta$ is a compactly supported function in $L(\log L)^{1/2}_{\text{loc}} \hookrightarrow L^{(1,2)}_{\text{loc}}$ and $\omega \in L^2 \cap L^1$, see the proof of Theorem 1. Moreover, it is easy to establish that $\Phi \mapsto \langle u\vartheta, \Phi\rangle$ is a continuous linear functional over $\mathcal{D}$.

We are now ready to state and prove our final result in this section.

Theorem 2. *Let $\omega \in L^\infty([0,T); L^2(\log L)^{1/4}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2))$ be a weak solution of the incompressible 2D Euler equations. Then the transport enstrophy defect $Z^T(\omega)$ exists (as a distribution). If ω is a viscosity solution with initial vorticity $\omega_0 \in (L^2(\log L)^{1/4})_c(\mathbb{R}^2)$ then $Z^T(\omega) \equiv 0$.*

Proof. Let j_ϵ be a radially symmetric, compactly supported Friedrichs mollifier. Recall the notation ω_ϵ, u_ϵ and $(u\omega)_\epsilon$ introduced in the beginning of Section 3.

Let $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^2)$. We multiply the equation (3.2) for $\vartheta_\epsilon = |\omega_\epsilon|^2/2$ by φ and integrate over $(0,T)\times\mathbb{R}^2$ to find:

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^2} \varphi_t \vartheta_\epsilon\,dxdt + \int_0^T \int_{\mathbb{R}^2} \nabla\varphi \cdot u_\epsilon \vartheta_\epsilon\,dxdt + \\ &\quad + \int_0^T \int_{\mathbb{R}^2} \nabla\varphi\,\omega_\epsilon \cdot ((u\omega)_\epsilon - u_\epsilon\omega_\epsilon)\;dxdt = \int_0^T \int_{\mathbb{R}^2} \varphi Z_\epsilon(\omega)\,dxdt.\end{aligned} \tag{4.12}$$

We wish to pass to the limit $\epsilon \to 0$. Let us begin by examining the first two terms above.

The integrand in the first term is $\varphi_t|\omega_\epsilon|^2/2$, which converges to $\varphi_t|\omega|^2/2$, as $\epsilon \to 0$, strongly in $L^1((0,T)\times\mathbb{R}^2)$. Indeed, by standard properties of mollifiers, $\omega_\epsilon(\cdot,t) \to \omega(\cdot,t)$ strongly in $L^2(\mathbb{R}^2)$ for each $0<t<T$, and also

$$\|\omega_\epsilon(\cdot,t)\|_{L^2} \leq \|\omega(\cdot,t)\|_{L^2} \equiv \|\omega_0\|_{L^2}.$$

Hence we may obtain the desired conclusion using the Dominated Convergence Theorem.

Next we consider the second term. We note that mollification is continuous in Δ-regular Orlicz spaces (see Theorem 8.20 in [1]) so that $\omega_\epsilon \in L^\infty([0,T);L^2(\log L)^{1/4}(\mathbb{R}^2)\cap L^1(\mathbb{R}^2))$. As convolutions are associative, we have that $u_\epsilon = K*\omega_\epsilon$. We are thus in position to write the second term in (4.12) using Definition 5:

$$\int_0^T\int_{\mathbb{R}^2}\nabla\varphi\cdot u_\epsilon\vartheta_\epsilon\,dxdt = -\int_0^T\int_{\mathbb{R}^2}\omega_\epsilon(y,t)[K\cdot*(\nabla\varphi\vartheta_\epsilon)](y,t)\,dydt. \quad (4.13)$$

It follows from Lemma 1 that the family $\nabla\varphi\vartheta_\epsilon$ is uniformly bounded in $L^\infty((0,T);L(\log L)^{1/2}(\mathbb{R}^2))$. Hence we find, as in (4.10), that

$$\{K\cdot*(\nabla\varphi\vartheta_\epsilon)\} \text{ is bounded in } L^\infty((0,T);L^2_{\text{loc}}(\mathbb{R}^2)).$$

Furthermore, $\|\nabla\varphi\vartheta_\epsilon(\cdot,t)\|_{L^1} \leq \|\nabla\varphi\vartheta_0\|_{L^1}$ and $\nabla\varphi\vartheta_\epsilon$ has compact support *uniformly* in t and ϵ, so that, as in (4.11),

$$\{K\cdot*(\nabla\varphi\vartheta_\epsilon)\} \text{ is bounded in } L^\infty((0,T)\times(\mathbb{R}^2\setminus B(0;2R))),$$

for R sufficiently large. Standard properties of mollifiers yield that $\omega_\epsilon \to \omega$ strongly in both $L^2((0,T)\times\mathbb{R}^2)$ and $L^1((0,T)\times\mathbb{R}^2)$. Thus we may conclude, as in the proof of Theorem 1, that the left hand side of (4.13) converges to

$$-\int_0^T\int_{\mathbb{R}^2}\omega(y,t)[K\cdot*(\nabla\varphi\vartheta)](y,t)\,dydt,$$

when $\epsilon \to 0$.

Finally, let us examine the third term. The key point in this proof is to show that it vanishes as $\epsilon \to 0$. We use the radial symmetry of the mollifier j_ϵ to obtain:

$$\int_0^T\int_{\mathbb{R}^2}\nabla\varphi\,\omega_\epsilon\cdot((u\omega)_\epsilon - u_\epsilon\omega_\epsilon)\;dxdt =$$

$$\int_0^T\int_{\mathbb{R}^2}[(\nabla\varphi\,\omega_\epsilon)*j_\epsilon]\cdot u\omega\,dxdt - \int_0^T\int_{\mathbb{R}^2}\nabla\varphi\cdot u_\epsilon|\omega_\epsilon|^2\;dxdt \equiv \mathcal{I}_\epsilon - \mathcal{J}_\epsilon.$$

We have already analyzed $\mathcal{J}_\epsilon$ in (4.13). We know that

$$\mathcal{J}_\epsilon \equiv -\int_0^T\int_{\mathbb{R}^2}\omega_\epsilon(y,t)[K\cdot*(\nabla\varphi|\omega_\epsilon|^2)](y,t)\,dydt$$

$$\to - \int_0^T \int_{\mathbb{R}^2} \omega(y,t)[K \cdot *(\nabla\varphi|\omega|^2)](y,t)\, dydt,$$

as $\epsilon \to 0$. We will now analyze $\mathcal{I}_\epsilon$. We start by observing that, using the antisymmetry of K, we can write:

$$\mathcal{I}_\epsilon = - \int_0^T \int_{\mathbb{R}^2} \omega(y,t)[\, K \cdot *(\, \omega(\, (\nabla\varphi\, \omega_\epsilon) * j_\epsilon\,)\,)\,](y,t)\, dydt.$$

Next we note that, by standard properties of mollification, $(\nabla\varphi\, \omega_\epsilon) * j_\epsilon \to \nabla\varphi\, \omega$ strongly in $L^2((0,T) \times \mathbb{R}^2)$ as $\epsilon \to 0$. In addition, $(\nabla\varphi\, \omega_\epsilon) * j_\epsilon$ is compactly supported, uniformly in t and ϵ, and it is uniformly bounded in $L^\infty((0,T); L^2(\log L)^{1/4}(\mathbb{R}^2))$. Using Lemma 1 we deduce that

$$\{\omega(\, (\nabla\varphi\, \omega_\epsilon) * j_\epsilon\,)\} \text{ is bounded in } L^\infty((0,T); L(\log L)^{1/2}(\mathbb{R}^2)).$$

