# REVIEW FOR MIDTERM 1

## MATH 312

## 1 NATURAL NUMBERS, INTEGERS

$\mathbb{N} = \{1, 2, 3, \ldots\}$ $\quad \mathbb{Z} = \{\ldots, -n, \ldots, -1, 0, 1, \ldots, n \ldots\}$

Peano axiom N5: $S \subseteq \mathbb{N}$, $1 \in \mathbb{N}$, $\{n \in S \Rightarrow n+1 \in S\}$
$\Rightarrow S = \mathbb{N}$.

N5 justifies Mathematical Induction:

$P_1, P_2, P_3, \ldots, P_n, \ldots$ collection of statements

All the $P_n$'s are true provided:

a) $P_1$ is true (Basis for induction)

b) $P_n$ true $\Rightarrow$ $P_{n+1}$ true (Induction Step).

## 2 RATIONAL NUMBERS

$\mathbb{Q}$ = set of equivalence classes $\frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$.

Say that $\frac{m}{n}$ equivalent to $\frac{k}{l}$ if $ml = kn$.

$\mathbb{Q}$ is an ordered field, with ordering $\leq$

$\mathbb{Q}$ is not complete, since

$$\sup\{r \in \mathbb{Q},\ r^2 \leq 2\} = \sqrt{2} \notin \mathbb{Q}.$$

the fact $\sqrt{2} \notin \mathbb{Q}$ is a consequence of the

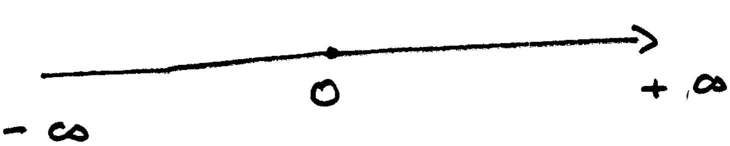
Rational Zeroes theorem: If $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ and $r \in \mathbb{Q}$ satisfies the polynomial equation:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \qquad n \geq 1,\ a_n \neq 0,\ a_0 \neq 0$$

then, if $r = \frac{p}{q}$, $q, p \in \mathbb{Z}$, $q \neq 0$, with $p$ and $q$ relatively prime, $q$ must divide $a_n$ and $p$ must divide $a_0$.

## ③ REAL NUMBERS

$\mathbb{R}$ can be seen as a directed line




$\mathbb{R}$ is an ordered field with ordering $\leq$.

$\mathbb{R}$ is complete.

If $S \subseteq \mathbb{R}$, non-empty set:

- $m = \text{Min}(S)$, the <u>minimum</u> of $S$, is the smallest element of $S$ (if it exists)
- $M = \text{Max}(S)$, the <u>maximum</u> of $S$, is the largest element of $S$ (if it exists)
- $a = \text{Sup}(S)$, the <u>supremum</u> of $S$, is the least upper bound, that is:
  - $a \geq s$, $\forall s \in S$
  - if $c < a$, then $\exists\, s \in S$ such that $a \geq s > c$.
- $b = \text{Inf}(S)$, the <u>Infimum</u> of $S$, is the largest lower bound, that is:
  - $b \leq s$, $\forall s \in S$
  - if $c > b$, then $\exists\, s \in S$ such that $b \leq s < c$.

<u>Completeness axiom</u>: Every non-empty set $S \subseteq \mathbb{R}$ that is bounded above has a supremum.

<u>Corollary</u>: Every non-empty set $S \subseteq \mathbb{R}$ that is bounded below has an infimum.

<u>Denseness theorem</u>: If $a, b \in \mathbb{R}$, $a < b$, then $\exists$ $r \in \mathbb{Q}$ such that $a < r < b$.

(Review proof of corollary 4.5 & theorem 4.7) 

Denseness of $\mathbb{Q}$ in $\mathbb{R}$ is a consequence of the <u>Archimedean Property</u>: If $a>0$, $b>0$, then $\exists\, n \in \mathbb{N}$ such that $na > b$.

(which follows from completeness axiom).

<u>Absolute Value</u>: $|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0 \end{cases}$

<u>Properties</u>: i) $|a| \geq 0$, $\forall\, a \in \mathbb{R}$

ii) $|ab| = |a||b|$, $a, b \in \mathbb{R}$

iii) $|a+b| \leq |a| + |b|$, $a, b \in \mathbb{R}$

Property iii) is called the <u>triangle Inequality</u>

<u>Distance</u> between two numbers $a, b \in \mathbb{R}$:

$$\text{dist}(a, b) = |a-b|$$

If $S \subseteq \mathbb{R}$ is unbounded above, by definition

$$\sup S = +\infty$$

If $S \subseteq \mathbb{R}$ is unbounded below, by definition

$$\inf S = -\infty$$

# 4 SEQUENCES

**Definition**: A sequence of real numbers is a function $s : \mathbb{N} \to \mathbb{R}$.

We write $s(n) = s_n$.

