MATH 502 - MIDTERM

REVIEW PROBLEMS

a) Show that the function $z \cdot \mathrm{Re}(z)$ is complex-differentiable at $z=0$, but non holomorphic.

b) Let f be a polynomial in z and assume

$$\int_{\partial D(0,1)} f(z)\,\bar{z}^j\,dz = 0 \qquad j=0,1,2,3,\ldots$$

Prove that $f \equiv 0$.

c) Find the power series expansion of $\dfrac{z^2}{(1-z^2)^3}$ about 0 and determine the radius of convergence. Do not use Taylor's theorem.

d) Prove that for all $\xi \in \mathbb{C}$, $e^{-\pi\xi^2} = \displaystyle\int_{-\infty}^{+\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx$

e) Suppose that f, g are <u>entire</u> functions and that g never vanishes. If $|f(z)| \leq |g(z)| \;\forall z$, prove that $\exists\, C$ constant such that $f(z) = C\,g(z)$.
What if g has zeroes?

f) If $f_j \in \mathrm{Hol}(\Omega)$, $|f_j(z)| \leq 2^{-j} \;\forall z \in \Omega$, then prove that $\displaystyle\sum_{j=0}^{\infty} f_j$ converges to $f \in \mathrm{Hol}(\Omega)$.

g) Let $z=0$. Classify each of the following as having a removable singularity, an essential singularity, or a pole at 0:

1) $\frac{1}{z}$, 2) $\sin\frac{1}{z}$, 3) $\frac{1}{z^3} - \cos z$, 4) $z \cdot e^{+\frac{1}{z}} e^{-\frac{1}{z^2}}$

5) $\frac{\sin z}{z}$.

h) Let f be holomorphic in a neighborhood of 0. Set $g_k(z) = z^{-k} f(z)$. If $\mathrm{Res}_0(g_k)$, $k = 0, 1, 2, 3, \dots$ Prove that $f \equiv 0$.

j) Compute the integrals $\int_{-\infty}^{+\infty} \frac{x \sin x}{1+x^2}\, dx$, $\int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^2 e^{ix}\, dx$.

k) Estimate the number of zeroes of the function $f(z) = z^2 e^z - z$ in the disk with center the origin, radius 2.

i) Prove that if $f : \Omega \to \mathbb{C}$ is holomorphic, $z_0 \in \Omega$, $f'(z_0) = 0$, then f is not one-to-one in any neighborhood of z_0.

l) Let h be meromorphic in a neighborhood of the closed unit disk. Assume that $|h(z)| = 1$ if $|z| = 1$. Show h is a rational function.