Therefore $\{K \cdot *(\, \omega(\, (\nabla\varphi\, \omega_\epsilon) * j_\epsilon\,)\,)\}$ is bounded in $L^\infty((0,T); L^2_{\text{loc}}(\mathbb{R}^2))$ and in $L^\infty((0,T) \times (\mathbb{R}^2 \backslash B(0;2R)))$, for R sufficiently large. From this observation we may conclude, as we have before, that

$$\mathcal{I}_\epsilon \to - \int_0^T \int_{\mathbb{R}^2} \omega(y,t)[K \cdot *(\nabla\varphi|\omega|^2)](y,t)\, dydt,$$

as $\epsilon \to 0$. Therefore the third term vanishes in the limit $\epsilon \to 0$. The proof that the transport enstrophy defect exists as a distribution is complete. In fact, we have established that

$$\begin{aligned} \langle Z^T(\omega), \varphi\rangle &= \int_0^T \int_{\mathbb{R}^2} \varphi_t \vartheta\, dxdt - \int_0^T \int_{\mathbb{R}^2} \omega(y,t)[K \cdot *(\nabla\varphi\vartheta)](y,t)\, dydt \\ &= \langle \vartheta, \varphi_t\rangle + \langle u\vartheta, \nabla\varphi\rangle \equiv -\langle \vartheta_t + \operatorname{div}(u\vartheta), \varphi\rangle, \end{aligned} \tag{4.14}$$

where the former identity follows from Definition 5.

Finally, in view of Theorem 1 we have that, if ω is a viscosity solution with initial vorticity $\omega_0 \in (L^2(\log L)^{1/4})_c(\mathbb{R}^2)$, then the enstrophy density balance equation holds in the sense of distributions, so (4.14) above implies $Z^T(\omega) \equiv 0$ in this case.

Remark 4. This result raises a few interesting questions. First, if one could find an example of a weak solution with initial vorticity in $L^2(\log L)^{1/4}$ and such that Z^T does not vanish, one would have established nonuniqueness of weak solutions. In fact, any example where Z^T exists and does not vanish would be quite interesting. Second, one naturally wonders how sharp is the regularity condition $L^2(\log L)^{1/4}$ on vorticity. This is the subject of the next section.

5. The Biot-Savart law in L^2-based Zygmund spaces

The purpose of this section is to illustrate the behavior of the term $u|\omega|^2$ through examples. We will be considering pairs (u, ω) related by the Biot-Savart law, but not necessarily solutions of the 2D Euler equations. We will not establish that the condition $\omega \in L^2(\log L)^{1/4}$ (or $|\omega|^2 \in L^{(1,2)}$) is necessary for making sense of $u|\omega|^2$, but we will exhibit an example showing that it is not possible to define $u|\omega|^2$ as a distribution for an arbitrary vorticity in L^2. Furthermore, the family of examples we will present also proves that the velocities associated to vorticities in $L^2(\log L)^{1/4}$ are not necessarily bounded, something which would trivialize the proofs in the previous section.

It would be natural to look for such examples in the class of radially symmetric vorticities, but we will see in our first Lemma that this approach is not useful.

Lemma 2. *Let $\omega \in L^2_c(\mathbb{R}^2)$, such that $\omega(x) = \phi(|x|)$. Let $u \equiv K * \omega$. Then u is bounded and $\|u\|_{L^\infty} \leq C\|\omega\|_{L^2}$.*

Proof. The reader may easily check that if the vorticity is radially symmetric, then the Biot-Savart law becomes:

$$u(x) = \frac{x^\perp}{|x|^2} \int_0^{|x|} s\phi(s)ds.$$

As $\omega \in L^2$, it follows that $\phi \in L^2(sds)$. We use the Cauchy-Schwarz inequality with respect to sds to obtain:

$$\left| \int_0^{|x|} s\phi(s)ds \right| \leq \left(\int_0^{|x|} sds \right)^{1/2} \left(\int_0^{|x|} s\phi^2(s)ds \right)^{1/2} \leq C\|\omega\|_{L^2}|x|.$$

This concludes the proof.

Recall that the velocity associated to an L^p vorticity is bounded if $p > 2$, but logarithmic singularities may occur when $p = 2$. The symmetry in a radial vorticity configuration implies a certain cancellation in the Biot-Savart law, and it is this cancellation which is responsible for the additional regularity observed in the lemma above. We will consider a family of examples given by breaking the symmetry in the simplest way possible.

Let $1/2 < \alpha < 1$. We will denote by ω^α_+ the function

$$\omega^\alpha_+(x) \equiv \frac{1}{|x||\log|x||^\alpha} \chi_{B^+(0;1/3)}(x),$$

where $B^+(0;1/3) = B(0;1/3) \cap \{x_2 > 0\}$.

Note first that $\omega^\alpha_+ \in L^2_c$. Indeed,

$$\|\omega^\alpha_+\|^2_{L^2} = \pi \int_0^{1/3} \frac{ds}{s|\log s|^{2\alpha}} = \frac{\pi}{2\alpha - 1}(\log 3)^{1-2\alpha},$$

as long as $\alpha > 1/2$. We can make a more precise characterization of the regularity of ω_+^α using the Zygmund class hierarchy.

We denote the radially symmetric extension of ω_+^α as

$$\omega^\alpha(x) \equiv (|x||\log|x||^\alpha)^{-1}\chi_{B(0;1/3)}. \tag{5.1}$$

Lemma 3. *If $1/2 < \alpha < 1$ then $\omega_+^\alpha \in L^2(\log L)^\kappa$, for all $0 \le \kappa < \alpha - 1/2$.*

Proof. We observe that $(|x||\log|x||^\alpha)^{-1}$ is a decreasing function of $|x|$ if $|x| \le e^{-\alpha}$. In particular, as $\alpha < 1$, it is decreasing in the ball $B(0;1/3)$. Hence ω^α has a positive lower bound, say c. Next, using the notation from Section 4, we estimate $\int A_{2,\kappa}(\omega_+^\alpha)\,dx$. Since $A_{2,\kappa}$ is nondecreasing we have

$$\int A_{2,\kappa}(\omega_+^\alpha)\,dx \le \int |\omega^\alpha|^2 \log^{2\kappa}(\omega^\alpha + 2)\,dx$$

$$\le C(\|\omega^\alpha\|_{L^2}) \int_{B(0;1/3)} \frac{1}{|x|^2|\log|x||^{2\alpha}} \left|\log^{2\kappa} \frac{1}{|x||\log|x||^\alpha}\right| dx,$$

using the fact that $|\omega^\alpha| \ge c > 0$ on $B(0;1/3)$,

$$\le C \int_{B(0;1/3)} \frac{1}{|x|^2|\log|x||^{2\alpha}} |\log|x||^{2\kappa}\,dx = C\int_0^{1/3} \frac{1}{r|\log r|^{2\alpha-2\kappa}}\,dr < \infty,$$

as long as $2\alpha - 2\kappa > 1$, i.e., $\kappa < \alpha - 1/2$. The last inequality is due to the fact that the double logarithm grows slower than the single logarithm.

The condition that $\kappa \ge 0$ arises from the definition of the Zygmund spaces.

