REVIEW FOR MIDTERM II

1 MONOTONE SEQUENCES

Definition: A sequence (s_n) of real numbers is nondecreasing if:

$$s_n \leq s_{n+1} \quad \forall n \in \mathbb{N}$$

A sequence (s_n) of real numbers is nonincreasing if:

$$s_n \geq s_{n+1} \quad \forall n \in \mathbb{N}.$$

In particular, constant sequences, $s_n = c \ \forall n \in \mathbb{N}$, are always nonincreasing and nondecreasing.

Theorem: All bounded monotone sequences converge to a real number.

(Review proof in the book).

Proof shows:

- If (s_n) nondecreasing, then $\lim_{n \to \infty} s_n = \sup\{s_n, n \in \mathbb{N}\}$
- If (s_n) nonincreasing, then $\lim_{n \to \infty} s_n = \inf\{s_n, n \in \mathbb{N}\}$

theorem : i) If (s_n) is an unbounded nondecreasing sequence, then :

$$\lim_{n \to \infty} s_n = +\infty$$

ii) If (s_n) is an unbounded nonincreasing sequence, then :

$$\lim_{n \to \infty} s_n = -\infty$$

(2) LIMSUP and LIMINF

Definition : Let (s_n) be a sequence in $\mathbb{R}$.
We define :

$$\liminf_{n \to \infty} s_n = \lim_{N \to \infty} \left(\operatorname{Inf} \{ s_n, n > N \} \right)$$

$$\limsup_{n \to \infty} s_n = \lim_{N \to \infty} \left(\operatorname{Sup} \{ s_n, n > N \} \right)$$

- lim Inf and lim Sup <u>always</u> <u>exist</u>, even if (s_n) does not converge or diverges to $+\infty$ or $-\infty$.
- In general, $\liminf s_n \neq \inf\{s_n, n\in\mathbb{N}\}$, $\limsup s_n \neq \sup\{s_n, n\in\mathbb{N}\}$
- If (s_n) unbounded above, then $\limsup_{n\to\infty}(s_n) = +\infty$.

  If (s_n) unbounded below, then $\liminf_{n\to\infty}(s_n) = -\infty$.

<u>theorem</u>: Let (s_n) be a sequence in $\mathbb{R}$.

i) If $\lim_{n\to\infty} s_n = s$, where $s = -\infty$, or $+\infty$, or $s \in \mathbb{R}$, then:

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \lim_{n\to\infty} s_n = s$$

ii) If $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = s$, where $s = -\infty$, or $+\infty$, or $s \in \mathbb{R}$, then

$\lim_{n\to\infty} s_n$ exists and:

$$\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = s$$

7 CAUCHY SEQUENCES

Definition: A sequence (s_n) is called a Cauchy sequence if:

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n, m > N$, then $|s_n - s_m| < \varepsilon$.

Ex: $s_n = \frac{2n+1}{n}$.

- The condition $|s_n - s_m| < \varepsilon$ must be true for all pairs of integers $n, m > N$. It is not enough that it holds for n and $n+1$ for example.

theorem: A sequence is convergent if and only if it is a Cauchy sequence.

(Review proof in the book, along with proof of following Lemma:

Lemma: Cauchy sequences are bounded

- In showing that a sequence is Cauchy, can assume $n > m$.
- Sometimes, it may be useful to use the following summation formula: $\boxed{\sum_{k=1}^{n} r^k = \frac{1-r^{n+1}}{1-r}}$ $r \neq 1$

3 SUBSEQUENCES

Def: Let $(s_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. A subsequence of (s_n) is a sequence of form $\boxed{t_n = s_{n_k}}$, where

$$n_1 < n_2 < n_3 < n_4 \cdots < n_k < n_{k+1} < \cdots$$

- Recall (s_n) is the set of values of the function $s : \mathbb{N} \to \mathbb{R}$, $s_n = s(n)$. then (s_{n_k}) is the set of values of the composite function $s \circ \sigma(k)$, where $\sigma : \mathbb{N} \to \mathbb{N}$, $\sigma(k) = n_k$, is strictly increasing

Ex: $s_n = (-1)^n \log n \Rightarrow s_{2k} = (-1)^{2k} \log 2k = (\log k)^2$

Theorem 1: If the sequence (s_n) converges, then every subsequence converges to the same limit.

