MATH 110: REVIEW PROBLEMS FOR MIDTERM 2

1. Differentiate $f(x) = (3x+1)^3(2x-1)^4$.

   Ans: $f'(x) = 9(3x+1)^2(2x-1)^4 + 8(3x+1)^3(2x-1)^3$.

2. Differentiate $f(x) = \sqrt{\dfrac{x+1}{x-1}}$.

   Ans: $f'(x) = -\dfrac{1}{(x-1)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}$.

3. Find the rate of change of $f(x) = \dfrac{x}{\sqrt{x^2-1}}$.

   Ans: $-\dfrac{1}{(x^2-1)^{\frac{3}{2}}}$.

4. Differentiate $f(x) = |x| = \sqrt{x^2}$.

   Ans: $f'(x) = \dfrac{|x|}{x}$.

5. Suppose $y = u^3 - 3u$, and $u = 2x+1$. Find $\dfrac{dy}{dx}$ when $x = 1$.

   Ans: 48

6. Suppose $y = \sqrt{u}$ and $u = \dfrac{x-1}{x+1}$. Find $\dfrac{dy}{dx}$ at $x = 2$.

   Ans: $\dfrac{\sqrt{3}}{9}$.

7. Find the rate of change of the marginal cost for a cost function of $C(x) = 200 + 50x - .01x^2$ dollars.

   Ans: $C''(x) = -.02$.

8. Evaluate the sixth derivative of $f(x) = ((x+1)^3 - 1)^2$.

   Ans: Since $f(x)$ is a 6^{th} degree polynomial, its derivative is the product $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$.

9. Find $f^{(6)}(x)$ for $f(x) = ((2x+1)^2 + 1))^3$.

   Ans: $2^6 6!$.

10. Find the fourth derivative of the function defined by $f(x) = \dfrac{x-2}{x+2}$.

    Ans: $y^{(4)} = -\dfrac{96}{(x+2)^5}$.

11. Use the second derivative test to show that every cubic ($y = ax^3 + bx^2 + cx + d \ \ (a \neq 0)$) has exactly one point of inflection. Find that inflection point in general terms.

    Ans: The second derivative is the linear function $6ax + 2b$; so that $x_0 = -b/3a$, is the point of inflection.

12. Show that $\lim_{x\to\infty} f(x) = \pm\infty$ and $\lim_{x\to-\infty} f(x) = \mp\infty$ for every cubic $f(x)$. Use this to explain why every cubic has at least one real root.

Ans: Since the cubic is a continuous function its graph can be drawn without removing pen from paper. But its graph goes from $\pm$ *the national debt* to $\mp$ *the national debt,* so that it must cross the $x-$axis at one or more places. Each of these crossing points is a root of the cubic.

13. How many relative extrema does a cubic whose derivative has two distinct real roots have? Explain your answer.

Ans: The graph of the derivative is a parabola which crosses the $x-$axis at two places, each of which is a relative extremum.

14. How many relative extrema does a cubic whose derivative has no real roots have? Explain.

Ans: If its derivative has no real roots, the cubic cannot have any critical numbers and therefore cannot have any relative extrema.

15. How many relative extrema does a cubic whose derivative has one real root have? Explain your answer.

Ans: None, since its derivative is a parabola which is tangent to the $x-$axis.

16. Suppose the first derivative of a function $f(x)$ is $f'(x) = (x-1)^3(x-2)^2(x-3)^3$. Identify those points x at which relative maxima and minima occur.

Ans: Relative maximum at the critical number $x = 1$, relative minimum at the critical number $x = 3$. No relative extremum occurs at the critical number $x = 2$.

17. Find all inflection points of $f(x) = x^4 - 2x^2 + 1$.

Ans: Two inflection points at $x = \pm 1/\sqrt{3}$, since $f''(x) = 12x^2 - 4$. The function is even and graphs as a big W with minima at $(\pm 1, 0)$ and local maximum at $(0, 1)$.

18. Suppose the second derivative of a function $f(x)$ is $f''(x) = (x-1)^3(x-2)^2(x-3)^3$. Determine all points of inflection.