Theorem 3. *If $\alpha < 1$ then $u_+^\alpha \equiv K * \omega_+^\alpha$ is unbounded. If $1/2 < \alpha \le 2/3$ then $u_+^\alpha|\omega_+^\alpha|^2$ is not locally integrable.*

Proof. We will show that the first component of u_+^α, which we denote by u_1, is greater than or equal to $C|\log|x||^{1-\alpha}$ in a suitably small neighborhood of the origin. It is easy to see that this result proves both assertions in the statement of the theorem.

First we compute u_1 on the horizontal axis. Note that ω_+^α is even with respect to $x_1 = 0$. Then u_1 has the same symmetry, due to the specific form of the Biot-Savart kernel, and in particular $u_1(x_1,0) = u_1(-x_1,0)$. Therefore, it is enough to compute $u_1(x_1,0)$ for $x_1 > 0$. We have

$$u_1(x_1,0) = \int_{B^+(0;1/3)} \frac{y_2}{2\pi|x-y|^2}\,\frac{1}{|y||\log|y||^\alpha}\,dy$$

$$= \frac{1}{2\pi}\int_0^{1/3}\int_0^\pi \frac{r\sin\theta}{(x_1 - r\cos\theta)^2 + (r\sin\theta)^2}\,d\theta \frac{1}{|\log r|^\alpha}\,dr,$$

after changing to polar coordinates. Explicitly evaluating the integral in θ and subsequently implementing the change of variables $s = r/x_1$ we find

$$2\pi u_1(x_1,0) = \int_0^{1/3} \frac{1}{|\log r|^\alpha}\,\frac{1}{x_1}\log\left|\frac{r+x_1}{r-x_1}\right|\,dr$$

$$= \int_0^{1/(3x_1)} \frac{1}{|\log sx_1|^\alpha} \log\left|\frac{s+1}{s-1}\right| ds$$

$$= \int_0^{2} \frac{1}{|\log sx_1|^\alpha} \log\left|\frac{s+1}{s-1}\right| ds + \int_2^{1/(3x_1)} \frac{1}{|\log sx_1|^\alpha} \log\left|\frac{s+1}{s-1}\right| ds$$

$$\equiv \mathcal{I} + \mathcal{J}.$$

We assume $0 \leq x_1 < 1/6$ and we estimate $\mathcal{I}$:

$$0 \leq \mathcal{I} \leq \frac{1}{(\log 3)^\alpha} \int_0^2 \log\left|\frac{s+1}{s-1}\right| ds \equiv C < \infty.$$

Next we estimate $\mathcal{J}$ from below. We begin with two observations. For $2 < s < 1/(3x_1)$ we have:

$$\frac{1}{|\log sx_1|^\alpha} \geq \frac{1}{|\log 2x_1|^\alpha};$$

and

$$\log\frac{s+1}{s-1} > \frac{1}{s}.$$

Therefore,

$$\mathcal{J} \geq \frac{|\log 6x_1|}{|\log 2x_1|^\alpha} \geq \frac{1}{2}|\log x_1|^{1-\alpha},$$

where the last inequality was derived assuming further that $x_1 \leq 1/36$.

In summary, we have shown that

$$u_1(x_1, 0) \geq C|\log|x_1||^{1-\alpha} \quad \text{if } |x_1| \leq \frac{1}{36}, \tag{5.2}$$

for some $C > 0$. In addition, it follows from the specific form of the Biot-Savart law that $u_1(x_1, 0) \geq 0$ for all x_1.

Recall the radially symmetric function ω^α, introduced in (5.1). Consider the vorticity $\omega^\alpha - \omega^\alpha_+$, supported in the lower half-plane. Let

$$v_1 = v_1(x_1, x_2) = \int \frac{y_2 - x_2}{2\pi|x-y|^2}(\omega^\alpha - \omega^\alpha_+)(y)\, dy,$$

be the first component of the associated velocity. Then v_1 is a harmonic function in the upper half-plane, whose boundary value, by symmetry, is equal to $-u_1(x_1, 0)$, since the horizontal velocity associated to ω^α vanishes on the horizontal axis. We may thus write, using the Poisson kernel for the upper half-plane,

$$v_1(x_1, x_2) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_2\, u_1(s, 0)}{(x_1 - s)^2 + x_2^2}\, ds, \ \text{ if } x_2 > 0. \tag{5.3}$$

Note that $u^\alpha \equiv v_1 + u_1$ is the velocity associated to ω^α. In view of Lemma 2 we have that u^α is bounded and there exists $C > 0$ such that:

$$\|u^\alpha\|_{L^\infty} \leq C\|\omega^\alpha\|_{L^2}.$$

In what follows we will show that $v_1 \leq -C|\log|x||^{1-\alpha}$ for sufficiently small $|x|$, with $x_2 > 0$. By virtue of the previous observation this is enough to conclude the proof.

Let $0 < \delta < 1/36$. Using (5.2) and the fact that u_1 is nonnegative on $x_2 = 0$, we find for $x_2 > 0$,

$$v_1(x_1, x_2) \leq -\frac{1}{\pi}\int_{-\delta}^{\delta} \frac{x_2\, u_1(s,0)}{(x_1 - s)^2 + x_2^2}\, ds$$
$$\leq -C|\log\delta|^{1-\alpha}\left\{\arctan\left(\frac{x_1+\delta}{x_2}\right) - \arctan\left(\frac{x_1-\delta}{x_2}\right)\right\},$$

by explicitly integrating the Poisson kernel in the interval $(-\delta, \delta)$.

Next, let $x = (\delta/2)(\cos\theta, \sin\theta)$ with $0 \leq \theta \leq \pi$. Then

$$v_1(x) \leq -C|\log 2|x||^{1-\alpha}\left[\arctan\left(\frac{\cos\theta + 2}{\sin\theta}\right) - \arctan\left(\frac{\cos\theta - 2}{\sin\theta}\right)\right]$$
$$\equiv -C|\log 2|x||^{1-\alpha} g(\theta).$$

It is easy to compute the minimum of $g(\theta)$, thereby verifying that $g(\theta) \geq 2\arctan 2 > 0$ for all $\theta \in [0,\pi]$. We have therefore shown that, for any $x = (x_1, x_2)$ with $x_2 > 0$ and $|x| \leq 1/72$, $v_1(x_1,x_2) \leq -C|\log 2|x||^{1-\alpha}$. The conclusion follows as $|\log 2|x|| \geq (1/2)|\log|x||$ for any $|x| < 1/4$.

Remark 5. We emphasize that we have proved above that there exist constants $C > 0$, $0 < r_0 < 1/72$ such that

$$u_1(x) \geq C|\log|x||^{1-\alpha}, \quad \text{for } x \in B^+(0; r_0). \tag{5.4}$$

We wish to use Lemma 3 and Theorem 3 to draw two separate conclusions. The first is that $L^2(\log L)^{1/4}$ contains vorticities whose associated velocities are unbounded. Indeed, it is enough to consider ω^α_+, for $3/4 < \alpha < 1$. The second conclusion is that there are difficulties in making sense, as a distribution, of $u|\omega|^2$ for an arbitrary vorticity in L^2. In fact, we have already shown that $u^\alpha_+|\omega^\alpha_+|^2$ is not locally integrable if $1/2 < \alpha \leq 2/3$. From Lemma 3 it follows that $\omega^\alpha_+ \in L^2(\log L)^\kappa$, for some $0 \leq \kappa < 1/6$ if $1/2 < \alpha \leq 2/3$. Although suggestive, the non-integrability of $u^\alpha_+|\omega^\alpha_+|^2$ does not exclude the possibility that $u^\alpha_+|\omega^\alpha_+|^2$ gives rise to a well-defined distribution. One may recall the way in which the non-integrable functions $1/s$ and $1/s^2$ can be identified with the distributions pv-$1/s$ and pf-$1/s^2$.