(Review proof in the book)

Theorem 2: For any sequence (s_n), there exists a monotone subsequence (s_{n_k}) such that

$$\lim_{k \to \infty} s_{n_k} = \liminf_{n \to \infty} s_n,$$

and a monotone subsequence (t_{n_k}) such that

$$\lim_{k \to \infty} t_{n_k} = \limsup_{n \to \infty} s_n.$$

Bolzano-Weierstrass theorem: Every bounded sequence has a convergent subsequence.

Remark: If (s_n) bounded, then $\liminf_{n \to \infty} s_n$ and $\limsup_{n \to \infty} s_n$ are real numbers.

So, the Bolzano-Weierstrass theorem follows from theorem 2.

Def: A subsequential limit of a sequence $(s_n) \subseteq \mathbb{R}$ is a real number, or $+\infty$, or $-\infty$, that is the limit of a subsequence of s_n (that is, $\exists (s_{n_k})$ such that $\lim_{k \to \infty} s_{n_k} = \alpha$)

Ex : $s_n = (-1)^n$. 1 is a subsequential limit as $1 = \lim_{k \to \infty} (-1)^{2k}$.

theorem 3 : Let S denote the set of all subsequential limits of a sequence (s_n).

then :

a) S is not empty.

b) $\sup S = \limsup_{n \to \infty} s_n$, $\inf S = \liminf_{n \to \infty} s_n$

c) $\lim s_n$ exists $\iff$ $S = \{\alpha\}$ where $\alpha = \lim s_n$.

- $\sup S \neq \sup \{s_n : n \in \mathbb{N}\}$
  $\inf S \neq \inf \{s_n : n \in \mathbb{N}\}$ in general.

In fact, $\inf\{s_n\} \leq \liminf_{n \to \infty} s_n \leq \limsup_{n \to \infty} s_n \leq \sup\{s_n\}$,

with strict inequality possible

(take, e.g., $s_n = (-1)^n + \frac{1}{n} \Rightarrow S = \{-1, 1\}$ but $\sup\{s_n\} = 2$, $\inf\{s_n\} = -1$).

- the set of all subsequential limits is closed, that is it contains all limits of sequences from S.

- the intervals $[a, b]$, $[a, +\infty)$, $(-\infty, b]$ are all closed sets.

  the intervals (a, b), $(a, +\infty)$, $(-\infty, b)$, $[a, b)$, $(a, b]$ are not closed.

<u>Thm 4</u> : Let (s_n) be any sequence of <u>nonzero</u> real numbers.
then :

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \le \liminf |s_n|^{1/n} \le$$

$$\limsup |s_n|^{1/n} \le \limsup \left|\frac{s_{n+1}}{s_n}\right| .$$

<u>Idea of proof</u> : If $\limsup \left|\frac{s_{n+1}}{s_n}\right| = L \Rightarrow \exists N \in \mathbb{N}$

$$\sup \left\{ \left|\frac{s_{n+1}}{s_n}\right| , n > N \right\} \le L$$

$\Rightarrow$ From $|s_n| = \left|\frac{s_n}{s_{n-1}}\right| \cdot \left|\frac{s_{n-1}}{s_{n-2}}\right| \cdot \dots \cdot \left|\frac{s_{N+1}}{s_N}\right| . |s_N|$,

it follows :

$$|s_n| \le L^{n-N} |s_N| \qquad n > N$$

or $$|s_n|^{1/n} \le L \left(\frac{|s_N|}{L^N}\right)^{1/n} = L a^{1/n}$$

a constant in n. take limit as $n \to \infty$.

Corollary: If $\lim_{n\to\infty} \left| \frac{s_{n+1}}{s_n} \right|$ exists, then also

$\lim_{n\to\infty} |s_n|^{1/n}$ exists and

$$\lim_{n\to\infty} \left| \frac{s_{n+1}}{s_n} \right| = \lim_{n\to\infty} |s_n|^{1/n}.$$

Ex: Find $\lim_{n\to\infty} \frac{1}{n} (n!)^{1/n} \Rightarrow \lim_{n\to\infty} \left| \frac{\frac{(1+n)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \right| =$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \left(\frac{\cancel{(n+1)}\, \cancel{n!}}{\cancel{(n+1)}\, \cancel{n!}} \right) = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}} \right)^n = \frac{1}{e}$$

(In fact, for n very large:

$$\boxed{n! \sim n^n \sqrt{2\pi n}\, e^{-n}}$$

which is known as Sterling's Formula)

- It is not true that if $\lim |s_n|^{1/n}$ exists, then $\lim \left| \frac{s_{n+1}}{s_n} \right|$ also exists.

