MATH 312 – REVIEW FOR FINAL

Study Reviews for Midterm I and II.

1 MORE on SERIES

Integral test: Compare $\sum_{n=m}^{N} a_n$ with a definite integral. If the integral can be computed or estimated easily, this test works well since it is a complete test:

$$\boxed{\sum_{n=m}^{N} f(n) \text{ converges} \iff \int_m^{+\infty} f(x)\,dx < +\infty.}$$

Can apply only if $a_n = f(n)$, where f is a positive function.

Ex: a) $\boxed{\sum_{n=1}^{+\infty} \frac{1}{n^p}}$ converges if and only if $p > 1$.

Proof: $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ converges $\iff$ $\int_1^{+\infty} \frac{1}{x^p}\,dx < +\infty$ by the integral test. But $\int \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p} x^{1-p} & p \neq 1 \\ \log(x) & p = 1 \end{cases}$

So $\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{N \to +\infty} \begin{cases} \frac{-1}{1-p}\left(1 - N^{1-p}\right) & p \neq 1 \\ +\log(N) & p = 1 \end{cases} = \begin{cases} +\infty & \text{if } p < 1 \\ \frac{1}{p-1} & p > 1 \\ +\infty & \text{if } p = 1 \end{cases}$

b) works well also for series of the form $\sum_{n=2}^{+\infty} \frac{1}{n(\log n)^p}$

⑥ Alternating Series Test (skipped proof):

If $a_1 \geq a_2 \geq \dots \geq a_n \geq \dots \geq 0$ and $\lim_{n \to \infty} a_n = 0$,

then the alternating series $\boxed{\sum (-1)^n a_n}$

converges.

Ex: $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ converges.

③ CONTINUITY

- Continuity is about functions

$f : D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}$

D domain of f (natural domain = largest possible domain)

$R = \{ f(x) : x \in D \}$ range of f , $R \subseteq \mathbb{R}$

- operations on functions:

$(f+g)(x) = f(x) + g(x)$ $\quad$ $(f \cdot g)(x) = f(x) \cdot g(x)$

$(kf)(x) = k f(x)$ $\quad k \in \mathbb{R}$ $\quad$ $(f/g)(x) = f(x)/g(x)$ $\quad (g(x) \neq 0)$.

$(f \circ g)(x) = f(g(x))$ composition of f and g

<u>$\varepsilon-\delta$ definition</u> : Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$.

Let $x_0 \in D$. We say that f <u>is continuous at x_0</u>, if : $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$x \in D, \ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

<u>Sequence Definition</u> : Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$.

Let $x_0 \in D$. We say that f <u>is continuous at x_0</u> if : for <u>every</u> sequence (x_n), $(x_n) \subseteq D$,

Such that $x_n \xrightarrow[n\to\infty]{} x_0$, $f(x_n) \xrightarrow[n\to\infty]{} f(x_0)$.

- <u>the two definitions are equivalent</u>.
- $\varepsilon-\delta$ definition is useful especially when proving properties of continuous functions
- Sequence definition is useful especially when showing that a function is continuous.

- $\varepsilon-\delta$ definition says that we can get <u>arbitrarily</u> close to $f(x_0)$ using $f(x)$ provided x is sufficiently close to x_0, in the domain of f.

- In the ε-δ definition, ε and the point x_0 are given <u>first</u>, <u>δ is a function of ε and x_0</u>

<u>Ex</u>: $f(x) = \frac{1}{x}$, $x_0 = 1$.

Given $\varepsilon > 0$, set $\delta \leq \dfrac{\varepsilon}{1+\varepsilon}$. then:

$$\left| \frac{1}{x} - 1 \right| = \frac{|x-1|}{x} < \frac{\delta}{x} \leq \frac{\delta}{1-\delta} \leq \varepsilon$$

- Recall

$$\boxed{\begin{aligned} &|x - x_0| < \delta \iff x_0 - \delta < x < x_0 + \delta \\ &|f(x) - f(x_0)| < \varepsilon \iff f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon. \end{aligned}}$$

- You can assume that <u>both ε and δ are</u> <u>small</u>, e.g. $\varepsilon < 1$, $\delta < 1/2$ (depending on the problem).