We must address more closely the problem of identifying $u|\omega|^2$ with a distribution. In view of Definition 5 one might suspect that by re-arranging the Biot-Savart law in a clever way and using the antisymmetry of the kernel, it would be possible to give meaning to $u|\omega|^2$ in a consistent manner, even if ω is only in L^2_c. The antisymmetry of the Biot-Savart kernel has been used on more than one occasion to prove results of this nature; for instance it was used to define the nonlinear term $u \cdot \nabla\omega$, when $\omega \in \mathcal{B}M_+ \cap H^{-1}_{\text{loc}}$, by S. Schochet in [17]. We will see that this strategy would not be successful in this case.

Ultimately, our purpose here is to examine the sharpness of the condition $\omega \in L^2(\log L)^{1/4}$, which we showed to be sufficient to define the term $u|\omega|^2$. This condition was used in Theorem 1 and Definition 5. We would like to argue through a counterexample that it is not possible to make sense of $u|\omega|^2$ for arbitrary $\omega \in L^2(\log L)^\kappa$, with $0 \le \kappa < 1/6$. If we wish to attribute meaning to $u|\omega|^2$ (as a distribution) for any $\omega \in X \subseteq L^2$, then the key issue is the nature of the nonlinear map $T : \omega \mapsto u|\omega|^2$, from X to $\mathcal{D}'$. First, note that T is well-defined for $X = L^p_c$, $p > 2$, since then $u = K * \omega$ is bounded. Next, note that Definition 5 actually consists of the continuous extension of T to $X = (L^2(\log L)^{1/4})_c$. We will show through the counterexample we present that there is no continuous extension of T from L^p_c, $p > 2$ to $X = (L^2(\log L)^\kappa)_c$, $0 \le \kappa < 1/6$, and hence, to $X = L^2_c$. In fact we will prove that our example ω^α_+, with $1/2 < \alpha \le 2/3$, can be approximated in $(L^2(\log L)^\kappa)_c$, $0 \le \kappa < \alpha - 1/2$, by a sequence $\omega^n_+ \in L^\infty_c$ for which $\int u^n_+|\omega^n_+|^2 \to \infty$ as $n \to \infty$, thereby reaching the desired conclusion.

Theorem 4. *Let $x = (x_1, x_2)$ with $x_2 \ge 0$. Fix $1/2 < \alpha \le 2/3$. For each $n \in \mathbb{N}$ we define the approximate vorticity by:*

$$\omega^n_+ = \omega^n_+(x) = \begin{cases} \omega^\alpha_+(x) & \text{if } |x| > 1/n, \\ \\ \frac{n}{|\log n|^\alpha} & \text{if } |x| \le 1/n. \end{cases} \tag{5.5}$$

Then $\omega^n_+ \to \omega^\alpha_+$, as $n \to \infty$, strongly in $L^2(\log L)^\kappa$ for all $0 \le \kappa < \alpha - 1/2$.

Denote u^n_1 the first component of $K * \omega^n_+$. Then it also holds that

$$\lim_{n\to+\infty} \int u^n_1 |\omega^n_+|^2 \, dx = +\infty. \tag{5.6}$$

Proof. Our first step is to show that u^n_1 is nonnegative in $B^+(0; r_0)$, if n is large enough, where r_0 is such that (5.4) holds. We require two different arguments, one for $|x| \le 2/n$ and another for $2/n < |x| < r_0$. We will begin with the latter.

Let $W_n = \omega^\alpha_+ - \omega^n_+ \ge 0$, which is a function with support in $B^+(0; 1/n)$. Let e_n be the first component of $K * W_n$, i.e., the error in the velocity induced by the truncation. Therefore, $u^n_1 = u_1 - e_n$. It follows from (5.4) that

$$u^n_1(x) \ge C|\log|x||^{1-\alpha} - e_n(x), \text{ for } x \in B^+(0; r_0). \tag{5.7}$$

We will prove that

$$|e_n(x)| \le C/(\log n)^\alpha, \text{ for } |x| > 2/n. \tag{5.8}$$

For $x \in B^+(0;r_0)$, $|x| > 2/n$ we estimate:

$$|e_n(x)| \le \int_{B^+(0;1/n)} \frac{1}{|x-y|} W_n(y)\,dy \le Cn \int_{B^+(0;1/n)} W_n(y)\,dy, \qquad (5.9)$$

as $|x-y| \ge 1/n$,

$$= Cn \int_0^{1/n} \left(\frac{1}{|\log r|^\alpha} - \frac{nr}{(\log n)^\alpha} \right) dr, \qquad (5.10)$$

after changing to polar coordinates, yielding (5.8).

As $|\log|x||^{1-\alpha}$ is decreasing with respect to $|x|$, it follows from (5.7) and (5.8) that one can choose n_0 sufficiently large so that if $n > n_0$ and $|x| > 2/n$, with $x \in B^+(0;r_0)$, then $u_1^n(x) \ge 0$.

Now we address the case $|x| \le 2/n$. We will show that

$$u_1^n(x) \ge C(\log n)^{1-\alpha} \qquad (5.11)$$

for x in this region. The proof closely parallels the proof of Theorem 3. We begin by estimating $u_1^n(x_1,0)$ if $|x_1| < 2/n$. We have:

$$\pi u_1^n(x_1,0) = \int_{1/n}^{1/3} \frac{1}{r|\log r|^\alpha} \frac{r}{|x_1|} \log\left| \frac{(r/|x_1|)+1}{(r/|x_1|)-1} \right| dr$$
$$+ \frac{n}{(\log n)^\alpha} \int_0^{1/n} \frac{r}{|x_1|} \log\left| \frac{(r/|x_1|)+1}{(r/|x_1|)-1} \right| dr$$
$$\ge \int_{2/n}^{1/3} \frac{1}{r|\log r|^\alpha} g(r/|x_1|)\,dr,$$

where $g(s) \equiv s \log|(s+1)/(s-1)|$. It can be easily verified that $g(s) > 1$ if $s > 1$. Therefore, as $r/|x_1| > 1$ for $r > 2/n$ and $|x_1| < 2/n$, we obtain

$$\pi u_1^n(x_1,0) \ge \int_{2/n}^{1/3} \frac{1}{r|\log r|^\alpha}\,dr \ge C|\log n|^{1-\alpha}, \qquad (5.12)$$

for n sufficiently large. We also know that $u_1^n(x_1,0) \ge 0$ for all x_1.

Let ω^n be the radially symmetric extension of ω_+^n and set v_1^n to be the first component of $K * (\omega^n - \omega_+^n)$. As in the proof of Theorem 3, we find that

$$\pi v_1^n(x_1,x_2) = -\int_{-\infty}^{+\infty} \frac{x_2\, u_1^n(s,0)}{x_1-s)^2 + x_2^2}\,ds$$
$$\le -\int_{-2/n}^{2/n} \frac{x_2\, u_1^n(s,0)}{x_1-s)^2 + x_2^2}\,ds$$
$$\le -C(\log n)^{1-\alpha} \left[\arctan\left(\frac{x_1 + (2/n)}{x_2} \right) - \arctan\left(\frac{x_1 - (2/n)}{x_2} \right) \right],$$

by (5.12). It is easy to see that, if $|x_1| < 2/n$ and $0 < x_2 < 2/n$, then the difference of arctangents above is bounded from below by $\arctan 1 =$

$\pi/4$. Therefore we deduce that, if n is sufficiently large, then $v_1^n(x) \leq -C(\log n)^{1-\alpha}$ for $x \in B^+(0;2/n)$. Then, as in the proof of Theorem 3, we obtain (5.11) as long as n is large enough. This completes the proof that u_1^n is nonnegative in $B^+(0;r_0)$ for n large enough.