4 SERIES

- Summation Notation:

$$\boxed{a_0 + a_1 + \cdots + a_{N-1} + a_N = \sum_{i=0}^{N} a_i}$$

$0, N$ summation limits, fixed

the index i is a <u>dumb index</u>:

$$\sum_{i=0}^{N} a_i = \sum_{k=0}^{N} a_k = \sum_{j=0}^{N} a_j$$

- <u>Infinite series</u>:

given a sequence of real numbers (a_n), $n \geq m$, form the sequence (s_n), $n \geq m$, of <u>PARTIAL SUMS</u>:

$$\boxed{s_n = \sum_{k=m}^{n} a_k}$$

<u>Def</u>: the series $\sum_{n=m}^{+\infty} a_n$ is said to <u>converge</u>, if the sequence of partial sums (s_n) converges to real number s. In this case, we define the value of $\sum_{n=m}^{+\infty} a_n$ to be s and write $\sum_{n=m}^{+\infty} a_n = s$.

Otherwise, we say that the series <u>diverges</u>.

- A series diverges if $\lim_{n\to\infty} s_n$ does not exists, or $\lim_{n\to\infty} s_n = +\infty$, in which case we write $\sum_{n=m}^{+\infty} a_n = +\infty$, or $\lim_{n\to\infty} s_n = -\infty$, in which case we write $\sum_{n=m}^{+\infty} a_n = -\infty$.

- the symbol $\sum_{n=m}^{+\infty} a_k$ has no meaning if $\lim_{n\to\infty} s_n$ does not exist.

- If a_n is always positive, $\underline{a_n \geq 0 \ \forall n \geq m}$, or always negative, $\underline{a_n \leq 0 \ \forall n \geq m}$, then the symbol $\sum_{n=m}^{+\infty} a_n$ <u>is always</u> <u>well-defined</u>:

$$a_n \geq 0 \quad \forall n \geq m \quad \Rightarrow \quad \sum_{n=m}^{+\infty} a_n = \begin{cases} S \in \mathbb{R}, \ S>0 \\ +\infty \end{cases}$$

$$a_n \leq 0 \quad \forall n \geq m \quad \Rightarrow \quad \sum_{n=m}^{+\infty} a_n = \begin{cases} S \in \mathbb{R}, \ S<0 \\ -\infty \end{cases}$$

$\Rightarrow$ For <u>any</u> sequence of real numbers (a_n), $\sum |a_n|$ always makes sense

Def: A series $\sum a_n$ is said to converge absolutely or be absolutely convergent if $\sum |a_n|$ converges.

- Absolutely convergent series always converge (by Cauchy criterion).

Special examples:

a) Geometric series: $\boxed{\sum_{n=0}^{+\infty} r^n}$, where $r \in \mathbb{R}$ fixed.

Since $\boxed{\sum_{k=0}^{n} = \frac{1-r^{n+1}}{1-r}}$ (if $r \neq 1$),

$\sum_{n=0}^{+\infty} r^n$ converges for $|r| < 1$ and

$$\boxed{\sum_{n=0}^{+\infty} r^n = \frac{1}{1-r}}$$

$\sum_{n=0}^{+\infty} r^n$ diverges if $|r| \geq 1$ (to $+\infty$ if $r \geq 1$), as $r^n \underset{n\to\infty}{\not\to} 0$ in this case.

b) $\boxed{\sum_{n=1}^{+\infty} \frac{1}{n^p}}$ $p>0$ fixed

this series converges if and only if $p>1$

(by the integral test). E.g.:

$$\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty \qquad \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Convergence Criteria

1) Cauchy Criterion: A series converges if and only if it satifies Cauchy Criterion, i.e.:

$$\boxed{\forall \varepsilon>0,\ \exists N \in \mathbb{N} \text{ such that if } n>m>N, \quad \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon.}$$

Proof: the series satisfies Cauchy Criterion if and only if the sequence of partial sums (S_n) is a Cauchy sequence. ∎

Cauchy Criterion is a necessary and sufficient condition for convergence. Note the absolute value is <u>outside</u> the summation symbol.