Ans: Points of inflection occur at $x = 1$ and $x = 3$. No point of inflection occurs at $x = 2$.

19. Find the equation of the line tangent at $(1, 2)$ to the graph of the function $y = f(x)$ defined implicitly by $x^2y - x^2 + y^3 = 9$.

Ans: $2x + 13y - 28 = 0$

20. Find the equation of the line tangent at $(1, 1)$ to the graph of the function defined implicitly by $y^2 + xy + x^2 = 3$.

Ans: $x + y - 2 = 0$

21. Write the equation of the line tangent to the graph of the function $y = f(x)$, defined implicitly by $x^2y = 1$, at the point $(2, 1/4)$.

Ans: $x + 4y - 3 = 0$

22. Suppose Q and R are functions of t which are related by $\dfrac{10}{Q} + \dfrac{12}{R} = 5$. Find $\dfrac{dQ}{dt}$ when $Q = 5, R = 4$ and $\dfrac{dR}{dt} = 4$.

Ans: $\frac{dQ}{dt} = -\frac{15}{2}$

23. A rectangular piece of cardboard, 15 in. by 8 in., is to be made into a topless box by cutting a square from each corner and turning up the sides. Find the maximum volume one can obtain.

Ans: 90.74 cubic inches.

24. Use the method of differentials to estimate $\sqrt[6]{65}$.

Ans: $2\frac{1}{192}$

25. Use the method of differentials to estimate $\sqrt{63}$.

Ans: $7\frac{15}{16}$

26. A moving body travels according to the law $s = 16t^2 - \frac{2}{3}t + 1$, where s is the distance travelled in feet and t is the time in seconds. What is the acceleration (rate of change of velocity with respect to time) after 3 seconds?

Ans: 32 ft./sec^2.

27. A brick comes loose from the top of a 144 foot building. Its distance $s(t)$ (in feet) from the street at the time t (in seconds) is given by $s(t) = 144 - 16t^2$. What is the acceleration of the brick when it hits the ground?

Ans: -32 ft./sec.^{2}.

28. Graph $f(x) = -x^3 + 3x - 2$ noting all maxima, minima and points of inflection.

Ans: Relative minimum at $(-1, -4)$, relative maximum at $(1, 0)$; a point of inflection at $(0, -2)$; intercepts at $(0, -2)$ and $(1, 0)$.

29. Graph the function $f(x) = 3x^{2/3} - x$.

Ans: Intercepts: $(0, 0)$, $(27, 0)$, increases on the interval: $(0, 8)$, decreases: on the intervals $(-\infty, 0)$, and $(8, \infty)$, relative minimum at the cusp $(0, 0)$, relative maximum at $(8, 4)$, concave downward everywhere except $(0, 0)$ where the first (and hence the second) derivative is undefined. Note that $(8, 4)$ is a critical point (of the first kind) and $(0, 0)$ is a critical point (of the second kind).

30. Graph $f(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x$ noting all maxima, minima and points of inflection.

Ans: No relative extrema and a point of inflection at $x = -1$

31. Graph $f(x) = \dfrac{1}{x^2 + 1}$ noting all maxima, minima, intervals of increase and intervals of decrease.

Ans: Relative and absolute maximum, intercept at $(0, 1)$, increases on $(-\infty, 0)$, decreases on $(0, \infty)$, points of inflection at $x = \pm\frac{\sqrt{3}}{3}$, symmetric with respect to the y-axis

32. Graph $f(x) = x - \dfrac{1}{x}$ noting all extrema, vertical asymptotes, intercepts and intervals of increase.

Ans: Symmetric with respect to the origin; increasing everywhere except at 0 where it is not defined; no extrema; $x = 0$ is a vertical asymptote.

33. Graph $f(x) = x\sqrt{x - 1}$ noting all critical numbers as well as intervals where $f(x)$ increases and decreases, vertical asymptotes and intercepts.

Ans: The function is only defined for $x \geq 1$, $f(1) = 0$ and the function is increasing wherever it is defined.

34. Graph $f(x) = x^3 - 3x^2 + 3x - 2$ noting all critical numbers, extrema, inflection points as well as intervals where $f(x)$ increases and decreases.