Let $\mathcal{U}_n \equiv B^+(0;r_0) \setminus B^+(0;1/\sqrt[3]{n})$. Recall that e_n is the error in the first component of velocity, due to truncation. We will show that there exists $C > 0$, such that for n sufficiently large we have

$$\left| \int_{\mathcal{U}_n} e_n |\omega_+^n|^2 \, dx \right| \leq \frac{C}{(\log n)^\alpha}. \tag{5.13}$$

In fact, we observe first that for $x \in \mathcal{U}_n$ we have

$$|e_n(x)| \leq C \sqrt[3]{n} \int_{B^+(0;1/n)} W_n(y) \, dy,$$

as $|x - y| \geq 1/(2\sqrt[3]{n})$ for n sufficiently large and $|y| \leq 1/n$, so that

$$|e_n(x)| \leq \frac{C}{n^{2/3}(\log n)^\alpha},$$

as in the proof of (5.8), see (5.9), (5.10). Additionally, for $x \in \mathcal{U}_n$,

$$|\omega_+^n(x)|^2 = \frac{1}{|x|^2 |\log |x||^{2\alpha}} \leq C(r_0) n^{2/3}.$$

Estimate (5.13) follows immediately from these two observations.

We now complete the proof of (5.6). We note that

$$\int_{B^+(0;r_0)} u_1^n |\omega_+^n|^2 \, dx \geq \int_{\mathcal{U}_n} u_1^n |\omega_+^n|^2 \, dx,$$

as $u_1^n \geq 0$ in $B^+(0;r_0)$,

$$= \int_{\mathcal{U}_n} u_1 |\omega_+^\alpha|^2 \, dx - \int_{\mathcal{U}_n} e_n |\omega_+^n|^2 \, dx \equiv \mathcal{I}_n + \mathcal{E}_n,$$

where we have used that $\omega_+^n = \omega_+^\alpha$ in $\mathcal{U}_n$ and $u_1^n = u_1 - e_n$.

By (5.13) we obtain that $\mathcal{E}_n \to 0$ as $n \to \infty$. Moreover, we have established in Theorem 3 that

$$\int_{B^+(0;r_0)} u_1(x) |\omega_+^\alpha(x)|^2 \, dx = \infty. \tag{5.14}$$

Therefore, using the Monotone Convergence Theorem, we find that $\mathcal{I}_n \to \infty$ as $n \to \infty$. We conclude that

$$\lim_{n \to \infty} \int_{B^+(0;r_0)} u_1^n |\omega_+^n|^2 \, dx = \infty.$$

To finish the proof of (5.6) we observe that arguments similar to those used above imply that u_1^n is bounded in $B^+(0;1/3) \setminus B^+(0;r_0)$; the same is true of ω_+^n by construction. This completes the proof of (5.6).

Finally, we turn to the convergence of ω_+^n to ω_+^α. Let $0 \le \kappa < \alpha - 1/2$. We estimate the difference in the Zygmund space $L^2(\log L)^\kappa$. We have that:

$$\int A_{2,\kappa}\left(\frac{W_n}{\|W_n\|_{2,\kappa}}\right) dx = 1, \tag{5.15}$$

since W_n does not vanish identically. We observe that $0 \le W_n \le \omega_+^\alpha \chi_{B^+(0;1/n)}$. By Lemma 3, $\omega_+^\alpha \in L^2(\log L)^\kappa$. Therefore, since $A_{2,\kappa}$ is nondecreasing, it follows that

$$\int A_{2,\kappa}(W_n)\, dx \le \int_{B^+(0;1/n)} A_{2,\kappa}(\omega_+^\alpha)\, dx \to 0, \tag{5.16}$$

as $n \to \infty$ by continuity of integrals. Now, recall that $A_{2,\kappa}$ is convex. Therefore,

$$A_{2,\kappa}\left(\frac{W_n}{\|W_n\|_{2,\kappa}}\right) \le \frac{1}{\|W_n\|_{2,\kappa}} A_{2,\kappa}(W_n). \tag{5.17}$$

By virtue of (5.15) and (5.17) we find

$$\|W_n\|_{2,\kappa} \le \int A_{2,\kappa}(W_n)\, dx.$$

Using (5.16) then implies that $\|W_n\|_{2,\kappa} \to 0$ as we wished.

We emphasize at this point that this section was concerned with the cubic nonlinearity $u|\omega|^2$ without reference to dynamics. Something strange might occur with enstrophy dissipation and with the transport enstrophy defect at the initial time for a weak solution of incompressible 2D Euler obtained with ω_+^α as initial data. We do not offer any prognosis, as the answer depends on how the initial snarl in the term $u|\omega|^2$ would resolve itself for positive time. It would be very interesting to determine what happens, but this problem does not seem tractable.

6. Counterexample for Eyink's conjecture

In this section we will present a counterexample to Eyink's conjecture, as formulated in Section 3. We will exhibit a family of solutions to the 2D Navier-Stokes equations, which converge, as viscosity vanishes, to a stationary solution of 2D Euler. This stationary solution is such that both Z^T and Z^V exist and Z^T vanishes identically while Z^V does not.

We consider ω_0 of the form:

$$\omega_0(x) = \phi(x)\frac{1}{|x|}, \quad \phi \in C_c^\infty(\mathbb{R}^2), \tag{6.1}$$

with ϕ radially symmetric, $\operatorname{Supp}\phi \subset B(0;1)$, $\phi \equiv 1$ on $B(0;1/2)$. Note that such ω_0 belongs to $L^{2,\infty} \cap L^p$, $1 \le p < 2$.

It is well known that any radially symmetric vorticity configuration $\omega = \omega(x) = \rho(|x|)$ gives rise to an exact steady solution u of the incompressible Euler equations, see [16]. As in Lemma 2, the 2D Biot-Savart law becomes:

$$u(x) = \frac{x^{\perp}}{|x|^2} \int_0^{|x|} s\rho(s)\, ds. \tag{6.2}$$

Such steady solutions are called Rankine vortices.

Remark 6. If ϕ is chosen instead so that $\int \omega_0(x)\, dx = 0$, then u defined in (6.2) is compactly supported, vanishing outside $\operatorname{Supp}\phi$ (see [7]). This observation would allow us to adapt the present example to the periodic case.

Similarly, if ω_ν is the solution of the *heat* equation

$$\partial_t \omega_\nu = \nu \Delta \omega_\nu, \tag{6.3}$$

with radially symmetric initial data ω_0, then $u_\nu \equiv K * \omega_\nu$ is a solution of the 2D Navier-Stokes equations with initial vorticity ω_0 and viscosity ν.

We will show that ω_0 belongs to $B^0_{2,\infty}$ and that the sequence ω_ν satisfies the hypothesis of the Eyink's Conjecture, however we postpone the proof of this fact to the end of this section,

see Proposition 4.