- To prove that a function is <u>not</u> <u>continuous</u> at a point:

  - using ε-δ definition, show $\exists\ \varepsilon > 0$ such that $\forall\ \delta > 0$, $\exists\ x_\delta \in D$, $|x_\delta - x_0| < \delta$ but $|f(x_\delta) - f(x_0)| \geq \varepsilon$.

  - using <u>sequence definition</u>, show $\exists\ (x_n) \subseteq D$, such that $x_n \xrightarrow[n\to\infty]{} x_0$, but $f(x_n)$ does not converge to $f(x_0)$ as $n \to \infty$ (<u>easier method</u>).

Definition of limit : We say that a function $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, has limit L as $x \to x_0$, if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that if $x \in D$, $|x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

- x_0 need not be in D. f is continuous at x_0 if 1) $f(x_0)$ exists, 2) $\lim_{x \to x_0} f(x)$ exists, 3) $f(x_0) = \lim_{x \to x_0} f(x)$.

Def : A function $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, is said to be continuous on a set $S \subseteq D$, if f is continuous at every $x_0 \in S$. f is simply continuous, if it is continuous on its domain.

theorem 5 : Let f, g be real-valued functions, continuous at x_0. then:

i) $f+g$, $f \cdot g$ are continuous at x_0

ii) kf, $|f|$, $k \in \mathbb{R}$, are continuous at x_0

iii) f/g is continuous at x_0 if $g(x_0) \neq 0$.

Proof : Reduce to the corresponding theorems for sequences and use sequence definition. ∎

Theorem 6 : If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

- All trigonometric functions are continuous where they are defined
- e^x ($\log x$) is continuous on $\mathbb{R}$ (on $(0, +\infty)$)
- polynomial functions are continuous on $\mathbb{R}$, rational functions are continuous except at points where denominator vanishes.
- $\max(f, g)$, $\min(f, g)$ are continuous if f, g are.

Properties of continuous functions

Theorem 7 : Let $f : [a, b] \to \mathbb{R}$ be continuous on a closed, bounded interval. Then, f is bounded and f has maximum value and a minimum value.

- f is bounded on $[a,b]$ if $\exists\ M>0$ such that $|f(x)| \leq M \ \forall\, x \in [a,b]$.

<u>Idea of proof</u> : Proceed by <u>contradiction</u> and assume f not bounded $\Rightarrow \forall\, n \in \mathbb{N}$, $\exists\ x_n \in [a,b]$ such that $|f(x_n)| > n$. (x_n) is bounded, so by Bolzano – Weierstrass $\exists\, (x_{n_k})$, $x_{n_k} \to x_0$. $x_0 \in [a,b]$ closed. By continuity at x_0:

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0) \quad \text{contradiction.}$$

To prove, f attains maximum value, consider $M = \sup\{f(x) : x \in [a,b]\} \Rightarrow \exists\ x_n \in [a,b]$ such that $f(x_n) \underset{n\to\infty}{\to} M$. Proceeding as above, $\exists$ (x_{n_k}), $x_{n_k} \underset{k\to\infty}{\to} x_1 \in [a,b]$ and by continuity

$$f(x_1) = \lim f(x_{n_k}) = \lim f(x_n) = M. \quad \blacksquare$$

- the conditions that the interval is closed and bounded are necessary.

<u>Ex</u> : $f(x) = \dfrac{1}{x}$, continuous and unbounded on $(0,1)$

$f(x) = x$, continuous on $(0,1)$, but with no maximum or minimum.

Intermediate Value Theorem:

If $f: I \to \mathbb{R}$, I interval, is continuous then f has the Intermediate Value Property (IVP):

if $a, b \in I$, $a < b$, and y lies in between $f(a)$ and $f(b)$, then $\exists$ at least one $x \in (a, b)$ such that $f(x) = y$.

