

Math 312, Fall 2004

Practice Midterm 2

Total 40 pts+ 10 extra credit

“Prove” means give a careful, well-explained proof.

Put your name on the exam.

Good luck!

The amount of work for this practice midterm is expected to be more than 2 hours. The real midterm will be shorter.

PROBLEM	SCORE
1	
2	
3	
4	
5	
TOTAL	

1a. Give the definition of supremum, infimum, maximum, minimum of $f(x)$ on an interval I .

1b. Is the product of increasing functions an increasing function? Either prove it, or give a counterexample.

2. State the absolute convergence theorem for series and prove it.

3. Determine where the following series converge

$$a) \sum \ln nx^n, b) \sum \frac{1}{n^n} x^n, c) \sum \frac{x^n}{\ln n}.$$

Use the n-th root test.

$$a) \lim |a_n|^{1/n} = |x|.$$

If $x = \pm 1$, the series $\sum (-1)^n \ln n$ diverges because $a_n \not\rightarrow 0$:

$$\lim_{n \rightarrow \infty} \ln n = \infty.$$

$$b) \lim |a_n|^{1/n} = 0, R = \infty.$$

$$c) \lim |a_n|^{1/n} = |x|, R = 1.$$

If $x = 1$, the series $\sum 1/\ln n$ diverges by the comparison test:

$$\frac{1}{n} \leq \frac{1}{\ln n}.$$

If $x = -1$, the series $\sum (-1)^n / \ln n$ converges by the Cauchy test.

4a. Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences. Let

$$s_n = \sum_{k=1}^n a_k$$

be the n -th partial sum of $\{a_n\}$. Prove the summation by parts formula: for $m \leq n$

$$\begin{aligned} \sum_{k=m}^n a_k b_k &= (s_n b_n - s_{m-1} b_m) + \sum_{k=m}^{n-1} s_k (b_k - b_{k+1}). \\ &= (s_n b_n - s_{m-1} b_m) + \sum_{k=m}^{n-1} s_k (b_k - b_{k+1}) \\ &= s_n b_n - s_{m-1} b_m + \sum_{k=m}^{n-1} s_k b_k - \sum_{k=m}^{n-1} s_k b_{k+1} \\ &= s_n b_n + \sum_{k=m}^{n-1} s_k b_k - s_{m-1} b_m - \sum_{k=m+1}^n s_{k-1} b_k = \sum_{k=m}^n s_k b_k - \sum_{k=m}^n s_{k-1} b_k \\ &= \sum_{k=m}^n (s_k - s_{k-1}) b_k = \sum_{k=m}^n a_k b_k. \end{aligned}$$

4b. Using the summation by parts formula prove that

$$\sum \frac{\sin 2nx}{n} \tag{1}$$

converges for all x . Hint: you may use that for $x \neq \pi m$, $m \in \mathbb{Z}$

$$\sum_{k=1}^n \sin 2kx = \frac{1}{2 \sin x} (\cos x - \cos(2n+1)x).$$

When $x = \pi m$ all $\sin 2kx = 0$. So the series converges to zero.

When $x \neq \pi m$, we will show that (1) is Cauchy. We have

$$b_k - b_{k+1} = \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}.$$

Using the summation by parts formula and the hint

$$\begin{aligned} \sum_{k=m}^n \frac{\sin 2kx}{k} &= \frac{1}{2 \sin x} \left((\cos x - \cos(2n+1)x) \frac{1}{n} \right. \\ &\quad \left. - (\cos x - \cos(2m-1)x) \frac{1}{m} + \sum_{k=m}^{n-1} (\cos x - \cos(2k+1)x) \frac{1}{k(k+1)} \right). \end{aligned}$$

Since $|\cos \alpha| \leq 1$ and $1/k(k+1) \leq 1/k^2$, by triangle inequality

$$\left| \sum_{k=m}^n \frac{\sin 2kx}{k} \right| = \frac{1}{|\sin x|} \left(\frac{1}{n} + \frac{1}{m} + \sum_{k=m}^{n-1} \frac{1}{k^2} \right).$$

Since

$$\sum \frac{1}{k^2}$$

converges and $1/n \rightarrow 0$, as $n \rightarrow \infty$, given $\varepsilon > 0$, there exists N , such that for any $m, n \geq N$

$$\frac{1}{n} < \frac{\varepsilon |\sin x|}{3}, \quad \frac{1}{m} < \frac{\varepsilon |\sin x|}{3}, \quad \sum_{k=m}^{n-1} \frac{1}{k^2} < \frac{\varepsilon |\sin x|}{3}.$$

Altogether

$$\left| \sum_{k=m}^n \frac{\sin 2kx}{k} \right| < \varepsilon.$$

5 (Extra credit). Prove

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \Rightarrow \lim_{n \rightarrow \infty} |a_n|^{1/n} = L.$$

Deduce that the n -th root test is stronger than the ratio test, in the sense that if a series satisfies the hypothesis of the ratio test, it will also satisfy the hypothesis of the n -th root test. Give an example of the series, for which the ratio test is inconclusive, whereas the n -th root test determines the convergence property of the series.