In what follows, we recall the notation used in Section 3. If j_ϵ is a (radially symmetric) Friedrichs mollifier, then we denote $j_\epsilon * \omega_0$ with ω_ϵ. We introduce the approximate transport defect $Z_\epsilon(\omega_0)$ and the approximate viscous defect $Z^\nu(\omega_\nu)$ as defined in Section 3.

We state below the main result of this section.

Theorem 5. *The enstrophy defects $Z^T(\omega_0)$ and $Z^V(\omega_0)$ both exist. Moreover,*

$$Z^T(\omega_0) \equiv 0 \quad \textit{while} \quad Z^V(\omega_0) = \frac{4\pi^3}{t}\delta_0,$$

where δ_0 is the Dirac measure supported at the origin.

Proof. To prove that $Z^T(\omega_0)$ exists and vanishes identically we observe that ω_ϵ remains radially symmetric by construction and the flow lines of $u_\epsilon = j_\epsilon * K * \omega_0$ are concentric circles centered at the origin. Therefore we find

$$Z_\epsilon(\omega_0) = -\nabla\omega_\epsilon \cdot ((u\omega_\epsilon)_\epsilon - u_\epsilon \omega_\epsilon) = 0,$$

so that $Z^T(\omega_0) \equiv 0$.

In the rest of the proof, we will discuss the viscous enstrophy defect. We begin by deriving sharp asymptotic estimates for $\nu\|\nabla\omega_\nu\|^2_{L^2} = \|Z^\nu(\omega_\nu)\|_{L^1}$. This is accomplished in the following proposition.

Proposition 3. *For each $t > 0$, the approximate viscous enstrophy defect satisfies:*

$$\|Z^\nu(\omega_\nu)\|_{L^1} = \frac{4\pi^3}{t} + o(1), \tag{6.4}$$

as $\nu \to 0^+$.

Proof (Proof of Proposition.). By Plancherel's Theorem we have

$$\|Z^\nu(\omega_\nu)\|_{L^1} = \nu \int_{\mathbb{R}^2} |\xi|^2 \, e^{-t\nu|\xi|^2} \, |\widehat{\omega}_0(\xi)|^2 \, d\xi. \tag{6.5}$$

We begin by estimating the Fourier transform of ω_0. Set

$$e = e(\xi) = |\xi||\widehat{\omega}_0(\xi)| - 2\pi.$$

We will show that e is a bounded function which vanishes along rays near ∞, i.e., for each $\xi \neq 0$ fixed, $|e(s\xi)| \to 0$ as $s \to \infty$. To this end, fix $\xi \neq 0$ and write $\xi = r\sigma$, with $|\sigma| = 1$ and $r = |\xi|$. We recall that

$$\widehat{\left(\frac{1}{|x|}\right)}(\xi) = \frac{2\pi}{|\xi|},$$

(see Lemma 1 of Chapter V of [19] for details) and hence

$$\widehat{\omega}_0(\xi) = \left(\frac{2\pi}{|z|} * \check{\phi}(z)\right)(-\xi), \tag{6.6}$$

by the usual properties of the Fourier transform. As $\phi \in C_c^\infty$ it follows that $\check{\phi} \in \mathcal{S}$, the Schwartz space of rapidly decaying smooth functions.

Using (6.6) now gives:

$$\begin{aligned} e(s\xi) = |s||\xi||\widehat{\omega}_0(s\xi)| - 2\pi &= |sr| \left| \left(\frac{2\pi}{|z|} * \check{\phi}(z)\right)(-s\xi)\right| - 2\pi \\ &= |sr| \left| \int \frac{2\pi}{|y|} \check{\phi}(-s\xi - y)\, dy \right| - 2\pi \\ &= |sr| \left| \int \frac{2\pi}{|sr||z|} \, \check{\phi}(-|sr|\sigma - |sr|z) \, |sr|^2 dz \right| - 2\pi, \end{aligned}$$

after making the change of variables $y = |sr|z$,

$$= \left| \int \frac{2\pi}{|z|} \, |sr|^2 \check{\phi}(|sr|(-\sigma - z))\, dz \right| - 2\pi$$

$$\equiv \left| \int \frac{2\pi}{|z|} \check{\phi}_{|sr|}(-\sigma - z)\, dz \right| - 2\pi = \left| \left(\frac{2\pi}{|z|} * \check{\phi}_{|sr|}(z)\right)(-\sigma) \right| - 2\pi,$$

where $\check{\phi}_M(\cdot) \equiv M^2 \check{\phi}(M \cdot)$.

It is easy to see that $|e(s\xi)|$ is uniformly bounded in both s and ξ, since $C/|z|$ is a locally integrable function, bounded near infinity, and $\check{\phi}_{|sr|}$ is small near infinity and integrable with constant integral with respect to sr. We simply estimate the convolution above by distinguishing points z near σ and points z far from σ. Then we use the fact that $|\sigma| = 1$.

Note that $\check{\phi}_{|sr|} \to \delta_0$ in $\mathcal{D}'$ as $s \to \infty$ so that the convolution above should, in principle, converge to $2\pi/|-\sigma| = 2\pi$. The difficulty in making this argument precise is that neither $2\pi/|z|$ nor $\check{\phi}$ are compactly supported.

Let $0 < \beta < 1/4$. Set $\eta_\beta = \eta_\beta(\xi)$ a radially symmetric smooth cut-off function of the ball of radius β, so that η_β is identically 1 in $B(0;\beta)$ and vanishes in $\mathbb{R}^2 \setminus B(0;2\beta)$. We use η_β to write:

$$\left(\frac{2\pi}{|z|} * \check{\phi}_{|sr|}(z)\right)(-\sigma) =$$

$$= \int \frac{2\pi}{|-\sigma - z|}\eta_\beta(z)\check{\phi}_{|sr|}(z)\,dz + \int \frac{2\pi}{|-\sigma - z|}(1-\eta_\beta(z))\check{\phi}_{|sr|}(z)\,dz$$
$$\equiv \mathcal{J}_1 + \mathcal{J}_2.$$

Note that, for each fixed σ, with $|\sigma| = 1$, the function $2\pi\eta_\beta(z)/|-\sigma - z|$ is smooth and compactly supported, which implies that $\mathcal{J}_1 \to 2\pi$ as $s \to \infty$. We show that $\mathcal{J}_2 \to 0$:

$$\left|\int \frac{2\pi}{|-\sigma - z|}(1-\eta_\beta(z))\check{\phi}_{|sr|}(z)\,dz\right| \leq$$

$$\leq \left|\int_{\beta<|z|<2} \frac{2\pi}{|-\sigma - z|}\check{\phi}_{|sr|}(z)\,dz\right| + \left|\int_{|z|>2} \frac{2\pi}{|-\sigma - z|}\check{\phi}_{|sr|}(z)\,dz\right|$$

$$\leq \|\check{\phi}_{|sr|}\|_{L^\infty(\{|z|>2\beta\})}\left|\int_{2<|z|<2\beta} \frac{2\pi}{|-\sigma - z|}\,dz\right| + 2\pi\left|\int_{|z|>2} \check{\phi}_{|sr|}(z)\,dz\right|.$$

Clearly each term above vanishes as $s \to \infty$.