- there can be many x corresponding to the same value y.
- A consequence of the IVP is that the graph of a continuous function on an interval (need not be closed, nor bounded) is a solid, continuous line:

Theorem 8: the range of $f: I \to \mathbb{R}$, I an interval, f continuous, $R = \{f(x) : x \in I\}$ is either a point or an interval.

SKIP PROOF.

Idea of proof : If R contains at least two points, then $\operatorname{Sup}(R) = \alpha > \operatorname{Inf}(R) = \beta$.

Show $R =$ interval with endpoints α & β, given $\alpha > y > \beta$, $\exists$ $y_0 \in R$, $y_1 \in R$ such that $y_0 < y < y_1$ by definition of Sup and Inf. then, apply IVP to conclude $y \in R$. ∎

- Application of theorem 8 to finding fixed points.

If $f: D \to \mathbb{R}$, a fixed point $x_0 \in D$ of f is such that $x_0 = f(x_0)$.

- There are discontinuous functions that have the Intermediate value Property.

Ex : $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$

- If a function is strictly increasing (or decreasing) and has IVP, it is continuous $\Rightarrow$

the inverse f^{-1} of a strictly increasing (decreasing) function is continuous. Recall $\boxed{f^{-1}(y) = x \iff f(x) = y.}$

UNIFORM CONTINUITY

- Stronger notion of continuity.
- the graph of a uniformly continuous function is a solid continuous line that changes slowly and does not escape to infinity.
- We have uniform continuity if we can choose δ in the $\varepsilon - \delta$ definition of continuity depending only on ε but not on x_0 :

Def : Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. We say f is uniformly continuous on S if :

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that if

$x, y \in S$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Ex : $f(x) = \frac{1}{x}$, $S = [a, +\infty)$.

δ can be taken in the ε-δ definition of continuity at x_0 as : $\delta \leq \frac{\varepsilon x_0^2}{1 + \varepsilon x_0} \Rightarrow \delta \leq \frac{\varepsilon a^2}{1 + \varepsilon a}$ $\forall x_0 \geq a$.

- δ may still depend on the set S, as well as ε.
- Uniform continuity is a notion about a function and a set
- to show that a function is not uniformly continuous, it is enough to show δ becomes arbitrarily small for given, fixed ε, as x_0 vary in the set.

Ex : $f(x) = \frac{1}{x}$, $S = (0, 1]$.

$$\delta \le \frac{\varepsilon x_0^2}{1 + \varepsilon x_0}, \text{ so } \delta \to 0 \text{ as } x_0 \to 0.$$

theorem 9 : If f is continuous on a closed, bounded interval $[a, b]$, then it is uniformly continuous.

Idea of proof : Assume by contradiction that f is not uniformly continuous $\Rightarrow \exists\, \varepsilon > 0$ such that $\forall n \in \mathbb{N}$, $\exists\, x_n, y_n \in [a,b]$, $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \ge \varepsilon$.

By Bolzano - Weierstrass, $\exists\, (x_{n_k})$, $x_{n_k} \xrightarrow[k\to\infty]{} x_0 \in [a, b]$

because $[a,b]$ closed and bounded.

By continuity, $f(x_{n_k}) \xrightarrow[k\to\infty]{} f(x_0)$, but

$|x_n - y_n| < \frac{1}{n} \implies y_{n_k} \xrightarrow[k\to\infty]{} x_0 \implies f(y_{n_k}) \xrightarrow[k\to\infty]{} f(x_0)$

by continuity $\implies |f(x_{n_k}) - f(y_{n_k})| \xrightarrow[k\to\infty]{} 0$.

Contradiction.

∎

- Uniformly continuous functions have nice properties:

  - f uniformly continuous on a bounded interval $I \implies \int_I f(x)\,dx$ exists, finite number.

  - f uniformly continuous on $(a,b) \iff$ f can be extended to a continuous function on $[a,b]$.

  - if (s_n) Cauchy sequence, $(f(s_n))$ also a Cauchy sequence.

- Another criterion for uniform continuity:

  $f: I \to \mathbb{R}$, I interval, f' exists & bounded on $I_0 = I$ with endpoints removed $\implies f$ uniformly continuous.