$(s_n) = \{ s_1, s_2, s_3, s_4, \dots, s_n, \dots \}$

The concept of limit tells us what the sequence does for large values of $n$.

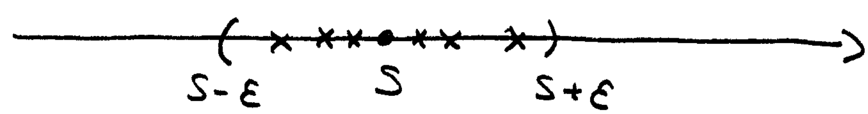
**Definition of Limit**: A sequence $(s_n)$ converge to $s \in \mathbb{R}$, $\lim_{n\to\infty} s_n = s$ or $s_n \xrightarrow[n\to\infty]{} s$, if:

$\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that if $n > N$, then $|s_n - s| < \varepsilon$ or

$$s - \varepsilon < s_n < s + \varepsilon.$$

- $\varepsilon$ represents an arbitrary small number.
- $\varepsilon$ is chosen first, $N$ is a function of $\varepsilon$
- Usually, $N$ becomes larger and larger as $\varepsilon$ becomes smaller and smaller.
- If limit exists it is unique (Review proof)

geometrically, $\lim_{n\to\infty} s_n = s$, if the elements $s_n$ of the sequence are getting closer and closer to $s$ as $n$ grows, that is, given an interval $(s-\varepsilon, s+\varepsilon)$ around $s$ on both sides, shrinking to the point $s$, we can always find all the elements of the sequence, $s_n$, starting from $N$, provided $N$ is large enough:




Limit Proofs using the definition:

- choose $\varepsilon$ first.
- See for which values of $n$, the inequality

  $$s-\varepsilon < s_n < s+\varepsilon \quad \text{is satisfied.}$$

  You will obtain $N$ as an expression involving $\varepsilon$.

# LIMIT THEOREMS

Theorem 1: Convergent sequences are bounded.

(Review proof in the book)

**theorem 2** : Let $(s_n), (t_n)$ be sequence of real numbers converging to $s, t \in \mathbb{R}$. Let $k \in \mathbb{R}$. then

i) $\lim_{n \to \infty} k s_n = k \lim_{n \to \infty} s_n = ks$

ii) $\lim_{n \to \infty} (s_n + t_n) = \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n = s + t$

iii) $\lim_{n \to \infty} (s_n t_n) = \left(\lim_{n \to \infty} s_n\right)\left(\lim_{n \to \infty} t_n\right) = st$

iv) If $\underline{s_n \neq 0}$ $\forall n$, $\underline{s \neq 0}$, then

$$\lim_{n \to \infty} \frac{1}{s_n} = \frac{1}{\lim_{n \to \infty} s_n} = \frac{1}{s}$$

$$\lim_{n \to \infty} \frac{t_n}{s_n} = \frac{\lim_{n \to \infty} t_n}{\lim_{n \to \infty} s_n} = \frac{t}{s}$$

( Review proofs of part iii))

**Special limits** : a) $\lim_{n \to \infty} \frac{1}{n^p} = 0$ if $p > 0$

b) $\lim_{n \to \infty} a^n = 0$ if $|a| < 1$

c) $\lim_{n \to \infty} n^{1/n} = 1$ d) $\lim_{n \to \infty} a^{1/n} = 1$ $a > 0$

- Unless specified otherwise, always try to use the special limits and the limit theorems in proving limits.

<u>Example</u> : Show $\lim\limits_{n \to \infty} \dfrac{n^3 + 4n}{2n^3 - 1} = \dfrac{1}{2}$

<u>Proof</u> : Write $\dfrac{n^3 + 4n}{2n^3 - 1} = \dfrac{\cancel{n^3}}{\cancel{n^3}} \cdot \dfrac{1 + \frac{4}{n^2}}{2 - \frac{1}{n^3}}$

So, $\lim\limits_{n \to \infty} \dfrac{n^3 + 4n}{2n^3 - 1} = \dfrac{1 + 4 \cdot \lim\limits_{n \to \infty} \frac{1}{n^2}}{2 - \lim\limits_{n \to \infty} \frac{1}{n^3}}$

using theorem 2, i), ii), iv)

