

Math 312, Fall 2004

Practice Midterm 2

Total 40 pts+ 10 extra credit

“Prove” means give a careful, well-explained proof.

Put your name on the exam.

Good luck!

The amount of work for this practice midterm is expected to be more than 2 hours. The real midterm will be shorter.

| PROBLEM | SCORE |
|---------|-------|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| TOTAL | |

1a. Give the definition of supremum, infimum, maximum, minimum of $f(x)$ on an interval I .

1b. Is the product of increasing functions an increasing function? Either prove it, or give a counterexample.

2. State the absolute convergence theorem for series and prove it.

3. Determine where the following series converge

$$a) \sum \ln nx^n, b) \sum \frac{1}{n^n} x^n, c) \sum \frac{x^n}{\ln n}.$$

4a. Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences. Let

$$s_n = \sum_{k=1}^n a_k$$

be the n -th partial sum of $\{a_n\}$. Prove the summation by parts formula: for $m \leq n$

$$\sum_{k=m}^n a_k b_k = (s_n b_n - s_{m-1} b_m) + \sum_{k=m}^{n-1} s_k (b_k - b_{k+1}).$$

4b. Using the summation by parts formula prove that

$$\sum \frac{\sin 2nx}{n}$$

converges for all x . Hint: you may use that for $x \neq \pi m$, $m \in \mathbb{Z}$

$$\sum_{k=1}^n \sin 2kx = \frac{1}{2 \sin x} (\cos x - \cos(2n+1)x).$$

5 (Extra credit). Prove

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \Rightarrow \lim |a_n|^{1/n} = L.$$

Deduce that the n-th root test is stronger than the ratio test, in the sense that if a series satisfies the hypothesis of the ratio test, it will also satisfy the hypothesis of the n-th root test. Give an example of the series, for which the ratio test is inconclusive, whereas the n-th root test determines the convergence property of the series.

Some further practice questions.

Suppose $\sum a_n$ converges $a_n \neq 0$. Decide whether the following statement is true or false and then prove that you are correct.

- a) $\sum (-1)^n a_n$ converges
- b) $\sum a_n x^n$ converges for $|x| < 1$.
- c) $\sum a_n^3$ converges.
- d) $\sum 1/a_n$ diverges.

Using power series representation for the geometric series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n,$$

find power series representation for $\ln(1+x)$, $0 < x < 1$. Prove it rigorously.

Hint: Review Chapter 4.2.