• **One Sided Limits.** One defines the *limit from the right* of a function $f(x)$ at a point a (not necessarily in the domain of $f(x)$ by $\lim_{x\to a^+} f(x) = L$ if $f(x)$ is *close* to L whenever x is *close* to but larger than a. Similarly, one defines $\lim_{x\to a^-} f(x) = L$ if $f(x)$ is *close* to L whenever x is *close* to but smaller than a. For example,

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1 \text{ while } \lim_{x\to 0^-} \frac{|x|}{x} = -1.$$

• **Two Sided Versus One Sided Limits.** For emphasis one frequently calls the limit defined previously *a two sided limit* and that defined above *a one sided limit.* Clearly, $\lim_{x\to a} f(x) = L$ if and only if

$$\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x).$$

Consequently, $f(x)$ is continuous at $x = a$ if and only if

$$\lim_{x\to a^+} f(x) = f(a) = \lim_{x\to a^-} f(x).$$

Problem. Find both of the one sided limits given by $\lim_{x\to 3^\pm} \frac{|x-3|}{x-3} + 2$ and use them to prove that the function $f(x) = \frac{|x-3|}{x-3} + 2$, cannot be defined at $x = 3$ in such a way as to make it continuous at that point. • **Limits at** $\pm\infty$. One extends the definition of limits by writing

$$\lim_{x\to\infty} f(x) = L,$$

if $f(x)$ can be made arbitrarily *close* to L for x sufficiently *large* (i.e. x is beyond the national debt). Analogously, one writes

$$\lim_{x\to-\infty} f(x) = L,$$

if $f(x)$ can be made arbitrarily *close* to L for x sufficiently *small* (i.e. x is to the left of the **negative** of the national debt).

• **Horizontal Asymptotes.** If either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ then one calls the horizontal line with equation $y = L$ a *horizontal asymptote* of the (graph of the) function. Thus a function can have at most two horizontal asymptotes. Rational functions (i.e. the quotient of two polynomials) can have at most one, and then only when the degree of the numerator polynomial is no greater than that of the denominator polynomial.

Problem. Find the horizontal asymptotes of

i) $f(x) = \dfrac{1-x^2}{1+x^2}$,

ii) $f(x) = \dfrac{1}{x} + 2$ and

iii) $f(x) = \dfrac{(x+1)^3}{1+x^2}$.