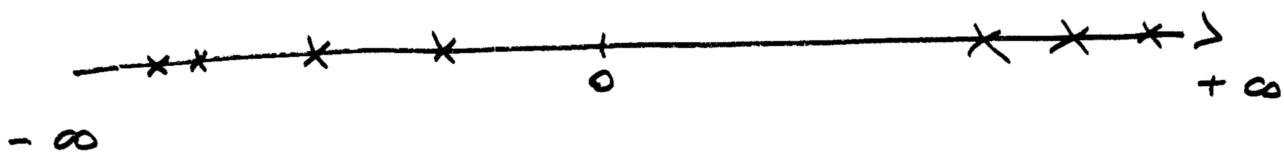
$= \dfrac{1 + 4 \cdot 0}{2 - 0} = \dfrac{1}{2}$ using special limit case a).

## <u>INFINITE LIMITS</u>

<u>Definition</u> : A sequence $(s_n)$ is said to diverge to $+\infty$, $\lim\limits_{n \to \infty} s_n = +\infty$ or $s_n \xrightarrow[n \to +\infty]{} +\infty$, if $\forall\, M > 0$, $\exists\, N \in \mathbb{N}$ such that if $n > N$, then $s_n > M$.

Similarly, a sequence $(s_n)$ is said to diverge to $-\infty$, $\lim_{n\to\infty} s_n = -\infty$ or $s_n \xrightarrow[n\to\infty]{} -\infty$, if: $\forall M>0$, $\exists N \in \mathbb{N}$ such that, if $n>N$, then $s_n < -M$.

- $M$ represents an <u>arbitrarily</u> <u>large</u>, positive number.
- $M$ is chosen first, $N$ is a function of $M$.
- there is no absolute value in the definition of infinite limit, "one-sided" inequalities, since we can only approach $+\infty$ from the left and $-\infty$ from the right:




- the limit theorems 1 & 2 CANNOT BE USED with infinite limits.

# LIMIT THEOREMS FOR INFINITE LIMITS

**theorem 3**: Let $(s_n), (t_n)$ be sequences of real numbers such that $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n = t$, with $t = +\infty$ or $t > 0$.

then: $\lim(s_n t_n) = +\infty$.

Symbolically: $+\infty \cdot +\infty = +\infty$

$+\infty \cdot t = +\infty$ if $t > 0$.

Similarly: $+\infty \cdot -\infty = -\infty$

$-\infty \cdot -\infty = +\infty$

$-\infty \cdot t = -\infty \quad t > 0$

$= +\infty \quad t < 0$

$+\infty \cdot t = -\infty$ if $t < 0$

- $+\infty \cdot 0$ or $-\infty \cdot 0$ is an INDETERMINATE FORM (that is, if $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n = 0$ then the value of $\lim_{n\to\infty} s_n t_n$ can be $\pm\infty$, $0$, or $t \in \mathbb{R}, t \neq 0$, depending on cases).

- Recall that $+\infty$ and $-\infty$ are SYMBOLS, NOT numbers.

**theorem 4** : Let $(s_n)$ be a sequence of positive real numbers. then:

$$\lim_{n\to\infty} s_n = +\infty \iff \lim_{n\to\infty} \frac{1}{s_n} = 0$$

- If $s_n < 0 \ \forall n$, then:

$$\lim_{n\to\infty} s_n = -\infty \iff \lim_{n\to\infty} \frac{1}{s_n} = 0$$

this theorem cannot be applied if $(s_n)$ has both positive and negative elements.

(Review proof of theorem 9.9 in book).

## Other Limit theorems for sequences

- **Squeeze Property** : If $(a_n), (b_n), (c_n)$ are sequences such that $a_n \leq b_n \leq c_n \ \forall n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then:

$$\lim_{n\to\infty} c_n = L$$

(this is Exercise 8.5a) )

- If $s_n \geq a$ $\forall n \geq N$ for some $N \in \mathbb{N}$, then $\lim_{n\to\infty} s_n \geq a$. Similarly, if $s_n \leq b$, $\forall n > N$, then $\lim_{n\to\infty} s_n \leq b$. (Exercise 8.9).

- Other special limits:

$$\boxed{\begin{aligned}
&\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a|<1 \\ 1 & \text{if } a=1 \\ +\infty & \text{if } a>1 \\ \text{DNE} & \text{if } a \leq -1 \end{cases} \\
&\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if } |a|\leq 1 \\ +\infty & \text{if } a>1 \\ \text{DNE} & \text{if } a<-1 \end{cases} \\
&\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad \forall a \in \mathbb{R}.
\end{aligned}}$$

(Exercise 9.13-14-15).