• **The Graph of a Function.** The set of all pairs $(x, f(x))$ is called the *graph* of a function $f(x)$. These pairs can be uniquely plotted obtaining a curve in the Cartesian plane. Such a curve satisfies the *vertical line criteria,* namely, that every vertical line intersects this graph at no more than one point. Note that the graph of the circle with center at $(0,0)$ and radius 1 as defined algebraically by $x^2 + y^2 = 1$ does **not** satisfy this vertical line criteria, and hence is **not** the graph of a function.

• **Even and Odd Functions.** A function f such that $f(x) = f(-x)$ for all x in its domain of definition, is called *even.* If $-f(x) = f(-x)$ (or equivalently $f(x) = -f(-x)$), the function is called *odd.* It follows that even functions are those whose graphs are *symmetric* with respect to the $y-$axis while odd functions are those whose graphs are symmetric with respect to the origin.

Problem Decide if each of the following functions is even, odd or neither.

i) $f(x) = x^n \ (n = \pm 1, \pm 2, \cdots)$

ii) $f(x) = |x^n| \ (n = \pm 1, \pm 2, \cdots)$ Recall that $|x| = \sqrt{x^2} = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{otherwise.} \end{cases}$

iii) $f(x) = x^3 + x,$

iv) $f(x) = x^4 + x^2 + 17$

v) $f(x) = x^3 + 2x^2$

• **Translates of a Function.** A function of the form $g(x) = f(x) + c \ (c \neq 0)$ is called a *vertical translate* of the function $f(x)$. The graph of $g(x)$ is simply the graph of $f(x)$ moved up (or down if $c < 0$) by c units. A function of the form $g(x) = f(x + c) \ (c \neq 0)$ is called a *horizontal translate* of the function $f(x)$. The graph of $g(x)$ is simply the graph of $f(x)$ moved to the right if $c < 0$ (or the left if $c > 0$) by $|c|$ units.

• **Symmetries of a Function.** Under translation, an axis of symmetry or a point of symmetry translates to an axis or point of symmetry. Specifically, the graph of $f(x)$ is *symmetric with respect to the vertical line* $x = c$ if its horizontal translate $f(x + c)$ is an even function. The vertical line $x = c$ is called an *axis of symmetry.* For example, the graph of $f(x) = x^2 - 2x + 2 = (x - 1)^2 + 1$ is symmetric with respect to the vertical line $x = 1$ since $f(x + 1) = x^2 + 1$ is even. More subtly, completing the square of the general quadratic function $f(x) = ax^2 + bx + c$ gives:

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a}\right).$$

From this, it follows that the graph of every quadratic $f(x)$ is but a translate of the graph of the graph of $f(x) = ax^2$. It is symmetric with respect the vertical line $x = -\frac{b}{2a}$. This graph is called a *parabola.* The point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right) = \left(-\frac{b}{2a}, f(-\frac{b}{2a})\right)$, is called the *vertex* of the parabola. It is the highest or lowest point on the graph of the parabola depending on whether a is negative or positive, respectively.

The graph of a function $f(x)$ is *symmetric with respect to a point* (c, k) if its translate $f(x + c) - k$ is an odd function. For example the graph of $f(x) = (x + 1)^3 + 2$ has $(-1, 2)$ as a point of symmetry since $f(x - 1) - 2 = x^3$ is an odd function. Graph this function by first graphing $f(x) = x^3$.

If $f(x) = -g(x)$ then each function $f(x)$ or $g(x)$ is called a *reflection* of the other.

Problems. Examine the following functions for symmetries and graph each.

i) $f(x) = x^2 + 4x - 2$, (*Hint: complete the square*)

ii) $f(x) = x^3 - 3x^2 + 3x - 1$ (*Hint: factor*)

iii) $f(x) = |x + 4| - 2$,

iv) $f(x) = \frac{|x|}{x}$,

v) $f(x) = \frac{1}{x}$,

vi) $f(x) = \frac{2x+1}{x-1}$ (*Hint: use long division*)

vii) $f(x) = \sqrt{x^2 + 4x + 4}$. (*Hint: complete the square under the radical*)

Problem. Use geometry (not algebra) to find all points of symmetry for the line $y = mx + b$.