Finally, we may now write:

$$|\xi|^2|\widehat{\omega}_0(\xi)|^2 = 4\pi^2 + 4\pi e(\xi) + |e(\xi)|^2, \tag{6.7}$$

so that, from (6.5), we find

$$\|Z^\nu(\omega_\nu)\|_{L^1} = \nu\left(\int_{\mathbb{R}^2} 4\pi^2\, e^{-t\nu|\xi|^2}\,d\xi + \int_{\mathbb{R}^2} (4\pi e(\xi) + |e(\xi)|^2)\, e^{-t\nu|\xi|^2}\,d\xi\right)$$

$$= \frac{1}{t}\int_{\mathbb{R}^2} 4\pi^2\, e^{-|z|^2}\,dz + \frac{1}{t}\int_{\mathbb{R}^2}\left(4\pi e\left(\frac{z}{\sqrt{t\nu}}\right) + \left|e\left(\frac{z}{\sqrt{t\nu}}\right)\right|^2\right) e^{-|z|^2}\,dz.$$

Since we have already shown that $e = e(z)$ is a bounded function and that $\lim_{\nu\to 0^+} e(z/\sqrt{t\nu}) = 0$, we deduce using the Dominated Convergence Theorem that

$$\lim_{\nu\to 0^+} \|Z^\nu(\omega_\nu)\|_{L^1} = \frac{4\pi^3}{t},$$

as we wished.

In view of the proposition above we find that, for each fixed $t > 0$, the set $\{Z^\nu(\omega_\nu),\ \nu > 0\}$ is uniformly bounded in L^1. Therefore, using the Banach-Alaoglu Theorem, for each $t > 0$ there is a sequence converging weakly to a Radon measure. Each of these measures is, in fact, a multiple of the Dirac measure, $C(t)\,\delta_0$, by virtue of the following claim, which we will prove later.

Claim. Any converging sequence of $\{Z^\nu(\omega_\nu),\ \nu > 0\}$ converges to a distribution supported at the origin.

Given the Claim we may conclude that $Z^\nu(\omega_\nu)$ is itself convergent (to a positive measure). To establish this result, it is enough to show that $C(t)$ is independent of the particular sequence $Z^{\nu_k}(\omega_{\nu_k})$. To this end, we fix a converging subsequence $Z^{\nu_k}(\omega_{\nu_k})$. We begin by observing that $Z^\nu(\omega_\nu)$ is a *tight* family of functions in L^1 with respect to the parameter ν. Indeed, $Z^\nu(\omega_\nu) = \nu|\nabla\omega_\nu|^2$ and ω_ν is the convolution of a compactly supported function with the heat kernel, so it is immediate to verify that $\int_{|z|>M} Z^\nu(\omega_\nu)\,dz \to 0$ as $M \to \infty$, uniformly in ν. Fix now $\epsilon > 0$ and choose M so large that $0 < \int_{|z|>M} Z^\nu(\omega_\nu)\,dz < \epsilon$. Then, if ψ_M is a smooth cut-off of the ball of radius $M+1$, we have

$$\left|\int_{\mathbb{R}^2} Z^{\nu_k}(\omega_{\nu_k})\,dx - \int_{\mathbb{R}^2} Z^{\nu_k}(\omega_{\nu_k})\,\psi_M\,dx\right| =$$

$$= \left|\int_{\mathbb{R}^2} Z^{\nu_k}(\omega_{\nu_k})\,(1-\psi_M)\,dx\right| < \epsilon,$$

By Proposition 3 and the Claim the left-hand side converges to $4\pi^3/t - C(t)$ as $k \to \infty$. As ϵ is arbitrary it follows that $C(t) = 4\pi^3/t$, independent of the sequence ν_k, as desired.

In summary, we have deduced that

$$\lim_{\nu\to 0^+} Z^\nu(\omega_\nu) = \frac{4\pi^3}{t}\delta_0, \text{ in } \mathcal{D}'. \tag{6.8}$$

It remains to establish the Claim.

Proof (Proof of Claim). We prove that, for any $\eta > 0$ and any $f \in C_c^\infty$ with $\mathrm{Supp}\, f \subset \mathbb{R}^2 \setminus B(0;\eta)$, we have

$$\lim_{\nu\to 0^+} \int_{\mathbb{R}^2} Z^\nu(\omega_\nu)\,f\,dx = 0. \tag{6.9}$$

The proof involves a simple estimate on ω_ν. Let $H_\nu = H_\nu(x,t) = (4\pi\nu t)^{-1}\,e^{-|x|^2/(4\nu t)}$ denote the heat kernel in $\mathbb{R}^2$. Recall that ω_ν satisfies (6.3) so that we may write $\omega_\nu = H_\nu * \omega_0$. Fix $\eta > 0$ and let $\varphi \in C_c^\infty$ be a cut-off of the ball of radius $\eta/2$ around the origin. We write

$$\omega_0 = \omega_0\,\varphi + \omega_0\,(1-\varphi) \equiv \omega_0^F + \omega_0^N.$$

We begin by observing that ω_0^N is a smooth function with compact support and hence

$$\nabla \omega_\nu = \nabla H_\nu * \omega_0^F + H_\nu * \nabla \omega_0^N.$$

Clearly $H_\nu * \nabla \omega_0^N$ is a bounded function, uniformly in ν. Next we estimate $\nabla H_\nu * \omega_0^F$ far from the origin. Let x be such that $|x| > \eta$. Then:

$$|\nabla H_\nu * \omega_0^F(x)| \leq \frac{1}{8\pi(\nu^2 t^2)} \left| \int_{|y|<\eta/2} |x-y| e^{-|x-y|^2/(4\nu t)} \omega_0(y)\, dy \right|$$

$$\leq \frac{C_1}{\nu^{3/2} t^{3/2}} \int_{|y|<\eta/2} e^{-C_2|x-y|^2/(\nu t)} \frac{1}{|y|}\, dy,$$

where we have used the fact that there exist constants C_1, $C_2 > 0$ such that $|z|e^{-|z|^2} \leq C_1 e^{-C_2|z|^2}$,

$$\leq \frac{2\pi C_1}{\nu^{3/2} t^{3/2}} e^{-C_2\eta/(\nu t)} \int_0^{\eta/2} e^{-r/(\nu t)}\, dr \leq C,$$

for some $0 < C < \infty$, C independent of ν. In summary we have shown that $\nabla \omega_\nu$ is bounded in the complement of $B(0;\eta)$ uniformly in ν. In view of this fact, since $Z^\nu(\omega_\nu) = \nu|\nabla \omega_\nu|^2$, (6.9) follows. This concludes the proof of the Claim.

The proof of Theorem 5 is complete.

We close by verifying that the sequence ω_ν satisfies the hypothesis of Eyink's conjecture.

Proposition 4. *We have that $\omega_0 \in B^0_{2,\infty}$ is a viscosity solution of the 2D Euler equations and*
$\omega_\nu \rightharpoonup \omega_0$ weak-$*$ in $L^\infty((0,T); B^0_{2,\infty})$, as $\nu \to 0^+$.

Proof. We begin by recalling the definition of the norm in $B^0_{2,\infty}$ (see e.g. [20], page 17):

$$\|f\|_{B^0_{2,\infty}} = \sup_{j \geq 0} \|\psi_j * f\|_{L^2},$$

where ψ_j are functions forming a Littlewood-Paley partion of unity. In particular, the Fourier transform of ψ_0, $\widehat{\psi_0}$, is smooth, compactly supported in the disk $B(0;1)$, $\widehat{\psi_0} \equiv 1$ on $B(0;2/3)$, while for $j > 0$, $\psi_j(x) = 2^{2j}\psi(2^j x)$, for a function ψ such that its Fourier transform $\widehat{\psi}$ is smooth, compactly supported in the shell $\{1/2 < |\xi| < 2\}$, $\widehat{\psi} \equiv 1$ on $\{2/3 < |\xi| < 4/3\}$.