Ans: The function can be rewritten as $f(x) = (x - 1)^3 - 1$ from which the graph easily follows. It increases on the entire real line and has intercepts at $(0, -2)$ and $(2, 0)$. It has an inflection point at $(1, -1)$.

35. Graph $f(x) = \dfrac{x+1}{x-1}$, noting all critical numbers, extrema, inflection points, intervals where $f(x)$ increases and decreases and asymptotes.

Ans: It has intercepts at $(-1, 0)$ and $(0, -1)$; horizontal and vertical asymptotes at $y = 1$ and $x = 1$ respectively; decreases on every interval which does not contain 1, concave down for $x < 1$ and concave up for $x > 1$.

36. Find y' if $y = x^{\sqrt{x}}$.

Ans: $y' = x^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \ln x + \frac{1}{\sqrt{x}} \right)$

37. Let u, v, w be differentiable functions of x. Use logarithmic differentiation to differentiate $y = \dfrac{uv}{w}$.

Ans: $y' = \frac{uv}{w} \left(\frac{u'}{u} + \frac{v'}{v} - \frac{w'}{w} \right)$

38. Let $P > 0,\ r > 0,\ m = 1, 2, \cdots$ Use compound interest considerations to verify that $P \left(1 + \dfrac{r}{m}\right)^m$ increases with m.

Ans: The formula given represents the return on an investment of P dollars for 1 year compounded m times a year at an interest rate of $i = 100r\%$. The larger m, the more the compounding. Thus the function given increases with m.

39. Determine the concavity of a polynomial over an interval (a, b) which is decreasing at an increasing rate

Ans: The word "increasing rate" implies y' increases, so that $y'' > 0$ on (a, b). Therefore the function is concave up on (a, b).

40. Find the rate of change of the derivative $f'(x)$ of the function $f(x) = \ln x$.

Ans: The problem merely asks for the second derivative of $f(x)$; $f''(x) = -\frac{1}{x^2}$.

41. Identify all relative extrema for $f(x) = \frac{x^2}{x-3}$.

a. $f(x)$ has a relative minimum at $x = 0$ and a relative maximum at $x = 6$.

b. $f(x)$ has a relative maximum at $x = 0$ and a relative minimum at $x = 6$.

c. $f(x)$ has a relative minimum at both $x = 0$ and $x = 6$ and a relative maximum at $x = 3$.

d. $f(x)$ has a relative maximum at both $x = 0$ and $x = 6$ and a relative minimum at $x = 3$.

e. The function has no relative extrema.

42. How many critical numbers does $f(x) = 3x^{1/3} - x$ have?

a. 0

b. 1

c. 2

d. 3

e. 4

43. How many real roots does the cubic $y = x^3 - 3x^2 + 1$ have? *Hint: Graph the function.*

a. 0

b. 1

c. 2

d. 3

e. 4

44. Suppose $f(x)$ is a function whose **derivative** is given by $f'(x) = x(x^2 - 1)^2$. How many relative extrema does its graph have?

a. 0

b. 1

c. 2

d. 3

e. 5

45. Locate all of the inflection points of $y = x^4 - 8x^3 + 24x^2 + 10x + 18$.

a. $x = 0$

b. $x = 2$

c. $x = -2$

d. $x = \pm 1$

e. The function has no inflection points.

46. Simplify $\ln\left(\frac{1}{\sqrt{e}}\right)$.

a. 1

b. $-e$

c. $\frac{1}{\sqrt{e}}$

d. $\sqrt{e}$

e. $-1/2$

47. Solve $2\ln e^x - \ln 1 + e^{\ln x^2} - \ln \frac{1}{e} = 0,$ for x.

a. 0

b. 1

c. -1

d. e

e. $\frac{1}{e}$

48. Every cubic $(y = ax^3 + bx^2 + cx + d, \ \ (a \neq 0))$ has exactly 1 inflection point.

a. True

b. False

49. If the derivative of a cubic $(y = ax^3 + bx^2 + cx + d, \ \ (a \neq 0))$ has exactly one real root then the cubic has exactly one relative extremum.

a. True

b. False