Math 312, Fall 2004
Practice midterm 1
Total 40 pts+ 10 extra credit
Please, show the details of your work.
Put your name on the exam.
Good luck!

1a. Give the definition of an increasing sequence.

1b. Formulate the negative statement without using "not":
Sequence $\{a_n\}$ converges to zero.

Solution There is a positive number C, and a subsequence $\{a_{n_i}\}$ so that

$$|a_{n_i}| \geq C.$$

2. State the uniqueness theorem for limits and prove it.

3. Evaluate the limits and prove you are correct:

$$a) a_{n+1} = a_n^2, a_1 = -1/2, n \in \mathbb{N},$$
$$b) b_n = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + (-1)^n \frac{1}{n+1}, n \in \mathbb{N},$$
$$c) c_n = -\frac{n}{n^2+1}, n \in \mathbb{N}.$$

Solution a) $a_n \to 0$. Let us show by induction that

$$a_n = \frac{1}{2^{2^{n-1}}}, \quad n \geq 2.$$

Basis: $n = 2$:

$$a_2 = \frac{1}{2^2} = \frac{1}{4}.$$

Inductive step: by induction

$$a_{k+1} = a_k^2 = \left(\frac{1}{2^{2^{k-1}}} \right)^2 = \frac{1}{2^{2^{k-1} \cdot 2}} = \frac{1}{2^{2^k}}.$$

Given $\varepsilon > 0$

$$|a_n - 0| < \varepsilon,$$

when

$$\frac{1}{2^{2^n}} < \varepsilon \Leftrightarrow 2^n \ln 2 > -\ln \varepsilon \Leftrightarrow n > \ln \left(-\frac{\ln \varepsilon}{\ln 2} \right) / \ln 2.$$

For b) see book. Geometric series.
c) $c_n \to 0$. Let us estimate the error

$$| - \frac{n}{n^2+1} - 0| \leq \frac{1}{n}$$

Hence

$$| - \frac{n}{n^2+1} - 0| \leq \epsilon, \text{ when } n \geq \frac{1}{\varepsilon}.$$

4. Suppose $a_n \to 0$ as $n \to \infty$, b_n is bounded. Prove that for any sequence $\{c_n\}$

$$\frac{a_n b_n}{1 + c_n^2} \to 0,$$

as $n \to \infty$.

Solution

$$\left| \frac{a_n b_n}{1 + c_n^2} - 0 \right| \leq |a_n| \left| \frac{|b_n|}{1 + c_n^2} \right|$$

Since b_n is bounded, there exists a constant $B \geq 0$, so that $|b_n| \leq B$ for all n. For any c_n we also have

$$1 + c_n^2 \geq 1 \Leftrightarrow \frac{1}{1 + c_n^2} \leq 1.$$

Hence

$$|a_n| \left| \frac{|b_n|}{1 + c_n^2} \right| \leq B|a_n|$$

Since $a_n \to 0$. Given $\varepsilon > 0$, $|a_n| < \varepsilon$, for $n \geq N$. Hence

$$|a_n| \left| \frac{|b_n|}{1 + c_n^2} \right| < B\varepsilon,$$

for $n \geq N$. By the $K - \varepsilon$ principle,

$$\frac{a_n b_n}{1 + c_n^2} \to 0,$$

as $n \to \infty$.

5 (Extra credit). If $a_n \to L$, and b_n lies between a_n and a_{n+1}, prove $b_n \to L$.
Remark: Note that "between" does not tell you in which direction the inequalities go.

Solution Define two new sequences

$$m_n = \min(a_n, a_{n+1}), \quad M_n = \max(a_n, a_{n+1}).$$

We have

$$m_n \le b_n \le M_n.$$

Hence, by squeeze theorem, we just need to show that

$$m_n \to L, M_n \to L,$$

as $n \to \infty$. The latter follows from the definition of the limit. We know that given $\varepsilon > 0$ $|a_n - L| < \varepsilon$ when $n > N$. Therefore, using the same N we also have

$$|m_n - L| \le \max(|a_n - L|, |a_{n+1} - L|) < \varepsilon,$$

and

$$|M_n - L| \le \max(|a_n - L|, |a_{n+1} - L|) < \varepsilon.$$