• **Exponential Growth and Decay.** The function $P(x) = P_0 b^x$ is called *exponential growth* or *exponential decay* according as $b > 1$ or $0 < b < 1$, respectively. In the former case it is an increasing function, whereas in the latter it is a decreasing function. Note that $P(0) = P_0$.

• **Inverse Functions.** Prior to defining the logarithm, we need to introduce the notion of an inverse function. A functions $g(x)$ defined on the range of a function $f(x)$ is called the *inverse* of $f(x)$ if $g(f(x)) = x$ for all x in the domain of $f(x)$. Briefly, the inverse $g(x)$ is that function which *undoes* what $f(x)$ *does.* The inverse of $f(x)$ can always be defined on either an interval where $f(x)$ increases or an interval where $f(x)$ decreases. It is constructed for $y = f(x)$ by interchanging the variable y and x (this is the *undoing* part) and then solving for y. This means that the graph of the inverse can be obtained from the graph of $f(x)$, by interchanging the y-axis with the x-axis, or simply flipping the graph of $f(x)$ about the line $y = x$. If $g(x)$ is the inverse of $f(x)$, then it follows that $f(x)$ is the inverse of $g(x)$. That is $g(x)$ also satisfies the functions equation, $f(g(x)) = x$ for all x in the domain of $g(x)$.

Problem. Find and graph the inverse, $g(x)$, of each of the following functions, $f(x)$. Show that each satisfies the functional equation $g(f(x) = x$.

i) $f(x) = x^3$
ii) $f(x) = \sqrt[3]{x}$
iii) $f(x) = x^2$ for $x > 0$
iv) $f(x) = \sqrt{x}$
v) $f(x) = \dfrac{1}{x}$

• **The Logarithm.** One defines $y = \log_b x$ as the unique solution y to the equation

$$b^y = x.$$

Thus the $\log_b x$ is that unique function satisfying $b^{\log_b x} = x$, for $x > 0$. The number b is called the base of the logarithm. When the base is e then $\log_e x$ is also denoted by $\ln x$ and is called the *natural logarithm.* It is convenient also to consider logarithms to the base 10 or so called *common logarithms.*

• **Properties of the Logarithm.** i) The domain of the function $\log_b x$ is the positive real numbers and its range is all of the real numbers.

ii) If $b > 1$ then $\log_b x$ is an increasing function such that $\lim_{x\to 0} \log_b x = -\infty$ and $\lim_{x\to\infty} \log_b x = \infty$.

If $0 < b < 1$ then $\log_b x$ is a decreasing function such that $\lim_{x\to 0} \log_b x = \infty$ and $\lim_{x\to\infty} \log_b x = -\infty$.

iii) If u and v are both positive and x is any real number then

$\log_b 1 = 0$, and $\log_b b = 1$,
$\log_b uv = \log_b u + \log_b v$,
$\log_b \frac{u}{v} = \log_b u - \log_b v$ and
$\log_b u^x = x \log_b u$.

Problem. Evaluate each of the following:

i) $\log_{10} 100$
ii) $\log_{10} .001$
iii) $\ln \sqrt{e}$
iv) $\log_{10} 25 + \log_{10} 4$
v) $\log_{\frac{1}{2}} 32$

Problem. Graph $f(x) = \log_b x$ for $b > 1$ and $0 < b < 1$.

Problem. Use natural logarithms to find k in terms of b so that $b^x = e^{kx}$ if $0 < b \neq 1$. Show that $k > 0$ if $b > 1$ and $k < 0$ if $0 < b < 1$.

The number k is called the *growth constant* if $k > 0$ and the *decay constant* if $k < 0$. More generally, if the variable t is time and if $P(t) = P_0 b^{kt}$ for $P_0 > 0, 0 < b \neq 1$ then k is called a *growth constant (with respect to b)* if $b^k > 1$ and a *decay constant* if $b^k < 0$.

In the case of growth, the *doubling time* is that unique t_d satisfying $P(t_d) = 2P_0$. For a decay function the *half life* is that unique t_h satisfying $P(t_h) = \frac{1}{2}P_0$.
Problem. Find the half life of each of the following.
i) $P(t) = \frac{P_0}{2^t}$.
ii) $P(t) = \frac{P_0}{3^t}$.
iii) $P(t) = P_0 e^{kt}$ where $k < 0$.

• **Important.** As an aid to understanding the properties of logarithms, strive to do all the problems (odd and even) in the text on Page 272 without pencil and paper. You might consider putting each of these on separate flash cards with the answers on the reverse side of the problem for your study.