• **Definition of a limit** Let $f(x)$ be defined at every point, except possibly a, on some interval $(a-\epsilon, a+\epsilon)$ containing a. Then

$$\lim_{x\to a} f(x) = L$$

means $f(x)$ is arbitrarily *close* to L whenever x is sufficiently *close* to but not equal to a. The above definition is only intuitive because the word *close* is undefined. However it should be clearly understood that $a-\epsilon$ is just as close to a as is $a+\epsilon$. That is, no preference is given between *close* from the right and *close* from the left. Moreover the value of the function at the point a has nothing to do with the limit. In fact, the point a need not even belong to the domain of $f(x)$.

Examples: i) $\lim_{x\to -1} f(x) = \begin{cases} x^2+2x+3, & \text{for } x > -1 \\ x+3, & \text{for } x < -1 \end{cases} = 2$, even though $f(-1)$ is not defined. It's what happens *close* to 1 that counts, not the actual value $f(1)$.

ii) $\lim_{x\to 1} \sqrt{x-1}$ does not exist since $f(x)$ is not defined anywhere to the left of 1.

• **Limit theorems.**

a) If $f(x) = C$, where C is a constant then $\lim_{x\to a} f(x) = C$.

b) If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then each of the following is true.

i) $\lim_{x\to a} (f(x) \pm g(x)) = L \pm M$.

ii) $\lim_{x\to a} (f(x)g(x)) = LM$.

iii) $\lim_{x\to a} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{L}{M}$, provided $M \neq 0$.

• **Limits of Polynomials and Rational Functions.** Recall that a polynomial is a function of the form $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, and a rational function is the quotient of two polynomials. If $f(x)$ is a rational function, the above theorem then implies

$$\lim_{x\to a} f(x) = f(a),$$

provided that neither the numerator nor the denominator polynomial has a zero at $x = a$. If one does have a zero at $x = a$, then, by cancellation, we can write

$$f(x) = (x-a)^m \frac{p(x)}{q(x)}, \text{ for all } x \neq a,$$

where a is not a root of either of the polynomials $p(x)$ or $q(x)$ and m is an integer. If m is a positive integer this limit is clearly zero. If $m = 0$, (i.e. the numerator and denominator polynomials have roots at a of the same order) then $\lim_{x\to a} f(x) = \dfrac{p(a)}{q(a)}$. The case for m negative will be discussed later.

Examples: i) $\displaystyle\lim_{x\to -1} \left(\frac{(x+2)(x^2+2x+1)}{(x+1)^2}\right) = \lim_{x\to -1} (x+2) = 1.$

ii) $\displaystyle\lim_{x\to -1} \left(\frac{(x+2)(x^2+2x+1)}{(x+1)}\right) = \lim_{x\to -1} (x+2)(x+1) = 0.$