• **Logarithmic Differentiation of positive functions.** Recall $D_x \ln x = 1/x$ for $x > 0$. Thus if u is a positive differentiable function of x, the chain rule implies

$$D_x \ln u = \frac{u'}{u}.$$

Rearranging this formula, one obtains the logarithmic differentiation formula

$$u' = uD_x \ln u.$$

• **Applications of Logarithmic Differentiation.**

i) Differentiation of the exponential (previously derived): Setting $u = b^x$ where $0 < b \neq 1$, then a direct application of the logarithmic differentiation formula reveals $D_x b^x = b^x \ln b$, which we've already observed.

ii) Let $y = u^v$, where u and v are a differentiable functions with $u > 0$. A direct application of the logarithmic differention formula shows:

$$y' = u^v \left(u' \frac{v}{u} + v' \ln u \right),$$

which reduces to the familiar power rule $vu^{v-1}u'$ when v is a constant.

iii) Extended product rule: Let u, v and w be (positive) differentiable functions. If $y = uvw$ then the logarithmic differention formula easily implies $y' = u'vw + uv'w + uvw'$, which we've observed directly from the product rule before. Equivalently,

$$y' = uvw \left(\frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w} \right).$$

iv) Extended quotient rule: More generally, if p, q and r are also (positive) differentiable functions and if $y = \frac{uvw}{pqr}$, then

$$y' = \frac{uvw}{pqr} \left(\frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w} - \frac{p'}{p} - \frac{q'}{q} - \frac{r'}{r} \right).$$

Setting $p = q = r = 1$, we obtain iii) above.

iv) Logarithmic differention, while not necessary, can often be used to simplify differention of rather complicated expressions. For example, if

$$y = \sqrt{(x^2 - 4)\sqrt{2x + 1}},$$

then an application of the logarithmic differention formula shows,

$$y' = \sqrt{(x^2 - 4)\sqrt{2x + 1}} \left(\frac{x}{x^2 - 4} + \frac{1}{4x + 2} \right).$$

Problem. Use logarithmic differentiation to verify each of the above. Establish each of the formulas iii) through v) without appealing to logarithmic differentiation.

• **Logarithmic Differentiation of non positive functions.** Note that $D_x \ln |x| = 1/x$. for $x \neq 0$. Thus if u is an arbitrary non zero differentiable function of x, the chain rule implies

$$D_x \ln |u| = \frac{u'}{u}.$$

Rearranging this formula gives the following more general version of the logarithmic differention formula:

$$u' = u D_x \ln |u|.$$

Problem. Use logarithmic differentiation to establish formulas iii) and iv) for functions which are nonzero but not necessarily positive.