We will estimate the low and high-frequency contribution to the $B^0_{2,\infty}$-norm of $\omega_\nu = H_\nu * \omega_0$ separately. Here again, H_ν is the heat kernel and the convolution is only in the space variable.

For the low-frequency part, we observe that $\omega_0 \in L^1_c(\mathbb{R}^2)$, the L^1-norm of H_ν as a function of x is uniformly bounded in t and ν, and that ψ_0 is smooth, rapidly decreasing. Consequently, by Young's inequality

$$\begin{aligned} \|\psi_0 * \omega_\nu(t)\|_{L^2} &\leq \|\psi_0\|_{L^2} \|\omega_\nu(t)\|_{L^1} \\ &\leq \|\psi_0\|_{L^2} \|H_\nu(t)\|_{L^1} \|\omega_0\|_{L^1} \leq C, \end{aligned} \tag{6.10}$$

C independent of ν and t.

To bound the high-frequency part we will employ the Fourier transform and knowledge of the behavior of $\widehat{w_0}$ gained in Proposition 6.4. In view of (6.7), we can write

$$\begin{aligned}\|\psi_j * \omega_\nu(t)\|_{L^2}^2 &= \int_{\mathbb{R}^2} |\widehat{\psi_j}(\xi)|^2\,|\widehat{\omega_\nu}|^2\,d\xi = \int_{\mathbb{R}^2} |\widehat{\psi}(2^{-j}\xi)|^2\,e^{-2\nu t\,|\xi|^2}\,|\widehat{\omega_0}(\xi)|^2\,d\xi \\ &\le \int_{\mathbb{R}^2} |\widehat{\psi}(2^{-j}\xi)|^2\,|\widehat{\omega_0}(\xi)|^2\,d\xi \\ &= \int_{\mathbb{R}^2} |\widehat{\psi}(2^{-j}\xi)|^2\,\frac{1}{|\xi|^2}(4\pi^2 + 4\pi e(\xi) + |e(\xi)|^2)\,d\xi.\end{aligned}$$

We now change variables from ξ to $\xi' = 2^{-j}\xi$, and use the support properties of $\widehat{\psi}$ to obtain

$$\|\psi_j * \omega_\nu\|_{L^2}^2 \le \int_{1/2<|\xi|<2} |\widehat{\psi}(\xi')|^2\,\frac{1}{|\xi'|^2}(4\pi^2 + 4\pi e(2^j\xi') + |e(2^j\xi')|^2)\,d\xi' \le C, \tag{6.11}$$

with C again independent of ν and t, since the function $e(2^j\xi)$ is bounded uniformly in j and ξ. We remark that this also shows that $\omega_0 \in B^0_{2,\infty}$.

Combining (6.10) and (6.11) finally gives

$$\sup_{\nu>0} \|\omega_\nu\|_{L^\infty((0,T);B^0_{2,\infty})} \le C < \infty.$$

Therefore, there exists a subsequence ω_{ν_k}, which converges weak-$*$ in $L^\infty((0,T);B^0_{2,\infty})$, to a weak-$*$ limit. But, since $H_\nu \rightharpoonup \delta_0$ in $\mathcal{S}'$ as $\nu \to 0^+$, we conclude that the whole family ω_ν converges weak-$*$ in $L^\infty((0,T);B^0_{2,\infty})$ and the weak-$*$ limit is ω_0.

What we have actually accomplished with Theorem 5 and Proposition 4 is to give a counterexample to the part of Eyink's conjecture identifying viscous and transport enstrophy defects. We have answered in the affirmative the part of the conjecture regarding the existence of a nontrivial enstrophy defect. Although we found such an example only for the viscous enstrophy defect, this is the physically meaningful one. Clearly, from the point of view of turbulence theory, one should attempt to understand better the viscous enstrophy defect. Informally, viscous dissipation of a quantity is enhanced the more complicated the spatial distribution of that quantity. Our radially symmetric, monotonic example is as simple a configuration as possible, and, as such, should have the least dissipation. We imagine that, in some sense, the viscous enstrophy defect should be greater for a generic configuration, and existence of the viscous enstrophy defect would be the more problematic issue.

7. Conclusions

We would like to add a few general remarks regarding the work presented here. First, the theory of viscous and transport enstrophy defects can be formulated in the more general setting of weak and renormalized solutions of linear transport equations and vanishing viscosity limits. The only instance where the specific form of the incompressible fluid flow equations was used is when we attributed meaning to the expression $u|\omega|^2$ for $\omega \in L^2(\log L)^{1/4}$. In particular, the counterexample presented in Section 6 is really a solution of the heat equation, of some interest even without mentioning the fluid dynamical context.

Our counterexample to Eyink's Conjecture is circularly symmetric, and as such, it corresponds to solutions of the Navier-Stokes equations for which the nonlinear term $P(u\cdot\nabla u)$ vanishes identically (P is the Leray projector). Since turbulence is regarded as coming from the interaction of nonlinearity and small viscosity, it is fair to ask what possible relevance would such an example have for the understanding of turbulence. If one looks at the cascade ansatz, the basic idea is that the nonlinearity produces a flow of enstrophy, from large to small scales across the inertial range, to be dissipated by viscosity. For flows with finite enstrophy, the nonlinearity must play a crucial role in sustaining the cascade because without the nonlinearity the viscosity would instantly make small scale enstrophy disappear. Now, for flows with infinite enstrophy, the nonlinearity is not needed for a sustained cascade because there already is an infinite supply of enstrophy at small scales. At this level, it is possible for the flow of enstrophy to small scales due to the nonlinearity to be small, or irrelevant. This would be a plausible explanation for why the viscous and transport enstrophy defects are not the same. It would be interesting to take a new look at the Kraichnan-Batchelor theory in light of this possibility.

It is not clear whether the notions of enstrophy defect will become useful in general issues of interest in PDE, but this is certainly possible and further research along this line is amply warranted. Due to the unexplored nature of this subject, it is easy to formulate a long list of open problems. We will single out a few that appear either particularly accessible or interesting. The main open problem is to prove that viscous enstrophy defects are well defined for some class of flows with infinite (local) enstrophy. Another important problem is to find an example of a solution to an inviscid transport equation, preferably given by a solution of the Euler equations, for which the transport enstrophy defect is nonzero. We have seen that the transport enstrophy density is a weak solution of the appropriate transport equation for initial vorticities in $L^2(\log L)^{1/4}$ if the weak solution comes from vanishing viscosity. It would be very interesting to find other properties of viscosity solutions that are not shared by general weak solutions. Although enstrophy plays a distinguished role among integrals of convex functions of vorticity due to its relevance to turbulence modeling, it is reasonable to ask to which extent similar defects might be usefully associated to other such first integrals. There is a certain arbitrariness in the definition of transport enstrophy defect that might be explored, as one could define another inviscid enstrophy defect by mollifying the initial data, for example. Finally, we state again a problem suggested in Section 2: determine whether viscosity solutions are renormalized solutions of the transport equations if initial vorticity is in L^p, $p < 2$. Note that nonuniqueness of weak solutions follows immediately if this is not the case.

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