<u>Corollary</u> : If a series $\sum a_n$ converges, then

$$\boxed{\lim_{n \to \infty} a_n = 0}.$$

<u>Proof</u> : Apply Cauchy Criterion with $n = m$. ∎

- this is is a <u>necessary</u>, but <u>not sufficient</u> condition for convergence.

<u>Ex</u> : $\lim_{n \to \infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges to $+\infty$.

It is most useful to verify that a series <u>does not</u> converge:

<u>if</u> $\lim_{n \to \infty} a_n \neq 0$, <u>then $\sum a_n$ diverges</u>

<u>Ex</u> : $\sum_{n=0}^{+\infty} (-1)^n$.

(2) <u>Comparison test</u> : Let <u>$a_n \geq 0 \; \forall n$</u>. then:

i) $\sum a_n$ converges, $|b_n| \leq a_n \; \forall n \Rightarrow \sum b_n$ converges

ii) $\sum a_n = +\infty$, $b_n > a_n \; \forall n \Rightarrow \sum b_n = +\infty$

Idea of proof : i) Compare $\left|\sum_{k=m+1}^{n} b_k\right| \leq \sum_{k=m+1}^{n} |b_k|$ to $\sum_{k=m+1}^{n} a_k$ and apply Cauchy Criterion.

ii) Compare the partial sum of $\sum b_n$, t_n, to the partial sum of $\sum a_n$, s_n:

$\lim_{n\to\infty} t_n \geq \lim_{n\to\infty} s_n = +\infty$, since $t_n \geq s_n \ \forall n$. ∎

- It is not true that if $|b_n| \geq a_n \ \forall n$, $\sum a_n = +\infty \Rightarrow \sum b_n = +\infty$.

Ex: $b_n = (-1)^n$, $a_n = 1^n$, $\sum a_n = +\infty$, but $\sum (-1)^n$ does not exist.

Corollary : Absolutely convergent series converges.

- the behavior of a series $\sum a_n$ is determined by the behavior of the partial sums, s_n, for large n. We can change any finite number of terms a_n in a series without changing its behavior (but we will change the value of $\sum a_n$).

So, in the comparison test we can take $|b_n| \leq a_n$ or $b_n > a_n$ only for $n > N$.

(3) **Root test** : Let (a_n) be a sequence of real numbers and set:

$$\boxed{\alpha = \limsup_{n \to \infty} |a_n|^{1/n}}$$

i) $\sum a_n$ converges if $\alpha < 1$ (absolutely).

ii) $\sum a_n$ diverges if $\alpha > 1$.

iii) No information if $\alpha = 1$.

Idea of proof : Compare $\sum a_n$ to the geometric series $\sum r^n$, where r is close (but perhaps not equal) to α.

i) Pick $\varepsilon > 0$ such that $\alpha + \varepsilon = r < 1$ (possible since $\alpha < 1$). By the definition of limsup, $\exists\, N \in \mathbb{N}$ such that $\sup\{ |a_n|^{1/n} : n > N \} < \alpha + \varepsilon = r$

$\Rightarrow \sum_{n=N}^{+\infty} |a_n|$ converges, since $\sum_{n=N}^{+\infty} (\alpha+\varepsilon)^n$ converges.

ii) From theorem 2, $\exists\, (a_{n_k})$ s.t $\lim_{k \to \infty} a_{n_k} = \alpha > 1$

$\Rightarrow a_{n_k} > 1 \;\; \forall\, k > N$. Hence, $\lim_{n \to \infty} a_n \neq 0$.

iii) consider the examples $\sum \frac{1}{n}$ & $\sum \frac{1}{n^2}$.

∎

④ <u>Ratio Test</u> : the series $\sum a_n$, <u>$a_n \neq 0 \ \forall n$</u>,

i) converges (absolutely) if $\boxed{\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1}$

ii) diverges if $\boxed{\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1}$

iii) No information if $\boxed{\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 < \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$

<u>Idea of proof</u> : Reduce to the Root test using theorem 4. ∎

- Ratio test generally easier to use, but Root test stronger test:
  - the Ratio test may fail when the Root test does not. <u>Ex</u> : $\sum_{n=0}^{+\infty} 2^{(-1)^n - n}$
  - If the Root test fails, the Ratio test will too. this is true, e.g., when $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow \lim_{n\to\infty} |a_n|^{1/n} = 1$.