• **Continuity at a Point.** A function $f(x)$ is *continuous* at a if the following three conditions are valid.

i) The function is defined at a. That is, a is in the domain of definition of $f(x)$.

ii) $\lim_{x\to a} f(x)$ exists.

iii) $\lim_{x\to a} f(x) = f(a)$.

Any function which fails any one of the above three conditions at a point a is said to be *discontinuous* at a.

The function

$$f(x) = \frac{x^2+x}{x^2-x}$$

fails i) but satisfies ii) for $a = 0$. The function $f(x) = \sqrt{x}$ fails ii) but satisfies i) for $a = 0$. The function

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases},$$

satisfies i) and ii) but fails iii) for $a = 0$.

• **Continuity Theorems.**

i) Let $f(x)$ and $g(x)$ be functions which are continuous at $x = a$. Then $f(x) \pm g(x)$ and $f(x)g(x)$ are both continuous at a. Moreover if $g(a) \neq 0$ then $\dfrac{f(x)}{g(x)}$ is also continuous at a.

ii) If $f(x)$ is continuous at a, then every translate $f(x+x_0)+y_0$, is continuous at the point $a + x_0$.

The proof of i) is a direct consequence of the second theorem on limits stated in the previous lecture.

• **Continuity on an Interval.** A function which is continuous at every point of an open interval I is called *continuous on* I.

Think of a continuous function on I as one that can be drawn over I without removing your chalk from the blackboard. Polynomials and rational functions are all continuous on each interval contained in their domain of definition.

Problems. Decide where each of the following functions are continuous and discontinuous.

i) $f(x) = |x|$, ii) $f(x) = |x+2| - 4$, iii) $f(x) = x$,iv) $f(x) = x^2 - 1$, v) $f(x) = \dfrac{x^2+1}{x^2-1}$ and

$$\text{vi) } f(x) = \begin{cases} x^2+4x+5 & \text{if } x \leq -2 \\ 2x+5 & \text{if } x > -2 \end{cases}.$$