Suppose $\lim_{n \rightarrow \infty} |a_{n+1}|/|a_n| = L$. It implies that for a given $\varepsilon > 0$, there exists N such that $|a_n|(L - \varepsilon) < |a_{n+1}| < |a_n|(L + \varepsilon)$, for all $n > N$. By induction we can prove that there exist two positive constants c_1 and c_2 so that for any $n > N$: $c_1(L - \varepsilon)^n < |a_n| < c_2(L + \varepsilon)^n$. Since for any c_1 and c_2

$$\lim_{n \rightarrow \infty} (c_1(L - \varepsilon)^n)^{1/n} = (L - \varepsilon) \lim_{n \rightarrow \infty} c_1^{1/n} = L - \varepsilon, \quad \lim_{n \rightarrow \infty} (c_2(L + \varepsilon)^n)^{1/n} = L + \varepsilon,$$

there exists N_1 so that for all $n > N_1$

$$(c_1(L - \varepsilon)^n)^{1/n} > L - 2\varepsilon, \quad (c_2(L + \varepsilon)^n)^{1/n} < L + 2\varepsilon.$$

Hence for all $n > N_1$

$$L - 2\varepsilon < |a_n|^{1/n} < L + 2\varepsilon.$$

Therefore by the $K - \varepsilon$ principle $\lim_{n \rightarrow \infty} |a_n|^{1/n} = L$.

Let us now construct a counterexample. Take

$$a_n = x_n^n, \quad \text{where } x_n = \sum_{k=1}^n (-1)^{k+1} \frac{1}{k}.$$

We know that (p.52 Example 4.2) $\sum_{n=1}^{\infty} (-1)^{n+1} 1/n = \ln 2 < 1$. Hence $\lim_{n \rightarrow \infty} |a_n|^{1/n} = \lim_{n \rightarrow \infty} x_n = \ln 2$. This guarantees that the series converges. The ratio test is, however, inconclusive:

$$\frac{|a_{n+1}|}{|a_n|} = x_{n+1} \left(1 + \frac{(-1)^{n+2}}{(n+1)x_n} \right)^n.$$

On the one hand, when $n = 2m$ is even

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{|a_{2m+1}|}{|a_{2m}|} &= \lim_{m \rightarrow \infty} x_{2m+1} \left(1 + \frac{1}{(2m+1)x_{2m}} \right)^{2m} \\ &= \ln 2 \lim_{m \rightarrow \infty} \left(\left(1 + \frac{1}{(2m+1)x_{2m}} \right)^{(2m+1)x_{2m}} \right)^{2m/(2m+1)x_{2m}} = \ln 2 \exp(1/\ln 2). \end{aligned}$$

On the other hand, when $n = 2m - 1$ is odd

$$\lim_{m \rightarrow \infty} \frac{|a_{2m}|}{|a_{2m-1}|} = \lim_{m \rightarrow \infty} x_{2m} \left(1 - \frac{1}{2mx_{2m-1}} \right)^{2m-1} = \ln 2 \exp(-1/\ln 2).$$

Some further practice questions.

Suppose $\sum a_n$ converges $a_n \neq 0$. Decide whether the following statement is true or false and then prove that you are correct.

a) $\sum (-1)^n a_n$ converges

b) $\sum a_n x^n$ converges for $|x| < 1$.

c) $\sum a_n^3$ converges.

d) $\sum 1/a_n$ diverges.

a) False $a_n = (-1)^n/n$.

b) True. Since $\sum a_n$ converges $|a_n| \leq K$. Hence

$$|a_n||x|^n \leq K|x|^n.$$

Since the geometric series $\sum |x|^n$ converges for $|x| < 1$, by comparison test $\sum a_n x^n$ converges for $|x| < 1$.

c) False

$$\sum a_n = 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{2^{1/3}} \left(1 - \frac{1}{2} - \frac{1}{2}\right) + \frac{1}{3^{1/3}} \left(1 - \frac{1}{2} - \frac{1}{2}\right) + \dots = 0.$$

$$\sum a_n^3 = 1 - \frac{1}{2^3} - \frac{1}{2^3} + \frac{1}{2} \left(1 - \frac{1}{2^3} - \frac{1}{2^3}\right) + \frac{1}{3} \left(1 - \frac{1}{2^3} - \frac{1}{2^3}\right) + \dots = \frac{3}{4} \sum \frac{1}{n} = \infty.$$

d) True. $a_n \not\rightarrow 0$.

Using power series representation for the geometric series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n,$$

find power series representation for $\ln(1+x)$, $0 < x < 1$. Prove it rigorously.

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}.$$

Let

$$S_k = \sum_{n=0}^k \frac{(-1)^n}{n+1} x^{n+1}, s_k = \sum_{n=0}^k (-1)^n x^n$$

Then

$$\begin{aligned} |\ln(1+x) - S_k| &= \left| \int_0^x \frac{dt}{1+t} + \int_0^k s_k(t) dt \right| \leq \int_0^x \left| \sum_{n=0}^{\infty} (-1)^n t^n - \sum_{n=0}^k (-1)^n t^n \right| dt \\ &= \int_0^x \left| \sum_{n=k+1}^{\infty} (-1)^n t^n \right| dt = \int_0^x t^{k+1} \left| \sum_{n=0}^{\infty} (-1)^n t^n \right| dt = \int_0^x \frac{t^{k+1}}{1+t} dt \leq \int_0^x t^{k+1} dt \leq \frac{x^{k+2}}{k+2}. \end{aligned}$$

When $0 < x < 1$

$$|\ln(1+x) - S_k| < \frac{x^{k+2}}{k+2} \rightarrow 0,$$

as $n \rightarrow \infty$.